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ANALYSIS OF NUCLEAR REACTOR SYSTEMS BY USE OF THE INTEGRATION METHOD

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Summary

The integration method is applied to the analysis of nuclear reactor systems. For second order systems this method is the same and even better than the isocline method; for third order systems, it offers some advantages in finding the characteristics of trajectories in state-space. Since stability as well as transient response characteristics can be realized directly from trajectories, the integration method is useful for system analysis.

1. Introduction

In the analysis of control systems, the phase-plane method is very useful. It gives the overall phase portrait of a system, from which the stability and response characteristics can be realized directly.[1] Unfortunately, however, this method can only be applied to second order systems. It becomes very complex for third order systems.[2]

The state-variable method is very powerful for finding system trajectories with given initial conditions. But it does not give an overall view about the characteristics of a system unless a great number of trajectories are calculated and plotted.

For stability analysis of nonlinear systems, the most powerful method is, up to now, the second method of Liapunov, which gives sufficient conditions for system stability and some information about the time domain response.[1,3] From the topological viewpoint, the second method of Liapunov can give a region in a phase-plane or phase-space such that all the trajectories in this region will approach the singular point asymptotically or with a limit cycle. The disadvantage of the second method of Liapunov is that there is no definite way of finding a Liapunov function and that the information obtained is relatively rough in comparison with a phase-portrait.

In short, the best method for the analysis of a second order system is to plot a phase portrait.[4,5] Thus, to analyze a third order system by finding its phase-portrait in a three dimensional space should be of interest to engineers.

The pioneer work of studying the trajectories of third order Nuclear reactor control systems was given by H.B. Smets.[6] In Smets' work several planes have been used for studying the characteristics in a certain region in a three dimensional phase-space. In addition, some special surfaces which contain a whole set of trajectories have been shown.

Another work dealing with trajectories of third order nonlinear systems was done by Chu and Han,[7] in which a method termed the "integration method" has been proposed. The integration method is similar to the method proposed by Nagaraja and Chalam.[8] The present work may be considered an extension of the works of references 6 to 8 with emphases on the analysis of nuclear reactor systems. First, a review of the integration method is given in the following section.

2. The integration method

Assume that a system is represented by a matrix equation as

$$[\dot{X}] = [A][X] \qquad (1)$$

where [X] and $[\dot{X}]$ are vectors, and [A] is a square matrix with state variables and constants as its elements. Eq.(1) can be rewritten as

$$[G] = [\dot{X}] - [A][X] = 0 \qquad (2)$$

where [G] is the vector with elements

$$g_i = \dot{x}_i - \sum_{j=1}^{n} a_{ij} x_j = 0 \qquad (3)$$

Multiplying Eq.(3) by x_k (k=1,2,..., n), yields

$$x_k g_i = 0 \qquad (4)$$

Integrating Eq.(4) with respect to time, one has

$$E_1 + \int(-\dot{E}_1)\, dt = 0 \qquad (5)$$

where E_1 consists of all the terms integrable to the closed forms, and $\dot{E}_1$ is the time derivative of E_1. If a square matrix B is selected with its elements representing the manipulation of the components of Eq.(4), then one has

$$\int [X]^T [B][G]\, dt = E_2 + \int(-\dot{E}_2)\, dt = 0 \qquad (6)$$

In addition, Eq.(2) can be integrated directly to give

$$\sum_{i=1}^{n} \int g_i\, dt = E_3 + \int(-\dot{E}_3)\, dt = 0 \qquad (7)$$

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or to write an equation such as

$$\int \frac{1}{x_i} g_i \, dt = E_4 + \int (-\dot{E}_4) \, dt = 0 \qquad (8)$$

to generate the required functions E_4 and $\dot{E}_4$.

All the above given functions E_i (i= 1,2,3,4,....) and any of the combinations of E_i are called the integration functions or simply the E-functions.[1] In case an E-function can meet the requirements that a Liapunov function does, it is a Liapunov function.[3,7,8] But the main objective of the integration method is to find the characteristics of trajectories in a state-plane or state-space. Once the trajectories in a state-space are found, the system characteristics can be realized directly. This will be shown in the following section.

