Constructions of DMT Optimal Vector Codes for Asynchronous Cooperative Networks Using Decode-and-Forward Protocols

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***Abstract*—An asynchronous cooperative network where different time delays exist among nodes is considered in this paper. Assuming the signals are OFDM modulated, it is first shown that the diversity-multiplexing tradeoff (DMT) achieved by the non-orthogonal selection decode-and-forward (NSDF) protocol for this network is the same as that for the synchronous one. In contrast to the complicated approximately universal "matrix" codes, where each relay uses a different codebook, a systematic construction of an extremely simple "vector" code is proposed. Given the transmitted codeword vector, this vector will be used by all nodes in the network for signal transmission; hence, the proposed coding scheme greatly reduces the complexity of relay deployment and decoding. Furthermore, it is proven that the proposed scheme is optimal in terms of the DMT of the NSDF protocol for this asynchronous network, provided all time delays are distinct. Finally, it is shown that the proposed code design can be extended to the orthogonal selection decode-and-forward protocol and remains to be DMT optimal.**

***Index Terms*—Asynchronous cooperative network, cyclic division algebra, decode-and-forward protocol, diversity-multiplexing gain tradeoff, multi-block space-time code.**

I. Introduction

RECENTLY, there has been a growing interest in the performance analyses and code designs of wireless cooperative relay networks [1]–[10], where multiple relays are used to help the source transmit coded information to the destination. In a way, the relays cooperate to form a virtual transmit array between the source and the destination. Communication in such a network is in general assumed to be half-duplex, i.e., at any time instant, a node can either transmit or receive, but not both. With the half-duplex assumption, several cooperative communication protocols have been proposed for relay networks. In [3], Laneman *et al.* proposed the orthogonal amplify-and-forward (OAF) and the orthogonal selection decode-and-forward (OSDF) protocols. In the OAF protocol,

1) the source first transmits to both relays and destination in the first half-frame, and
2) in the second half-frame, the relays transmit to the destination an amplified version of the signals received during the first half-frame.

The OSDF protocol, on the other hand, means that instead of amplification, each relay will decode the signal at the end of the first half-frame. If it succeeds in decoding, it will (possibly, but not necessarily in asynchronous relay networks as will be seen in the subsequent part of the paper) re-encode the message using a different codebook and transmit in the second half-frame. Some other protocols with improved performance, including the non-orthogonal amplified-and-forward (NAF), non-orthogonal selection decode-and-forward (NSDF), and dynamic decode-and-forward (DDF) protocols are proposed in [4]. By non-orthogonal it means that the source continues to transmit in the second half-frame. Following the notion of diversity-multiplexing tradeoff (DMT) proposed by Zheng and Tse [11], the performances of these protocols are explicitly characterized in [4], assuming that each node is equipped with only one antenna. For the case of multiple antennas, the DMT of NAF protocol can be found in [8]. By varying the numbers of channel uses in the first and second frames, a class of variable NSDF protocols is proposed in [9], and the corresponding DMTs are also given. Furthermore, explicit constructions of space-time codes that achieve the optimal DMTs of these protocols are also provided in [9], [12]. These codes are matrix codes in nature and are constructed based on some cyclic division algebra.

However, it should be noted that all the aforementioned works assume that the relay network is perfectly synchronized in time. Specifically, they assume the following.

1) The relays must know the exact timing to begin the transmission of the first and the second frames.
2) Most importantly, the path delays between the relays and the destination, and between the source and the destination are all of the same value.

Unfortunately, both assumptions are very hard to satisfy in practice since the relays have respective local oscillators and are usually geographically dispersed. Thus, it is unlikely for the synchronous assumptions to hold in general.

Taking account of the asynchronous nature of wireless cooperative relay networks, a slightly modified version of the two-phase OSDF protocol has been considered in [5]–[7] for the frame synchronized two-hop network. In such a network, the source transmits to the relays in the first half-frame, then the relays transmit to the destination in the second half-frame.

Manuscript received January 1, 2010; revised March 9, 2010; accepted May 4, 2010. The associate editor coordinating the review of this paper and approving it for publication was S. Wei.

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The material in this paper was presented in part at the 2008 IEEE International Symposium on Information Theory, Toronto, Ontario, Canada. The work was supported in part by Taiwan National Science Council under Grants NSC 96-2628-E-009-172-MY3 and NSC 98-2221-E-009-045-MY3.

Digital Object Identifier 10.1109/TWC.2010.07.100002

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To achieve full cooperative diversity for all possible path delays in the second phase, several coding schemes based on pairwise error probability (PEP) analysis have been proposed in [5]–[7], [13], [14]. In [6], the authors proposed a variation of the threaded algebraic space-time (TAST) codes. By viewing the time delay at each relay node as a shift of column indices in a row of a matrix, Shang and Xia proposed a systematic construction of shift-full-rank matrices in [5] and showed that these matrices can be used to produce fully-diverse space-time trellis codes. In [10] Wei studied the DMT performance of an asynchronous relay network in a continuous waveform channel with two relays. Apart from the time-domain based modulations, the technique of orthogonal-frequency-division-multiplexing (OFDM) is known to be easily applicable to communication in asynchronous scenarios without the need of equalization. We remark that while OFDM easily avoids the need of equalization, it introduces other system issues such as carrier frequency synchronization and high peak-to-average power ratio. A simple Alamouti transmission scheme using OFDM for asynchronous cooperative systems was proposed by Li and Xia [13] and was shown to achieve a diversity gain of order 2. Following the same idea, a high-rate space-frequency code was proposed in [14] based on the PEP criterion.

In contrast to the PEP-based criterion, in this paper we will focus on the DMT analysis and code design for the asynchronous cooperative networks using NSDF protocol. Besides, for practical reasons we will assume that each node knows only the time delay of its incoming channel, but not others'. Following [13], we will consider frequency-domain coding, and the codes will be OFDM modulated to eliminate the need of equalization, thereby reducing the hardware complexity.

This paper is organized as follows. In Section II we will briefly introduce the system model of the OFDM-based asynchronous cooperative network as well as the corresponding NSDF protocol. The DMT performance of such asynchronous NSDF protocol is analyzed in Section III. It will be seen that the resulting DMT is the same as that for the synchronous one; hence, it is expected there be no performance loss due to the asynchronous nature of the underlying network at high SNR regime, provided that an appropriate asynchronous OFDM-based coding scheme is used. While the conventional approximately universal, cyclic-division-algebra-based space-time codes for the synchronous network [9] can be easily modified to fit into the OFDM modulation and to achieve the optimal DMT in the asynchronous case, these codes are matrix codes in nature and are used such that each relay is associated with a specific row of a code matrix. As a result, the relays are distinguished by their places of rows in a matrix, and each relay requires a distinct encoder. This greatly complicates the fabrication and the deployment of relays in practice. In view of this, a systematic construction of an extremely simple vector code that operates on the asynchronous NSDF protocol will be given in Section IV. It will be seen that given the transmitted codeword vector, every node, including both source and relays, in the network will transmit the same vector; hence, it greatly reduces the complexity of relay deployment and decoding, compared to the complicated matrix coding schemes. The proposed construction is a variation of the multi-block space-time codes proposed by the author in [15]. These multi-block codes are approximately universal [16] and DMT optimal in MIMO point-to-point communication. They are modified such that a SISO multi-block vector code is used by all nodes in the asynchronous network. The proof of DMT optimality of the proposed vector coding scheme will be given in Section IV-A. Finally, in Section V we show that the DMT for the OSDF protocol for asynchronous networks is also the same as that for synchronous networks, and the same vector code proposed in Section IV can be used to achieve the DMT optimality. In Section VI we conclude the paper.