3. Applications of the Integration Method to Second Order Nuclear Reactor Systems

Consider a nuclear reactor with effective-life-time model and with constant power removal.[9] The dynamic equations are

$$\dot{n} = -\left(\frac{\alpha}{\ell}\right) T n \qquad (9)$$

$$\dot{T} = K_0 (n - n_0) \qquad (10)$$

There is one equilibrium point at $n=n_0$ and $T=0$. Introducing the new state variables

$$x_1 = \frac{(n-n_0)}{n_0} \qquad (11)$$

and

$$x_2 = \left(\frac{\alpha}{\ell K_0 n_0}\right)^{\frac{1}{2}} T \qquad (12)$$

one has

$$n = n_0 x_1 + n_0 \qquad (13)$$

and

$$T = \left(\frac{\alpha}{\ell K_0 n_0}\right)^{-\frac{1}{2}} x_2 \qquad (14)$$

Choosing the values of the parameters as

$$K_0 = 0.00001 \ ^\circ C/watt\text{-}sec, \ n_0 = 10^5 \ watt,$$

$$\alpha = 0.0001 \ ^\circ C^{-1} \ and \ \ell = 0.1 \ sec,$$

Eqs.(9) and (10) become

$$\dot{x}_1 = -0.0316 \, x_1 x_2 - 0.0316 \, x_2 \qquad (15)$$

and

$$\dot{x}_2 = 0.0316 \, x_1 \qquad (16)$$

respectively. The new equilibrium point is the origin of the state-plane (x_2 versus x_1 plane). By choosing

$$[B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (17)$$

the E-function and its first time derivative are

$$E_1 = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} x_2^2 \qquad (18)$$

and

$$\dot{E}_1 = -0.0316 \, x_1^2 x_2 \qquad (19)$$

respectively. Note that, for various values of E_1, Eq.(18) represents a family of concentric circles as shown in Fig.1, where the contours for $\dot{E}_1=0$, i.e., the x_1 and x_2 axes, are also shown. This result shows that all the trajectories of the system, while crossing the x_1 and x_2 axes, are tangent to the circles difined by the E_1-functions. Although the directions of the trajectories in any place other than the x_1 and x_2 axes can not be clearly defined, their relations with respect to the E_1-circles can be defined by the polarities of $\dot{E}_1$. For example, at point A in the first guadrant of Fig.1, the trajectory is going into the circle of $E_1=8$ (because $\dot{E}_1<0$) while at point B in the fourth guadrant the trajectory is going out from the circle (because $\dot{E}_1>0$). In order to know more about the characteristics of trajectories, additional E and $\dot{E}$-functions should be used. For

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (20)$$

one has

$$E_2 = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} x_2^2 - x_1 x_2 \qquad (21)$$

and

$$E_2 = 0.0316 (x_1^2 + x_1^2 x_2 - x_1 x_2^2 - x_2^2) \qquad (22)$$

The contours for various values of E_2 and $\dot{E}_2=0$ are plotted in Fig.2, where the regions for positive or negative $\dot{E}$ are also indicated. Same as before, the trajectories are parallel to the E-contours while crossing the contours of $\dot{E}_2=0$. But note that the line (A) should be excluded because it represents the minimum value of E_2 (i.e. $E_2=0$), and that the polarity of $\dot{E}_2$ are different on each side of line A; therefore, the trajectories will cross line A instead of staying on it.

In addition, by direct integration of Eqs.(15) and (16), one has

$$E_3 = x_1 + x_2 \qquad (23)$$

and

$$\dot{E}_3 = 0.0316 (x_1 - x_1 x_2 - x_2) \qquad (24)$$

Similarly, one may choose

$$E_4 = x_1 + 3x_2 \tag{25}$$

and

$$E_5 = x_1 - 3x_2 \tag{26}$$

then one has

$$\dot{E}_4 = 0.0316(3x_1 - x_1x_2 - x_2) \tag{27}$$

and

$$\dot{E}_5 = 0.0316(-3x_1 - x_1x_2 - x_2) \tag{28}$$

respectively. Finally, all the contours of the E-functions and the functions of $\dot{E}_i=0$ (i=1,2,3,4,5) are plotted in one plane as shown in Fig.3. This can be done easily by a computer with a plotter. Following the slope marks in Fig.3, the phase portrait of the system can be sketched.

Note that, letting $E=x_1+Nx_2$ (where N is a constant), the integration method is exactly the isocline method. However, the integration method has at least four advantages as stated in the following:

The first advantage of the integration method is that it can provide the contours for $E=ax_1^2+bx_1x_2+cx_2^2$ (where a, b, and c are constants) and for $\dot{E}=0$; thus it is more flexible than the isocline method.

Second, the integration method gives the direction of the trajectory at any point in the state-plane by checking with all the polarities of the $\dot{E}$-functions at that point. For example, the trajectory passing through the point P in Fig.3 is from P to Q but not from P to R, since only the direction from P to Q can meet all the requirements defined by the polarities of the $\dot{E}$-functions.

Third, for certain phase portraits, the integration method can give the equation of the trajectories, which is the E-function with its first time derivative always equal to zero. In other words, the trajectory itself is the contour of $\dot{E}=0$. For example, for the system considered, by the manipulations,

$$\frac{1}{(x_1+1)} \times \text{Eq.(15)},$$

$$x_2 \times \text{Eq.(16) and } 1 \times \text{Eq.(15)},$$

the results are

$$\frac{\dot{x}_1}{x_1+1} + 0.0316\, x_2 = 0 \tag{29}$$

$$x_2\dot{x}_2 - 0.0316\, x_1x_2 = 0 \tag{30}$$

$$\dot{x}_1 + 0.0316\, x_1x_2 + 0.0316\, x_2 = 0 \tag{31}$$

Respectively. Adding Eqs.(29) to (31) together and taking the integration with respect to time, one has

$$-\ln(x_1+1) + \tfrac{1}{2}x_2^2 + x_1 = 0 \tag{32}$$

Therefore, the E-function is

$$E = -\ln(x_1+1) + \tfrac{1}{2}x_2^2 + x_1 \tag{33}$$

and the $\dot{E}$-function is

$$\dot{E} = 0 \tag{34}$$

Since the region for $x_1<-1$ has no physical meaning, Eq.(33) represents all the trajectories in the useful region of the state-plane. For various values of E, the trajectories are plotted as shown in Fig.3.

Note that the method of finding the E-function with $\dot{E}=0$ is to formulate an E-function containing all the terms which are integrable, and then to formulate additional E-functions to cancel all the terms in $\dot{E}$, i.e., to make $\dot{E}=0$.

The fourth advantage of the integration method is that it can be applied to third order systems as shown in the next section.

Before the ending of this section, it is interesting to consider another second order nonlinear system analyzed in references 1 and 9. The system equations are

$$\dot{x}_1 = -2x_1 + x_1x_2 \tag{35}$$

$$\dot{x}_2 = -x_2 + x_1x_2 \tag{36}$$

There are two equilibrium points at (0,0) and (1,2), respectively. Using

$$[B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{37}$$

and

$$[B] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{38}$$

The results are

$$E_1 = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 \tag{39}$$

$$\dot{E}_1 = -2x_1^2 - x_2^2 + x_1x_2^2 + x_1^2x_2 \tag{40}$$

$$E_2 = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + x_1x_2 \tag{41}$$

$$\dot{E}_2 = -2x_1^2 - x_2^2 + 3x_1x_2 \tag{42}$$

By direct integration, one has

$$E_3 = x_1 + x_2 \tag{43}$$

$$\dot{E}_3 = -2x_1 - x_2 + 2x_1x_2 \tag{44}$$

The contours for E_1, E_2, E_3 and $\dot{E}_1=\dot{E}_2=\dot{E}_3=0$ are plotted in Fig.4, where some of the

system trajectories are also plotted. It is interesting to note that the trajectories can be sketched approximately by use of these three sets of E-contours and the polarities of $\dot{E}_1$, $\dot{E}_2$, and $\dot{E}_3$.[1]

4. Application of the Integration Method to a Third Order Nuclear Reactor System

In this section, an adbatic model of a third order nuclear reactor is considered. Two approaches for applying the integration method are presented. The first approach is to formulate an E-function with its first time derivative equal to zero; the second approach is to set up several E-functions and to find the directions of trajectories in state-space.