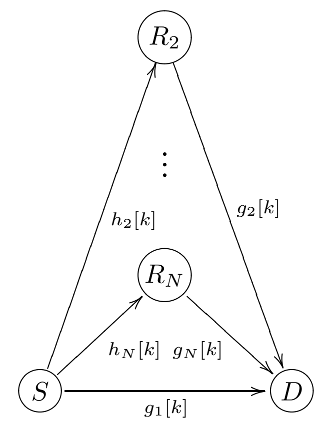

Fig. 1. A relay network consisting of a source node S, a destination node D, and a collection of $(N-1)$ intermediate relay nodes.

II. System Model

Consider an asynchronous relay network consisting of a source node S, $(N-1)$ relay nodes $\{R_2, \cdots, R_N\}$, and a destination node D (see Fig. 1). Signal transmissions from S to R_n, S to D, and R_n to D are OFDM modulated using Q subcarriers. In the first phase of the NSDF protocol, given the source message $\underline{x} = [x_0, \cdots, x_{Q-1}]^\top$, the transmitter at S first applies to $\underline{x}$ a Q-point inverse fast Fourier transform (FFT), where by $\underline{a}^\top$ we mean the transpose of the column vector $\underline{a}$. After attaching to this vector a cyclic prefix (CP) of length L_1, the resulting length-$(Q+L_1)$ signal vector is broadcasted from S to all the remaining nodes. The sampled discrete-time channel impulse response from S to relay R_n is modeled as

$$h_n[k] = h_n \delta_K[k-\nu_n], \quad n = 2, \cdots, N, \tag{1}$$

and the one from S to D is given by

$$g_1[k] = g_1 \delta_K[k-\tau_1], \tag{2}$$

where g_1 and h_n, $n = 2, \cdots, N$, are i.i.d. circularly symmetric, $\mathbb{C}\mathcal{N}(0,1)$ complex Gaussian random variables with zero mean and unit variance. $\delta_K[k]$ denotes the Kronecker delta function, i.e. $\delta_K[k] = 1$ if $k = 0$, and $\delta_K[k] = 0$ otherwise. The parameters ν_n's and τ_1 are used to capture the discrete-time delay from S to R_n, and from S to D, respectively. To avoid inter-channel interference in the frequency domain, we require $L_1 \geq \max\{\tau_1, \nu_2, \cdots, \nu_N\}$, which can be safely determined from field measurements. At the receiver end (relays and destination), after removing the CP from the received signal and taking a Q-point FFT, the resulting received signal at the qth subcarrier at relay R_n is given by

$$r_{n,q} = h_n \zeta_Q^{-q\nu_n} x_q + z_{n,q}, \quad (3)$$

and the one at node D is

$$y_{1,q} = g_1 \zeta_Q^{-q\tau_1} x_q + w_{1,q}, \quad (4)$$

where $\zeta_Q := \exp\left(\imath \frac{2\pi}{Q}\right)$, and $\imath = \sqrt{-1}$. The $z_{n,q}$'s and $w_{1,q}$'s are i.i.d. $\mathbb{C}\mathcal{N}(0,1)$ complex Gaussian random variables that are used to model the additive noises.

Having received the signal vector $\underline{r}_n = [r_{n,0} \cdots, r_{n,Q-1}]^\top$, the relay R_n proceeds to decode the vector. Assuming $(M-1)$ relays, say $R_{i_1}, \cdots, R_{i_{M-1}}$, where $\{i_1, \cdots, i_{M-1}\} \subseteq \{2, \cdots, N\}$, have successfully decoded the message sent by S, these relays then independently use a different codebook to re-encode the message. Let $\underline{s}_{i_m} = [s_{i_m,0}, \cdots, s_{i_m,Q-1}]^\top$ be the codeword produced by relay R_{i_m}. Then together with the source S, the $(M-1)$ relays will participate in the second-phase communication. Let

$$g_{i_m}[k] = g_{i_m} \delta_K[k - \tau_{i_m}] \quad (5)$$

be the sampled discrete-time channel impulse response between relay R_{i_m} and D, where τ_{i_m} is the corresponding discrete-time delay. Following the same OFDM modulation approach as in the first-phase, the received signal at the qth subcarrier at destination node D is given by

$$y_{2,q} = g_1 \zeta_Q^{-q\tau_1} x'_q + \sum_{m=1}^{M-1} g_{i_m} \zeta_Q^{-q\tau_{i_m}} s_{i_m,q} + w_{2,q}, \quad (6)$$

provided that the CP of the OFDM symbol in the second half-frame has length $L_2 \geq \max\{\tau_1, \tau_2, \cdots, \tau_N\}$. The g_{i_m}'s and $w_{2,q}$'s are again, modeled as i.i.d. $\mathbb{C}\mathcal{N}(0,1)$ complex Gaussian random variables. The vector $\underline{x}' = \left[x'_0 \cdots x'_{Q-1}\right]^\top$ is the codeword sent by S to D during the second phase. In both phases, the codewords sent by either the source or the relays are required to satisfy the following power constraint:

$$\mathbb{E}\|\underline{x}\|_F^2, \mathbb{E}\left\|\underline{s}_{i_m}\right\|_F^2, \mathbb{E}\|\underline{x}'\|_F^2 \leq Q \cdot \rho, \quad (7)$$

where by $\|\underline{x}\|_F$ we mean the Frobenius norm of the vector $\underline{x}$. ρ is the signal-to-noise ratio at each receive antenna.

III. DMT Performance of NSDF Protocol

In this section, we will analyze the DMT performance of the NSDF protocol discussed in the previous section. We first briefly review some notations defined in [11] that will be used in the subsequent analyses. Let $f(\rho)$ be a positive-valued function in ρ. We say $f(\rho) \doteq \rho^b$ if $\lim_{\rho\to\infty} \frac{\log f(\rho)}{\log \rho} = b$. Notations of $\dot{\geq}$ and $\dot{\leq}$ are defined similarly. Secondly, let $\mathcal{X}(\rho)$ be a family of codes, one for each SNR value, taking places in U channel uses. We say $\mathcal{X}(\rho)$ achieves multiplexing gain at value r if and only if $|\mathcal{X}(\rho)| \doteq \rho^{rU}$.