(1) The first approach

Consider the dynamic equations of the adbatic model as

$$\dot{n} = \frac{1}{\ell}(k_0 - r\,T - \beta)n + \bar{\lambda}C \tag{45}$$

$$\dot{C} = \frac{\beta}{\ell}\,n - \bar{\lambda}C \tag{46}$$

$$\dot{T} = a\,n \tag{47}$$

where n is the reactor power, ℓ is the neutron generation time, k_0 is the constant amount of reactivity necessary to maintain operating equilibrium at a given flux level, r is the temperature coefficient of feedback reactivity. β is total fraction of delayed neutrons, $\bar{\lambda}$ is delayed neutron precusor decay constant, C is precusor nuclei/cm^3, T is incremental temperature quantity, and a is the receiprocal of the reactor heat capacity.

The equilibrium points are on the line of n=0 and C=0. Introducing a new set of variables, such as x_1=n, x_2=C, x_3=T, Eqs. (45) and (46) become

$$\dot{x}_1 - \frac{1}{\ell}(k_0 - r x_3 - \beta)x_1 - \bar{\lambda}x_2 = 0 \tag{48}$$

$$\dot{x}_2 - \frac{\beta}{\ell}\,x_1 + \bar{\lambda}x_2 = 0 \tag{49}$$

respectively. Taking the manipulation, $-\frac{k_0}{a\ell}$ × Eq.(47), the result is

$$-\frac{k_0}{a\ell}\,\dot{x}_3 + \frac{k_0}{\ell}\,x_1 = 0 \tag{50}$$

Similarly, taking the manipulation, $\frac{r x_3}{a\ell}$ × Eq.(47), one has

$$\frac{r x_3}{a\ell}\,\dot{x}_3 - \frac{r x_3}{\ell}\,x_1 = 0 \tag{51}$$

Adding Eqs.(48) to (51) together, and integrating with respect to time, the result is

$$x_1 + x_2 + (k_0 - r x_3)^2/2ra\ell = 0 \tag{52}$$

which gives

$$E = x_1 + x_2 + (k_0 - r x_3)^2/2ra\ell = 0 \tag{53}$$

and

$$\dot{E} = 0 \tag{54}$$

Eq.(53) shows that the E-function does not represent a closed surface because it is not a positive definite function, so it is not a conventional Liapunov function. However, for the system considered, it is impossible that the neutron power (n) and the precusor (C) are negative. In other words, the useful subspace is defined by $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. Therefore, the surface represented by the E-function and limited by the planes x_1=0 and x_2=0 will contain all the trajectories originally started in this surface. This surface is called the "E-surface" in the latter part of this paper.

For illustration, the parameters are chosen at

$$r = 0.00001, \quad k_0 = 0.0005, \quad \ell = 0.0001,$$

$$a = 50, \quad \beta = 0.001, \text{ and } \bar{\lambda} = 0.031.$$

Letting E=4.9, and choosing several sets of initial conditions in the E-surface for E= 4.9, such as

$$x_1(0) = 1.5, \quad x_2(0) = 2.5, \quad x_3(0) = 20;$$

$$x_1(0) = 3.0, \quad x_2(0) = 1.8, \quad x_3(0) = 60;$$

$$x_1(0) = 3.0, \quad x_2(0) = 1.0, \quad x_3(0) = 80;$$

and

$$x_1(0) = 1.2, \quad x_2(0) = 1.2, \quad x_3(0) = 100,$$

the E-surface and the trajectories starting from the above given initial conditions are plotted as shown in Fig.5. It can be seen that every trajectory is staying in the E-surface and reach the point of equilibrium on the x_3-axis. The same system has been considered in reference 6. Here, the systematic procedure for obtaining the E-function with $\dot{E}$=0, and the construction of the trajectories staying in the E-surface are useful extensions.

From Fig.5, it can be seen that, by plotting the E-surface containing several trajectories, one can have a clear understanding of the characteristics of the considered system. This approach is better than the conventional Liapunov method, which can only offer sufficient conditions of stability along with some rough idea on transient response.