To perform the DMT analysis, we use Gaussian random codebooks, i.e., the entries of codewords $\underline{x}$, $\underline{x}'$, and $\underline{s}_{i_m}$, are i.i.d. $\mathbb{C}\mathcal{N}(0,\rho)$ variables so that the power constraint (7) is satisfied. First note that at the end of the first phase, relay R_n fails to decode the message sent by S if it is in outage. Assuming the source transmits at multiplexing gain r, such outage occurs with probability

$$\begin{aligned}
&\Pr\{R_n \text{ is in outage}\} \\
&= \Pr\left\{\sum_{q=0}^{Q-1} \log\left(1 + \rho\left|h_n \zeta_Q^{-q\nu_n}\right|^2\right) \leq 2Q \cdot r \log \rho\right\} \\
&= \Pr\left\{\log\left(1 + \rho|h_n|^2\right) \leq 2r \log \rho\right\} \doteq \rho^{-(1-2r)^+}, \quad (8)
\end{aligned}$$

where the factor $2Q$ is because the communication takes places in 2 OFDM symbols, each of length Q, and $(x)^+ := \max\{0, x\}$. Note that in (8) we have neglected the rate loss due to the insertion of CP.

Let $\mathcal{J}_M$ denote the event that there are $(M-1)$ relays, $R_{i_1}, \cdots, R_{i_{M-1}}$, participating in the second phase transmission. As the h_n's are i.i.d., the probability of $\mathcal{J}_M$ is given by

$$\begin{aligned}
&\Pr\{\mathcal{J}_M\} \\
&\doteq \binom{N-1}{M-1}\left(1 - \rho^{-(1-2r)^+}\right)^{M-1} \rho^{-(N-M)(1-2r)^+} \\
&\doteq \begin{cases} \rho^{-(N-M)(1-2r)}, & 0 \leq r \leq \frac{1}{2}, \\ 0, & r \geq \frac{1}{2}, M \geq 2, \\ 1, & r \geq \frac{1}{2}, M = 1. \end{cases} \quad (9)
\end{aligned}$$

Next, given event $\mathcal{J}_M$, let $\underline{y}_1 = [y_{1,0} \cdots y_{1,Q-1}]^\top$ and $\underline{y}_2 = [y_{2,0} \cdots y_{2,Q-1}]^\top$ be the signal vectors received by the destination node D during the first and the second phases, respectively. Then we can rewrite the channel input-output relation in the following matrix form

$$\underline{y}_1 = G_1 \underline{x} + \underline{w}_1, \quad \underline{y}_2 = G_1 \underline{x}' + \sum_{m=1}^{M-1} G_{i_m} \underline{s}_{i_m} + \underline{w}_2,$$

where $\underline{w}_1 := [w_{1,0} \cdots w_{1,Q-1}]^\top$ and $\underline{w}_2 := [w_{2,0} \cdots w_{2,Q-1}]^\top$. The $(Q \times Q)$ matrices G_1 and G_{i_m} are respectively given by

$$\begin{aligned}
G_1 &:= \text{diag}\left(g_1 \zeta_Q^{-0\cdot\tau_1}, \cdots, g_1 \zeta_Q^{-(Q-1)\cdot\tau_1}\right), \\
G_{i_m} &:= \text{diag}\left(g_{i_m} \zeta_Q^{-0\cdot\tau_{i_m}}, \cdots, g_{i_m} \zeta_Q^{-(Q-1)\cdot\tau_{i_m}}\right).
\end{aligned}$$

Note that $G_1 G_1^\dagger = |g_1|^2 I_Q$ and $G_{i_m} G_{i_m}^\dagger = |g_{i_m}|^2 I_Q$, where by $\dagger$ we mean the Hermitian transpose of a matrix. Thus, given $\mathcal{J}_M$, the probability that D is in outage and therefore fails to recover the message is

$$\begin{aligned}
&P_{\text{out}}(r|\mathcal{J}_M) \\
&= \Pr\Bigg\{\log\left(1 + \rho|g_1|^2\right) \\
&\quad + \log\left(1 + \rho\left(|g_1|^2 + \sum_{m=1}^{M-1}|g_{i_m}|^2\right)\right) \leq 2r \log \rho\Bigg\} \\
&\doteq \rho^{-d_M(r)}, \quad (10)
\end{aligned}$$

where $d_M(r)$ has been computed for the synchronous case in [9, Eq. (115)]

$$d_M(r) = \begin{cases} M(1-2r) + 2r, & 0 \leq r \leq \frac{1}{2} \text{ and } M > 1, \\ 2(1-r), & \frac{1}{2} \leq r \leq 1 \text{ and } M > 1, \\ (1-r), & 0 \leq r \leq 1 \text{ and } M = 1. \end{cases} \quad (11)$$

Combining results of (9) and (11) gives the following theorem.

Theorem 1: In an asynchronous cooperative network with $(N-1)$ relays, the diversity gain achieved by the NSDF protocol at multiplexing gain r is for $0 \le r \le 1$

$$\begin{aligned} d^*(r) &:= -\lim_{\rho\to\infty} \log_\rho \sum_{M=1}^{N} P_{\text{out}}\left(r|\mathcal{J}_M\right) \Pr\left\{\mathcal{J}_M\right\} \\ &= (N-1)(1-2r)^+ + (1-r). \end{aligned} \tag{12}$$

■

We remark that by neglecting the rate loss due to the insertion of CP, the DMT $d^*(r)$ in (12) is exactly the same as that for synchronous relay networks [4], [9].

Theorem 1 is indeed very surprising because of the following. Intuitively speaking, there are two kinds of diversity gains to be expected in the asynchronous relay network. The first is the cooperative diversity gain resulting from the cooperation among $(N-1)$ relays and the source node. The second is the frequency-selective diversity gain due to the different time delays associated with the paths from the $(N-1)$ relays to the destination node. It is then very surprising that these two different kinds of diversity gains do not multiply in the DMT expression (12), and the overall diversity gain remains the same as that of the synchronous network. A simple explanation of this can be seen from the PEP analysis at the point $r=0$ given by Shang and Xia in [5] where the time delay is interpreted as a shift of column indices in a row of the code matrix of a space-time trellis code. Since there are N rows in the matrix and the shifts cannot increase the rank of the different matrix that is already at full rank N, the maximal diversity gain equals N, which agrees with (12). It means the extra frequency diversity gain cannot benefit the fully-cooperated system. A similar observation can be obtained from [17]. In particular, we remark that all the $2Q$ subcarriers must be jointly coded. Otherwise, for example if the subcarriers are coded independently, then it can be seen from the PEP analysis presented in [17] that the resulting diversity gain can be as low as 1, instead of N, when the difference matrix is of rank 1.

IV. Proposed Construction of DMT Optimal Codes

In the previous section we have seen the very surprising result that the cooperative diversity gain and frequency-selective diversity gain do not multiply in the asynchronous NSDF protocol. Such a result suggests a certain means of simplification in code design. Recall that in the synchronous relay network, where one obtains only the cooperative diversity gain, approximately universal space-time matrix codes [9], [12] are used to achieve the optimal DMT performance. While the same approximately universal matrix codes still can be extended and applied to the asynchronous relay network to achieve the same DMT optimality, it is worth thinking how to obtain the other kind of diversity gain, namely, the frequency-selective diversity gain, provided by the asynchronous nature of the network. In [18], we reported that the DMT for a SIMO multi-path fading channel with L paths and n_r receive antenna is $Ln_r(1-r)$, which somewhat resembles the DMT of (12) (and is exactly in the same form as the DMT for asynchronous OSDF protocol given in Section V). This gives us some hint that by seeking only the naturally-provided frequency diversity, one might be able to fully recover the optimal DMT (12). One merit of this approach is that we can use an extremely simple vector code that operates on the asynchronous NSDF protocol to achieve the same DMT optimality (12) as the complicated matrix code does.

The proposed vector code is much simpler and easier to implement than the matrix code because of the following. In the matrix code, each relay is associated with a specific row of a code matrix. Hence the relays are distinguished by their places of rows in a matrix, and each relay requires a distinct encoder. This greatly complicates the fabrication and the deployment of relays in practice. On the contrary, when using the vector code, every node in the network will transmit the same code vector. Therefore, there is no need to distinguish the relays as well as the corresponding encoders during fabrication. The relays will be distinguished naturally by their locations of deployment through the different time delays. This approach greatly reduces the complexity of relay deployment and decoding, compared to the complicated approximately universal matrix coding schemes.

Below, we present a systematic construction of vector codes that can achieve the optimal DMT (12) promised by the NSDF protocol in asynchronous relay networks. Given the number of subcarriers Q and the multiplexing gain r, let $\mathbb{L}$ be a number field that is a cyclic Galois extension of $\mathbb{F}=\mathbb{Q}(\imath)$ of degree $2Q$. By viewing $\mathbb{L}$ as a vector space over $\mathbb{F}$ with integral basis $\{e_1,\cdots,e_{2Q}\}$, every element x in $\mathbb{L}$ can be represented as $x=\sum_{i=1}^{2Q} a_i e_i$ for some $a_i\in\mathbb{F}$. Let $\mathcal{A}_{\text{QAM}}(S^2)\subset\mathbb{Z}[\imath]$ be a set consisting of S^2-QAM symbols, i.e., it is given by

$$\mathcal{A}_{\text{QAM}}(S^2) = \{a+b\imath : -S\le a,b\le S, a,b \text{ odd integers}\}$$

with set size equal to S^2. As $\mathbb{L}/\mathbb{F}$ is cyclic Galois, let σ be the generator of the Galois group $\text{Gal}(\mathbb{L}/\mathbb{F})$; then the proposed code is the following

$$\mathcal{X} := \left\{ \underline{x} = \theta\left[x, \sigma(x), \cdots, \sigma^{2Q-1}(x)\right]^\top : \right. \\ \left. x=\sum_{i=1}^{2Q} a_i e_i \text{ and } a_i\in\mathcal{A}_{\text{QAM}}(\rho^r) \right\}. \tag{13}$$

Obviously, $|\mathcal{X}| = \rho^{2rQ}$ achieves multiplexing gain r in $2Q$ channel uses, or equivalently, in two OFDM symbols, when neglecting the rate-loss due to CP. The parameter θ should be set to meet the power constraint (7). In particular, for the DMT purpose it suffices to set $\theta^2 \doteq \rho^{1-r}$ at high SNR regime. With the above, we propose the following transmission scheme.

1) Given $\underline{x}\in\mathcal{X}$, the source S transmits $\theta\cdot\sigma^q(x)$ at the qth subcarrier, $q=0,1,\cdots,Q-1$, during the first half-frame, or equivalently, the first OFDM symbol of length Q.
2) For the second half-frame, S transmits $\theta\cdot\sigma^{q+Q}(x)$ at the qth subcarrier, $q=0,1,\cdots,Q-1$. Also, the relays which have successfully decoded the transmitted message based on the first half of $\underline{x}$ at the end of the first half-frame will send the second half of $\underline{x}$, i.e., $\theta\left[\sigma^Q(x)\cdots\sigma^{2Q-1}(x)\right]^\top$ to D during the second half-frame.

Thus, the relays are literally helping the source S complete the transmission of the last Q symbols of $\underline{x}$. Unlike all the previously proposed codes [8], [9], [12], here the same vector code $\mathcal{X}$ is used by the source S and by all the relays $\{R_n\}$. This gives a great reduction in the decoding complexity at the destination D, compared to that of the matrix codes. Furthermore, our scheme completely removes the difficulty in implementing the NSDF protocol since all the relays now use the same codebook for re-encoding, and there is no need to distinguish the relays.

It turns out that the proposed vector coding scheme is optimal in terms of the DMT for asynchronous cooperative networks. Specifically, we have the following theorem. The proof will be given in the next section.

Theorem 2: In an asynchronous cooperative network with $(N-1)$ relays, given the desired multiplexing gain r, $0 \le r \le 1$, let $\mathcal{X}$ be the vector code defined in (13). Then, following the NSDF transmission scheme proposed above, the resulting diversity gain equals

$$d(r) \;=\; (N-1)(1-2r)^+ + (1-r), \tag{14}$$

at high SNR regime, provided that the time delays from the relays to the destination are all distinct. ■

The condition of distinct time delays is crucial to Theorem 2. It is because in applying the vector code to the asynchronous NSDF protocol, the code does not seek the cooperative diversity gain, but seeks the diversity gain resulting from frequency selectivity. Should some of the relays have the same time delay, the available frequency-selective diversity gain decreases. It then follows that the vector code cannot achieve the same DMT (12) in Theorem 1. Nevertheless, the possibility of having the same time delay in practical cooperative networks is highly unlikely as the relays are sparsely deployed and each has its own local oscillator. In this sense, the condition of distinct time delays required by Theorem 2 can be easily met. On the other hand, we remark that the space-time codes presented in [5], [7], [12] always achieve full diversity no matter the time delays are distinct or not. So such codes may be helpful to relax the assumption of the distinct time delays.

In Fig. 2 we present a simulation result to compare the outage performance of the approximately universal matrix code given in [8], [9], [12] and that of the proposed vector code at the rate of two bits per channel use with $(N-1)=3$ relays and $Q=16$ subcarriers. We remark that the simulation result holds for all approximately universal codes, regardless of the underlying algebra and its basis. It is seen that the performance degradation resulting from the proposed vector code which seeks only the frequency diversity gain is negligible. It justifies the DMT optimality of the proposed code given in Theorem 2.