(2) The second approach

From Eqs.(45) to (47), after the substitutions of the state variables and the values of parameters as defined before, one has

$$\dot{x}_1 = (-5.0 - 0.1x_3)x_1 + 0.031x_2 \tag{55}$$

$$\dot{x}_2 = 10x_1 - 0.031x_2 \tag{56}$$

$$\dot{x}_3 = 50x_1 \tag{57}$$

Multiplying Eqs.(55) to (57) by x_1, x_2, and x_3, respectively, the results are

$$x_1\dot{x}_1 + (5+0.1x_3)x_1^2 - 0.031x_1x_2 = 0 \quad (58)$$

$$x_2\dot{x}_2 - 10x_1x_2 + 0.031x_2^2 = 0 \quad (59)$$

$$x_3\dot{x}_3 - 50x_1x_3 = 0 \quad (60)$$

Then adding Eqs.(58) to (60) together and integrating with respect to time, one has

$$\tfrac{1}{2}(x_1^2+x_2^2+x_3^2) + \int[(-5-0.1x_3)x_1^2+10.031x_1x_2 - 0.031x_2^2+50x_1x_3]dt = 0 \quad (61)$$

which gives

$$E_1 = \tfrac{1}{2}(x_1^2+x_2^2+x_3^2) \quad (62)$$

and

$$\dot{E}_1 = (-5-0.1x_3)x_1^2 + 10.031x_1x_2 - 0.031x_2^2 + 50x_1x_3 \quad (63)$$

Similarly, integrating Eq.(56) with respect to time, one has

$$E_2 = x_2 \quad (64)$$

and

$$\dot{E}_2 = 10x_1 - 0.031x_2 \quad (65)$$

In addition, taking the manipulations such as Eq.(55)+Eq.(56)+Eq.(57)×0.1; Eq.(55)+Eq.(56)+Eq.(57)×0.2; and Eq.(55)+Eq.(56); and then integrating them with respect to time, one has

$$E_3 = x_1 + x_2 + 0.1x_3 \quad (66)$$

$$\dot{E}_3 = 10x_1 - 0.1x_1x_3 \quad (67)$$

$$E_4 = x_1 + x_2 + 0.2x_3 \quad (68)$$

$$\dot{E}_4 = 15x_1 - 0.1x_1x_3 \quad (69)$$

$$E_5 = x_1 + x_2 \quad (70)$$

$$\dot{E}_5 = 5x_1 - 0.1x_1x_3 \quad (71)$$

Selecting E_i(i=1,2,..,5) as arbitrary constants, the surfaces for E_i and $\dot{E}_i$=0 can be plotted as shown in Fig.6. Thus the directions of all the trajectories while passing through the planes of $\dot{E}_i$=0 must be parallel to the corresponding E-surface. As shown in Fig.6, for the sake of simplicity, dashed lines are used to represent the relation between the E_i-surfaces and the planes for $\dot{E}_i$=0. It can be seen that the trajectories started in the region $x_1\geq 0$, $x_2\geq 0$, $x_3\geq 0$ must converge to the line of equilibrium; i.e., the x_3-axis. A typical trajectory with initial condition (1.5,450,50) is sketched as shown in Fig.6. This trajectory should be in the same shape as those shown in Fig.5, but the scale change of the x_2-axis in Fig.6 makes it look different.

Note also that, although there is no E-function which is positive definite except E_1, and that there is no $\dot{E}$-function which is negative definite, the system trajectories can be obtained roughly from the surfaces of E_i=constant and $\dot{E}_i$=0.

5. Consideration of High Order Systems

For 4-th order or higher order systems, it is impossible to plot the trajectories in a three dimensional state-space. Nevertheless, it is useful to find the E-surface with $\dot{E}$=0 since it can give an overall view of the characteristics of trajectories. For those systems, if it is impossible to find the E-surface with $\dot{E}$=0, the E-functions can be used to generate Liapunov functions.[1,8]

6. Conclusions

In this paper, the methods of using E-functions to study the trajectories of nuclear reactor systems have been presented. These methods, which provide a better understanding of the trajectories in state-space, can be considered as an extension of the phase-plane method to third order systems.