A. Proof of DMT Optimality of the Proposed Code

Below we will provide a proof to the DMT optimality achieved by the coding scheme $\mathcal{X}$ using the proposed transmission scheme under the assumption of distinct time delay. We will analyze the diversity gains achieved by $\mathcal{X}$ in the first and the second phases, respectively. We remark that our proof

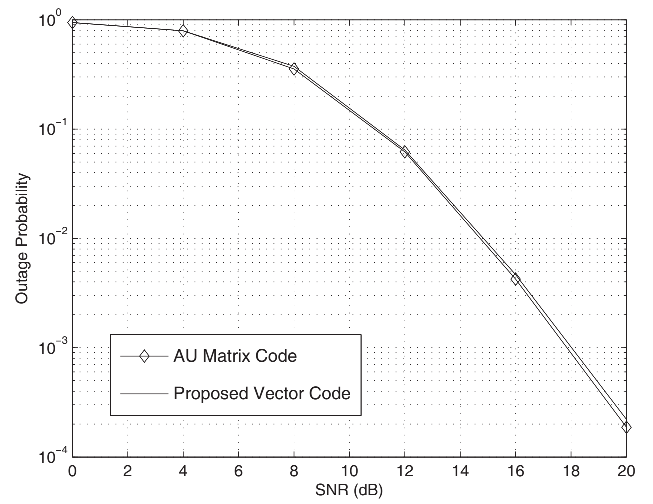

Fig. 2. Outage Performance comparison between the approximately universal (AU) matrix code given in [8], [9], [12] and the proposed vector code at rate 2 bits per channel use with $(N-1)=3$ relays and $Q=16$ subcarriers using asynchronous NSDF protocol.

of DMT optimality of the proposed vector coding scheme is very different from those of cyclic-division-algebra-based matrix codes [9]. The latter relies solely on the property of being approximately universal [16]. The proof of Theorem 2 calls for more complicated arguments, especially in the second phase. This is because here we have pushed the vector codes to their extremes so that the resulting coding scheme is much simpler and requires little complexity not only in the fabrication and deployment of the relays, but also in decoding.

Diversity Gain in the First Phase: Given $\underline{x} = \left[\underline{x}_1^\top \ \underline{x}_2^\top\right]^\top \in \mathcal{X}$, where $\underline{x}_1$ and $\underline{x}_2$ are both of length Q, during the first transmission phase the source S broadcasts to all the remaining nodes the OFDM modulated signal of $\underline{x}_1$. Below we will show that in the first phase the codeword error probability $P_{\text{cwe}}(r, R_n)$ of code $\mathcal{X}$ at relay R_n is

$$P_{\text{cwe}}(r, R_n) \doteq \rho^{-(1-2r)^+}, \tag{15}$$

which agrees exactly with (8). Such a result follows naturally from the fact that the proposed vector code $\mathcal{X}$ is approximately universal [15] when it is seen by an individual relay. Thus, simply by the property of approximately universal the DMT performance of the proposed code $\mathcal{X}$ in this phase should be exactly the same as that achieved by the Gaussian random codebooks.

Having seen the above insight, below we briefly outline the DMT analysis in this phase. Using the signal model (3), at relay node R_n the squared Euclidean distance between the noise-free received signals associated with any pair of distinct codewords $\underline{x} \neq \underline{x}' \in \mathcal{X}$ is given by

$$d_1^2(\underline{x}, \underline{x}') \;=\; \theta^2 \left|h_n\right|^2 \sum_{q=0}^{Q-1} \left|\sigma^q(x - x')\right|^2$$

for some $x \neq x' \in \mathcal{O}_{\mathbb{L}}$ associated with $\underline{x}$ and $\underline{x}'$, respectively. By $\mathcal{O}_{\mathbb{L}}$ we mean the ring of algebraic integers in $\mathbb{L}$. Repeatedly using the arithmetic-mean geometric-mean (AM-GM) inequality and the fact that $\left|N_{\mathbb{L}/\mathbb{F}}(x-x')\right| \ge 1$ as in

[15], [19] it can be shown that

$$\begin{aligned} d_1^2(\underline{x},\underline{x}') &\dot{\geq} \theta^2 |h_n|^2 \left[\prod_{q=0}^{Q-1} |\sigma^q(x-x')|^2\right]^{1/Q} \\ &\geq \theta^2 |h_n|^2 \left[\frac{1}{\prod_{q=Q}^{2Q-1} |\sigma^q(x-x')|^2}\right]^{1/Q} \\ &\dot{\geq} \theta^2 |h_n|^2 \left[\sum_{q=Q}^{2Q-1} |\sigma^q(x-x')|^2\right]^{-\frac{Q}{Q}} \\ &\dot{\geq} \theta^2 |h_n|^2 \rho^{-r}. \end{aligned} \quad (16)$$

Set $|h_n|^2 = \rho^{-\gamma_n}$; then using arguments similar to those in [15], [19] it can be shown that the codeword error probability $P_{\text{cwe}}(r, R_n)$ is upper bounded by

$$P_{\text{cwe}}(r, R_n) \ \dot{\leq} \ \Pr\{\gamma_n \geq 1-2r\} \doteq \rho^{-(1-2r)^+}.$$

It agrees exactly with (8). Thus, we conclude that $\mathcal{X}$ achieves the optimal DMT performance in the first phase. Furthermore, let $\mathcal{J}_M$ denote the event of $(M-1)$ relays participating in the second-phase transmission, and it is easy to see that the probability of event $\mathcal{J}_M$ is the same as (9).

Diversity Gain in the Second Phase: The DMT analysis of the second phase transmission is more complicated than that of the first as the previous approximately-universal insight cannot be carried over to the present case completely. To elaborate, the conventional approximately universal matrix codes [9], [12] seek to achieve both the cooperative and frequency selective diversity gains at the same time. But, the vector code $\mathcal{X}$, though being approximately universal as well, seeks only the latter. Nevertheless, from the remark of Theorem 1 we have already seen that the two diversity gains do not multiply in the asynchronous network. It is, therefore, possible to rely on only the frequency selective diversity gain to achieve the same DMT optimality. Such a concept is realized by the proposed coding scheme $\mathcal{X}$ as we will see in subsequent discussions.

In a nutshell, the DMT performance analysis provided below is an investigation of the frequency-selective diversity gain achieved by $\mathcal{X}$. Let $\{R_{i_1}, \cdots, R_{i_{M-1}}\}$ be the $(M-1)$ relays which have successfully decoded the first half of $\underline{x}$ during the first phase and are participating in the transmission of the second phase. According to our proposed scheme, the same second half of $\underline{x}$ will be used by all the relays R_{i_m} as well as by the source node S for information transmission to the destination node D in this phase; hence, there is no need to re-encode using different codebooks. Now using the signal model (6), the received signal at the qth subcarrier at destination node D in the second phase is given by

$$y_{2,q} \ = \ \left[g_1\zeta_Q^{-q\cdot\tau_1} + \sum_{m=1}^{M-1} g_{i_m}\zeta_Q^{-q\cdot\tau_{i_m}}\right] x_{2,q} + w_{2,q},$$

where $\underline{x}_2 = [x_{2,0}\cdots x_{2,Q-1}]^\top$ and $x_{2,q} = \theta\cdot\sigma^{Q+q}(x)$. Again, for any pair of distinct codewords $\underline{x} \neq \underline{x}' \in \mathcal{X}$, the squared Euclidean distance between the noise-free signals $\underline{y} = \left[\underline{y}_1^\top \ \underline{y}_2^\top\right]^\top$ associated with $\underline{x}$ and $\underline{x}'$ is given by

$$\begin{aligned} d_2^2(\underline{x},\underline{x}') &= \theta^2 |g_1|^2 \sum_{q=0}^{Q-1} |\sigma^q(x-x')|^2 \\ &\quad + \theta^2 \sum_{q=0}^{Q-1} |\ell_{q+Q}|^2 \left|\sigma^{q+Q}(x-x')\right|^2, \end{aligned} \quad (17)$$

where we have set

$$\ell_{q+Q} \ := \ g_1\zeta_Q^{-q\cdot\tau_1} + \sum_{m=1}^{M-1} g_{i_m}\zeta_Q^{-q\cdot\tau_{i_m}}. \quad (18)$$