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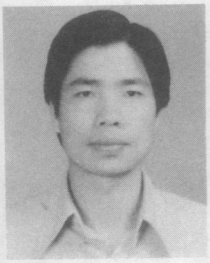

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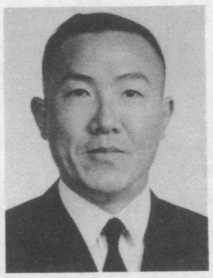

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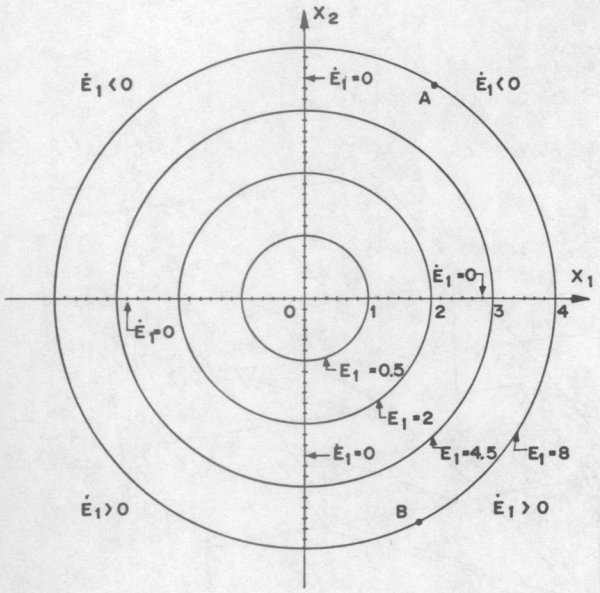

Fig. 1 THE E_1— CONTOURS & THE CONTOURS FOR $\dot{E}_1 = 0$.

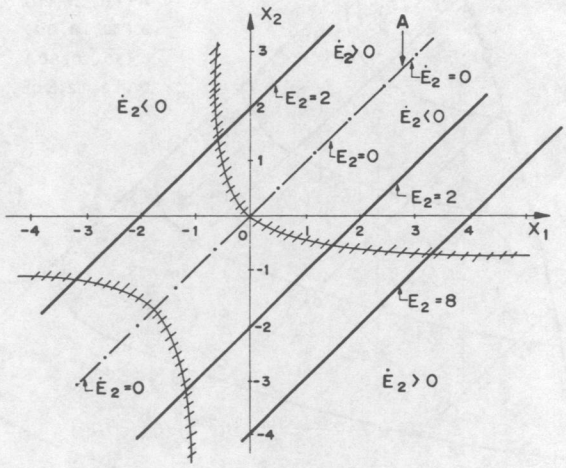

Fig. 2 THE CONTOURS FOR $\dot{E}_2 = 0$ & FOR VARIOUS VALUES OF E_2.

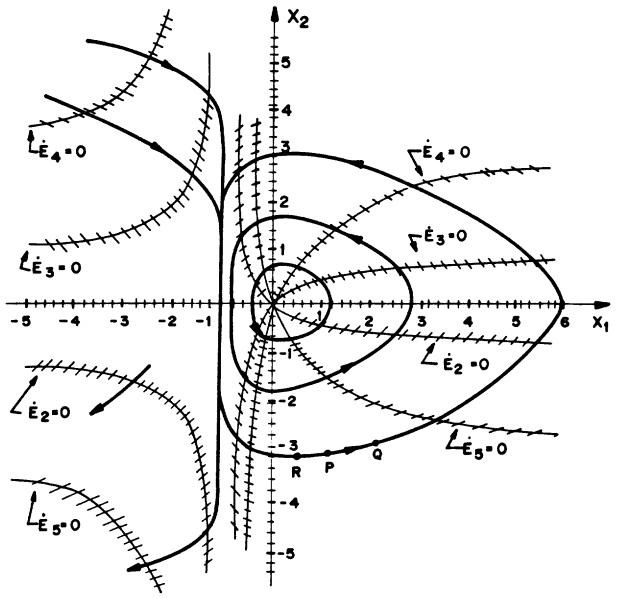

Fig. 3 CONTOURS FOR VARIOUS FORMS OF E & Ė=0.