It should be noted that the ℓ_q's are statistically correlated with each other due to the frequency-selectiveness through (18).

a) The Case when $M=1$: First we consider the case when no relays participate in the second phase, i.e., it is the case of $M=1$ and corresponds to the event $\mathcal{J}_1$ discussed in Section III. In this case, we see from (18) that $\ell_q = g_1\zeta_Q^{-q\cdot\tau_1}$; hence, the same AM-GM type arguments show

$$\begin{aligned} d_2^2(\underline{x},\underline{x}') &= \theta^2 |g_1|^2 \left[\sum_{q=0}^{2Q-1} |\sigma^q(x-x')|^2\right] \\ &\dot{\geq} \theta^2 |g_1|^2 \left[\prod_{q=0}^{2Q-1} |\sigma^q(x-x')|^2\right]^{\frac{1}{2Q}} \\ &\dot{\geq} \theta^2 |g_1|^2 \ \doteq \ \rho^{1-r-\alpha}, \end{aligned} \quad (19)$$

where the second inequality follows again from $\left|N_{\mathbb{L}/\mathbb{F}}(x-x')\right| \geq 1$ and where we have set $|g_1|^2 = \rho^{-\alpha}$. Now following an argument similar to (15), it can be shown that $P_{\text{cwe}}(r|\mathcal{J}_1) \ \dot{\leq} \ \rho^{-(1-r)^+}$, which agrees exactly with the third case of $d_M(r)$ in (11).

b) The Case when $M>1$: The proof for the case of $M>1$ is more complicated. For ease of understanding, we break the proof into two lemmas. We first show the following lemma.

Lemma 3: Given the event $\mathcal{J}_M$, the codeword error probability in the second phase is upper bounded by the channel outage probability of the corresponding frequency-selective channel, i.e.,

$$\begin{aligned} P_{\text{cwe}}(r|\mathcal{J}_M) &\leq P_{\text{FS,out}}(r|\mathcal{J}_M) \\ &:= \Pr\left\{\sum_{q=0}^{2Q-1} \log\left(1+\rho|\ell_q|^2\right) \leq 2rQ\log\rho\right\}, \end{aligned} \quad (20)$$

where we have set $\ell_q = g_1$ for $q = 0, 1, \cdots, Q-1$.

Proof: See Appendix A. ■

Intuitively, the result of (20) is more-or-less to be expected since the proposed code scheme $\mathcal{X}$ is approximately universal [15]. Hence, $\mathcal{X}$ should be able to achieve an error performance that is upper bounded by the outage probability of underlying channel, which turns out to be frequency-selective due to our proposed transmission scheme.

The next step is to analyze the channel outage probability (20), and we aim to show the following.

Lemma 4: Given event $\mathcal{J}_M$, the channel outage probability is upper bounded by

$$P_{\text{FS,out}}(r|\mathcal{J}_M) \ \dot{\leq}\ \rho^{-d_M(r)},$$

where $d_M(r)$ is given in (11).

Proof: For ease of reading, the proof is relegated to Appendix B. ■

Lemma 4 shows the frequency-selective diversity gain achieved by the code $\mathcal{X}$ is sufficient to compensate for its loss in cooperative diversity gain. Once Lemma 4 is obtained, combining it with Lemma 3 gives $P_{\text{cwe}}(r|\mathcal{J}_M) \ \dot{\leq}\ \rho^{-d_M(r)}$, which agrees with (10). Then the proof of Theorem 2 is complete.

V. OFDM-based Asynchronous Cooperative Network using OSDF Protocol

In this section, we will extend our results to the asynchronous OSDF protocol. Borrowing notations from Section II, the signals at the qth subcarrier received respectively by the relays and the destination during the first and the second phases of the OSDF protocol are given by

$$\begin{aligned} r_{n,q} &= h_n \zeta_Q^{-q\nu_n} + z_{n,q}, \\ y_{1,q} &= g_1 \zeta_Q^{-q\tau_1} x_q + w_{1,q}, \\ y_{2,q} &= \sum_{m=1}^{M-1} g_{i_m} \zeta_Q^{-q\tau_{i_m}} s_{i_m,q} + w_{2,q}. \end{aligned}$$

As the only difference lies in $y_{2,q}$, the diversity gain achieved in the first phase is the same as that in the NSDF protocol, i.e., $\Pr\{\mathcal{J}_M\}$ is the same as (9). With regard to the second phase, the probability that the destination D is in outage and therefore fails to recover the message given $\mathcal{J}_M$ is

$$\begin{aligned} P_{\text{out}}(r|\mathcal{J}_M) = \Pr\Bigg\{ & \log\left(1+\rho|g_1|^2\right) \\ & + \log\left(1+\rho\sum_{m=1}^{M-1}|g_{i_m}|^2\right) \leq 2r\log\rho \Bigg\} \\ \doteq\ & \rho^{-d_M^o(r)}. \end{aligned}$$

It can be shown

$$d_M^o(r) = \begin{cases} M(1-2r)^+ + 2r, & \text{if } 0 \leq r \leq \frac{1}{2} \text{ and } M > 1, \\ 2(1-r), & \text{if } \frac{1}{2} \leq r \leq 1 \text{ and } M > 1, \\ (1-2r)^+, & \text{if } 0 \leq r \leq 1 \text{ and } M = 1. \end{cases} \tag{21}$$

Thus, combining (9) and (21) yields the following theorem.

Theorem 5: In an asynchronous cooperative network with $(N-1)$ relays, the diversity gain achieved by the OSDF protocol at multiplexing gain r is $d^o(r) = N(1-2r)^+$ for $0 \leq r \leq \frac{1}{2}$. ■

Again, we note that the above conclusion is drawn under the neglect of rate loss due to CP. Compared with the NSDF asynchronous DMT given in (12), we see the maximal multiplexing gain achieved by asynchronous OSDF protocol is $\frac{1}{2}$ while a full multiplexing gain of 1 can be maintained in asynchronous NSDF protocol. The decrease in maximal multiplexing gain in OSDF protocol is because the source ceases to transmit new information to destination in the second phase transmission. A comparison between these two DMTs is given in Fig. 3.

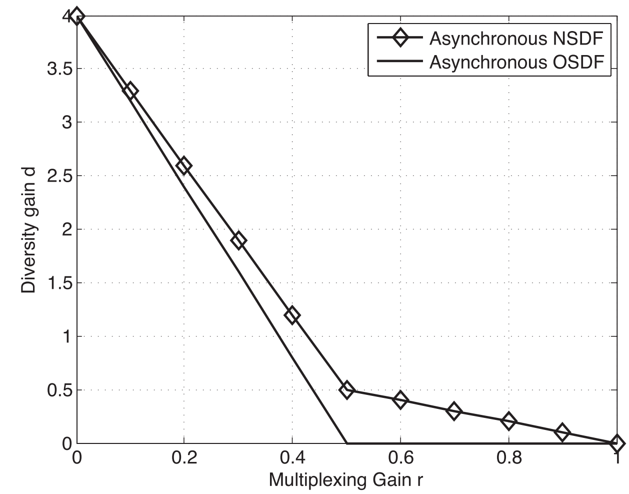

Fig. 3. Comparison of the DMTs of the asynchronous NSDF and OSDF protocols with $(N-1) = 3$ relays.

For code construction, below we will show that the same code $\mathcal{X}$ given in (13) can be used for OSDF protocol and achieves the optimal DMT $d^o(r)$. Specifically, given $\underline{x} = \theta[x, \sigma(x), \cdots, \sigma^{2Q-1}(x)]^\top \in \mathcal{X}$, the source S broadcasts the OFDM symbol $\theta[x, \sigma(x), \cdots, \sigma^{Q-1}(x)]^\top$ to all the relays and the destination in the first phase. The relays which have successfully decoded $\underline{x}$ based on the observations in the first phase will transmit the second half of $\underline{x}$, i.e., $\theta[\sigma^Q(x), \cdots, \sigma^{2Q-1}(x)]^\top$ to D in the second phase.

As the difference lies only in the second phase, the diversity result of the first phase holds the same as (15), or equivalently, (9). Regarding the second phase, it is clear that in event $\mathcal{J}_1$, i.e., $M = 1$, no relays will participate in the second-phase transmission as they all fail to decode the message; hence, the resulting codeword error probability is the same as that in the NSDF protocol, and we have

$$P_{\text{cwe}}(r|\mathcal{J}_1) \ \doteq\ \rho^{-(1-2r)^+}. \tag{22}$$

For $M > 1$, we redefine the ℓ_q's as

$$\ell_q = \begin{cases} \ell_q = g_1 \zeta_Q^{-q\cdot\tau_1}, & \text{if } 0 \leq q < Q, \\ \sum_{m=1}^{M-1} g_{i_m} \zeta_Q^{-q\cdot\tau_{i_m}}, & \text{if } Q \leq q < 2Q. \end{cases}$$

Following similar arguments as in Section IV-A we get

$$\begin{aligned} P_{\text{cwe}}(r|\mathcal{J}_M) &\leq \Pr\left\{ \sum_{q=0}^{2Q-1} \log_\rho\left(1+\rho|\ell_q|^2\right) \leq 2rQ \right\} \\ &\dot{\leq} \sum_{\substack{r_1+r_2 \leq 2r \\ r_1, r_2 \geq 0}} \sum_{S \in \mathcal{S}} \rho^{-(1-r_1)^+ - (M-1)(1-r_2)^+} \\ &\doteq \sup_{r_1+r_2 \leq 2r} \rho^{-(1-r_1)^+ - (M-1)(1-r_2)^+}. \end{aligned} \tag{23}$$

For general M, after combining (22) and (23) it is straightforward to show $P_{\text{cwe}}(r|\mathcal{J}_M) \dot{\leq} \rho^{-d_M^o(r)}$, where $d_M^o(r)$ is defined in (21). We summarize the above in the following theorem.

Theorem 6: The proposed vector code $\mathcal{X}$ (13) is DMT optimal for asynchronous OSDF protocol. ■

VI. Conclusion

In this paper we had presented the DMTs for the asynchronous cooperative network using NSDF and OSDF protocols, respectively. It was seen that using OFDM modulation and neglecting the rate loss due to CP, the DMTs are the same as those in the synchronous network. By seeking only the frequency selective diversity naturally provided by the asynchronous network, two optimal vector codes were proposed and were shown to achieve the optimal DMTs of these two asynchronous protocols, respectively. As the same vector code is used by all transmitting nodes, the proposed scheme is extremely simple compared to the complicated matrix codes reported in [8], [9], [12]. These vector codes offer a significant complexity reduction in not only the fabrication and deployment of replays, but also the decoding at destination.

Appendix

A. Proof of Lemma 3

To quickly outline a proof of Lemma 3, we set $\ell_q = g_1$ for $q = 0, \cdots, Q-1$ and we will be working with the ℓ_q's. First, we re-order the ℓ_q's such that $|\ell_{j_0}|^2 \leq |\ell_{j_1}|^2 \leq \cdots \leq |\ell_{j_{2Q-1}}|^2$, where $\{j_0, j_1, \cdots, j_{2Q-1}\} = \{0, 1, \cdots, 2Q-1\}$. We then rewrite (17) as $d_2^2(\underline{x}, \underline{x}') = \sum_{q=0}^{2Q-1} \theta^2 |\ell_{j_q}|^2 |\sigma^{j_q}(x - x')|^2$. It should be noted that $|\ell_{j_{2Q-1}}|^2 > 0$ with probability 1. Set $|\ell_{j_q}|^2 = \rho^{-\alpha_{j_q}}$. Following similar techniques as in (16) it can be shown that for any $0 \leq K \leq 2Q-1$,

$$P_{\text{cwe}}(r|\mathcal{J}_M) \leq \Pr\left\{ \sum_{q=2Q-K}^{2Q-1} \left(1 - \alpha_{j_q}\right) \leq 2\,rQ, \right.$$
$$\left. 1 \leq K \leq 2Q-1,\ \alpha_{j_0} \geq \cdots \geq \alpha_{j_{2Q-1}} \right\}.$$

Note the above event is the same as the event $\{\alpha_{j_0} \geq \cdots \geq \alpha_{j_{2Q-1}} \geq 0 : \sum_{q=0}^{2Q-1} (1 - \alpha_{j_q})^+ \leq 2rQ\}$ [15] and this proves Lemma 3.