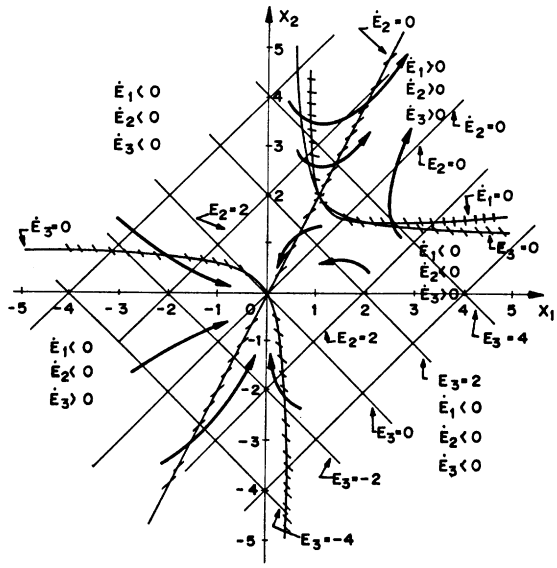

Fig. 4 TO SKETCH THE TRAJECTORIES BY USE OF THE CONTOURS FOR $\dot{E}_1=0$ & FOR VARIOUS VALUSE OF E_1.

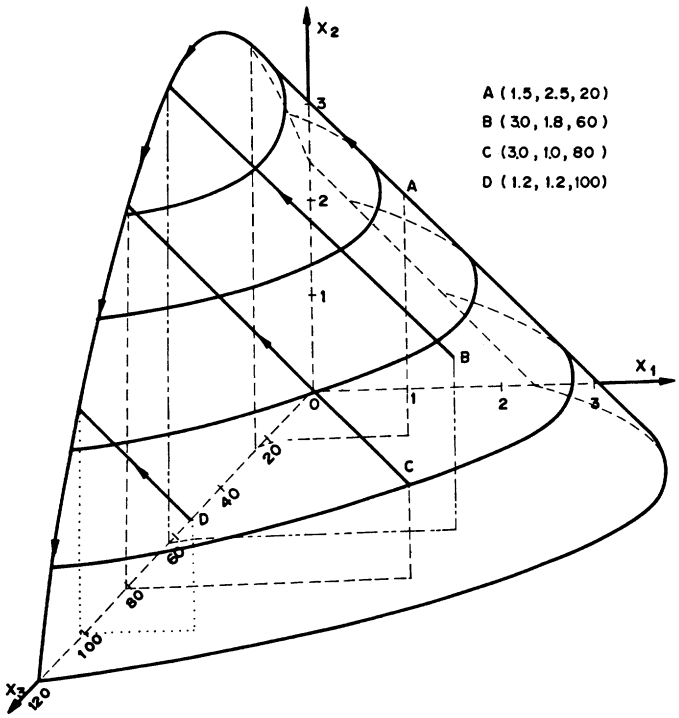

Fig. 5 AN E-SURFACE CONTAINING A FAMILY OF TRAJECTORIES.

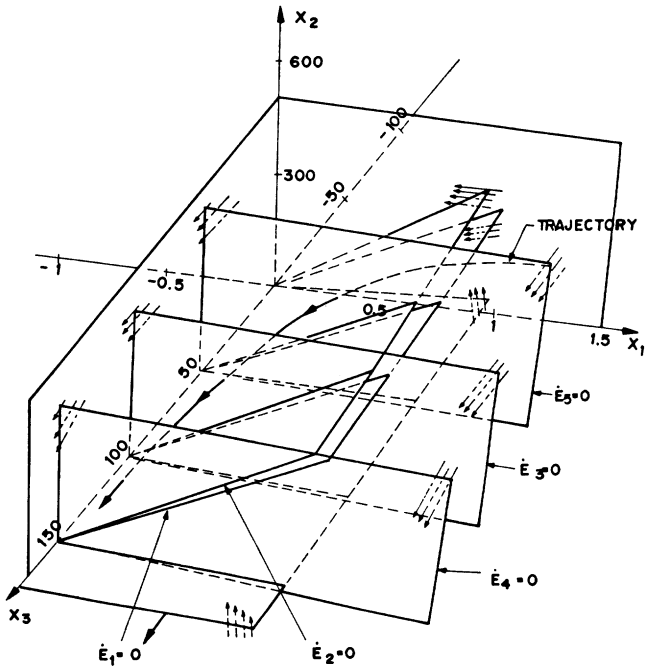

Fig. 6 CONSTRUCTION OF TRAJECTORY BY USE OF E & Ė - FUNCTIONS.