B. Proof of Lemma 4

For simplicity, we first define the following two random variables,

$$T_1 := \sum_{q=0}^{Q-1} \log\left(1 + \rho |\ell_q|^2\right) = Q \log\left(1 + \rho |g_1|^2\right),$$
$$T_2 := \sum_{q=Q}^{2Q-1} \log\left(1 + \rho |\ell_q|^2\right).$$

With the above, we have the following series of upper bounds on the outage probability.

$$P_{\text{FS,out}}(r|\mathcal{J}_M) = \Pr\{T_1 + T_2 \ \leq\ 2\,rQ \log \rho\}$$
$$\leq \sum_{\substack{r_1+r_2 \leq 2r \\ r_1, r_2 \geq 0}} \Pr\{T_1 \leq r_1 Q \log \rho,\ T_2 \leq r_2 Q \log \rho\}$$
$$\overset{(a)}{\leq} \sum_{\substack{r_1+r_2 \leq 2r \\ r_1, r_2 \geq 0}} \Pr\left\{ T_1 \leq r_1 Q \log \rho, \text{ and} \right.$$
$$\left. \bigcup_{S \in \mathcal{S}} \left\{ \sum_{i=1}^{M-1} \log\left(1 + \rho |\ell_{s_i}|^2\right) \leq r_2(M-1) \log \rho \right\} \right\}$$
$$\overset{(b)}{\leq} \sum_{\substack{r_1+r_2 \leq 2r \\ r_1, r_2 \geq 0}} \sum_{S \in \mathcal{S}} \Pr\left\{ T_1 \leq r_1 Q \log \rho, \text{ and} \right.$$
$$\left. \sum_{i=1}^{M-1} \log\left(1 + \rho |\ell_{s_i}|^2\right) \leq r_2(M-1) \log \rho \right\}, \quad (24)$$

where $\mathcal{S} := \{S = \{s_1, \cdots, s_{M-1}\} : S \subseteq \{Q, \cdots, 2Q-1\}\}$ is the collection of all $(M-1)$-subsets. (a) follows from if $T_2 \leq r_2 Q \log \rho$, then there must exist an $(M-1)$-subset of the Q summands in T_2 whose sum is $\leq r_2(M-1)\log \rho$. (b) is the standard union-bound inequality.

To analyze each summand in (24), given any $S = \{s_1, \cdots, s_{M-1}\} \in \mathcal{S}$ we define

$$\underline{\ell}_S := \left[\ell_{s_1} \cdots \ell_{s_{M-1}}\right]^\top,$$
$$\underline{g} := \left[g_{i_1} \cdots g_{i_{M-1}}\right]^\top,$$
$$\underline{\mu}_S := \left[\zeta_Q^{-s_1 \tau_1} \cdots \zeta_Q^{-s_{M-1}\tau_1}\right]^\top.$$

Then by (18) the random variables ℓ_{s_i}'s involved in (24) can be rewritten as a random vector

$$\underline{\ell}_S \ = \ C_S\, \underline{g} + g_1 \underline{\mu}_S,$$

where the (p,q)th entry of the $((M-1) \times (M-1))$ matrix C_S is $(C_S)_{p,q} = \zeta_Q^{-(s_{i_p}-Q)\tau_{i_q}} = \zeta_Q^{-s_{i_p}\tau_{i_q}}$, $1 \leq p, q \leq M-1$. Clearly, C_S is a matrix of Vandermonde type. C_S has full rank of $(M-1)$ since the time delays τ_n are all distinct. Furthermore, the complex Gaussian random vector $\underline{\ell}_S$ has mean $\underline{0}$ and covariance matrix

$$K_{\underline{\ell}_S} = \mathbb{E}\underline{\ell}_S \underline{\ell}_S^\dagger = C_S C_S^\dagger + \underline{\mu}_S \underline{\mu}_S^\dagger = U_S \Lambda_S U_S^\dagger,$$

where $U_S = [u_{m,m'}]$ is an $(M-1) \times (M-1)$ unitary matrix and $\Lambda_S = \operatorname{diag}(\lambda_1, \cdots, \lambda_{M-1})$ are the eigenvalues of $K_{\underline{\ell}_S}$ with $0 < \lambda_m \doteq \rho^0$ for all $m = 1, \cdots, M-1$. By the simulation of complex Gaussian random vectors we set $\underline{\ell}_S = U_S \sqrt{\Lambda_S}\ \underline{v}$, where $\underline{v} = [v_1, \cdots, v_{M-1}]^\top$ is a length-$(M-1)$ random vector with i.i.d. $\mathbb{CN}(0,1)$ entries. It then follows that

$$\sum_{i=1}^{M-1} \log\left(1 + \rho |\ell_{s_i}|^2\right)$$
$$= \sum_{i=1}^{M-1} \log\left(1 + \left|\left(U_S \sqrt{\rho \Lambda_S}\ \underline{v}\right)_i\right|^2\right)$$
$$= \sum_{i=1}^{M-1} \log\left(1 + \left|\left(U_S \sqrt{\rho}\ \underline{v}\right)_i\right|^2\right) + O(1),$$

where the $O(1)$ term follows from the eigenvalues $\lambda_m \doteq \rho^0$. Substituting the above result into (24) yields

$$\begin{aligned}
&\Pr\Bigg\{ \left(1+\rho\,|g_1|^2\right) \le r_1 \log\rho, \text{ and} \\
&\qquad \sum_{i=1}^{M-1} \log\left(1+\rho\,|\ell_{s_i}|^2\right) \le r_2(M-1)\log\rho \Bigg\} \\
&\overset{(a)}{\doteq} \Pr\Bigg\{ \left(1+\rho\,|g_1|^2\right) \le r_1 \log\rho, \text{ and} \\
&\qquad \sum_{i=1}^{M-1} \log\left(1+\rho\,|(U_S\, \underline{v})_i|^2\right) \le r_2(M-1)\log\rho \Bigg\} \\
&\overset{(b)}{\doteq} \Pr\Bigg\{ \left(1+\rho\,|g_1|^2\right) \le r_1 \log\rho, \text{ and} \\
&\qquad \sum_{i=1}^{M-1} \log\left(1+\rho\,|v_i|^2\right) \le r_2(M-1)\log\rho \Bigg\} \\
&\overset{(c)}{\doteq} \rho^{-(1-r_1)^+ - (M-1)(1-r_2)^+}, \qquad (25)
\end{aligned}$$

where (a) follows from dropping the $O(1)$ term, (b) is because $\underline{v}$ and $U_S\,\underline{v}$ are of the same statistical distribution, and (c) is due to the point-to-point multi-block DMT [11], [15]. Finally, combining results of (24), and (25), we see that given $\mathcal{J}_M$, the channel outage probability is upper-bounded by

$$P_{\text{FS,out}}(r|\mathcal{J}_M) \le \sum_{\substack{r_1+r_2\le 2r \\ r_1,r_2\ge 0}} \sum_{S\in\mathcal{S}} \rho^{-(1-r_1)^+ - (M-1)(1-r_2)^+}.$$

Hence the corresponding diversity gain at the second phase is given by

$$\begin{aligned}
&\inf_{r_1+r_2\le 2r} \left\{ (1-r_1)^+ + (M-1)(1-r_2)^+ \right\} \\
&= \begin{cases} M(1-2r)^+ + 2r, & \text{if } 0\le r\le \frac{1}{2}, \text{ and } M>1, \\ 2(1-r), & \text{if } \frac{1}{2}\le r\le 1, \text{ and } M>1. \end{cases}
\end{aligned}$$

This agrees exactly with the first two cases in (11). Now the proof of Lemma 4 is complete. ■

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