1 Introduction

In many scientific models, discrete reaction diffusion equations play an important and in some cases essential roles. In this report, we study the basic problem of stationary patterns and spatial structures for discrete reaction diffusion equations. The system we consider takes the form

$$\frac{du_i}{dt} = \beta\Delta u_i + \alpha f(u_i), \quad i \in \mathbb{Z}^1, \tag{1.1}$$

in one dimension and it takes the form

$$\frac{du_{i,j}}{dt} = \beta^+\Delta^+u_{i,j} + \beta^\times\Delta^\times u_{i,j} + \alpha f(u_{i,j}), \text{ where } (i,j) \in \mathbb{Z}^2, \tag{1.2}$$

in two dimension, where

$$\begin{aligned} \Delta u_i &= u_{i+1} + u_{i-1} - 2u_i \\ \Delta^+u_{i,j} &= u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \\ \Delta^\times u_{i,j} &= u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j}. \end{aligned} \tag{1.3}$$

Herein, for simplicity, we consider f as the cubic function in the form

$$f(\xi) = \xi^3 - \xi. \tag{1.4}$$

(1.1) and (1.2) have been considered in Chow et.al. [3]. The nonlinearity f therein is not a smooth function, but instead a set-valued function

$$f(\xi) = \begin{cases} (-\infty, -\gamma] & \text{if } \xi = -1, \\ \gamma\xi & \text{if } |\xi| < 1, \\ [\gamma, \infty) & \text{if } \xi = 1, \\ \emptyset & \text{if } |\xi| > 1. \end{cases} \tag{1.5}$$

As mentioned in [3], other nonlinearity f for such a reaction diffusion equation include cubic polynomial such as $f(\xi) = \gamma\xi + \xi^3$ or $f(\xi) = (\xi^2 - 1)(\xi - a)$, or more generally a smooth function with $f(\xi) \to \pm\infty$ as $\xi \to \pm\infty$ at a rate faster than linear. The logarithmic nonlinearity

$$f(\xi) = \gamma\xi + \ln(\frac{1+\xi}{1-\xi})$$

is also interesting which restricts the range of the argument to $-1 < \xi < 1$. Thus, we take a typical nonlinearity $f(\xi) = \xi^3 - \xi$. Our results can be adopted to (1.1), (1.2) with all of the above mentioned nonlinearity.

Chow et.al. [3] discussed mosaic solutions, their stability, pattern formation, spatial entropy, and spatial chaos. The mosaic patterns and solutions therein takes the value u_i or $u_{i,j} = -1, 0, 1$. In this paper, we use the basic pattern formation [7, 10] to discuss formations of mosaic patterns, spatial entropy and effect of boundary conditions for the discrete reaction diffusion equations (1.1), (1.2) with (1.3). Our treatments are motivated by numerical sense as well as the sense from real-world pattern formations. In comparing with the definition in [3], our patterns should be named as *mosaic patterns*, since we consider the component of the stationary solutions to lie in a small range, instead of just some single values, namely,

$$u_i \text{ or } u_{i,j} \in [-1-\sigma, -1+\sigma] \cup [-\sigma, \sigma] \cup [1-\sigma, 1+\sigma],$$

where σ is a small number. Stability and basins of attraction for these stationary solutions have been studied in [4]. We shall also discuss the question of the influence of boundary conditions on the spatial entropy, raised in [1]:

$$h = h_N = h_P = h_D?$$

Herein, h denotes the spatial entropy; and h_N, h_P and h_D respectively represent the spatial entropy for the same class of patterns that satisfy Neumann, periodic and Dirichlet boundary conditions.

In [7], [10], [11], [2], they construct the basic mosaic patterns and use translation-invariant property discuss the patterns formations and spatial chaos in cellular neural networks. [11] also discussed the question "$h = h_N = h_P = h_D$?" for cellular neural networks. We continue to use their formations in this report.

In the following, we write the spatially discrete reaction diffusion equations as "SD-RDE" for abbreviation. The rest of this paper is organized as follows. In Section 2, we definite the *mosaic pattern* for SD-RDE. And we use the concept of pseudo basic pattern to partition the parameter space in one dimension. Corresponding to each parameter region, there are sets of pseudo basic patterns and possible pseudo basic patterns. In Section 3, according to the fixed point theorem, we can use the pseudo basic patterns to span the global patterns in one dimension. In infinite lattices, we use the Schauder fixed point theorem to proof the result. In finite lattices, we use the Brouwer fixed theorem to prove the result. We divide Section 4 into three subsections. In subsection 4.1, we discuss mosaic patterns on one-dimensional infinite lattice. We use the transition matrices to span the global patterns. In subsection 4.2, we analyze the case of two-dimensional lattices.

We use the same method to partition the parameter space and the ideas from [10] to estimate lower and upper bounds of the spatial entropy. In subsection 4.3, We discuss the entropy in infinite lattices and effect of the boundary conditions on spatial entropy. We use the Proposition in [11] to discuss the question "$h = h_N = h_P = h_D$?" In Section 5, we provide some numerical illustrations. Those patterns in the illustrations also appear in [3].

2 Partitioning Parameter Space and Basic Patterns

In [3], $\mathbf{u} = \{u_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d}$ is called a *mosaic solution*, if $u_{\mathbf{i}} \in \{-1, 0, 1\}$, for all $\mathbf{i} \in \mathbb{Z}^d$, where $d = 1$ or 2. In this work, we want to extend such a notion to mosaic solutions for SD-RDE (1.1), (1.2). Their corresponding patterns have the same appearance as mosaic patterns.

Definition 2.1. *For a small fixed positive number σ, we say that a stationary solution $\mathbf{u} = \{u_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d}$ of (1.1) or (1.2), with (1.4), is a mosaic solution if*

$$u_{\mathbf{i}} \in [-1-\sigma, -1+\sigma] \cup [-\sigma, \sigma] \cup [1-\sigma, 1+\sigma], \ \textit{for all } \mathbf{i} \in \mathbb{Z}^d,$$

where $d = 1$ or 2.

For sure that with a small $\sigma > 0$, there associate different groups of similar solutions, corresponding to different parameters of the system. However, their appearances as patterns are similar, despite of small differences among the values of σ. We introduce $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d}$ to represent the corresponding pattern of $\{u_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d}$ by

$$\begin{aligned} s_i &= - \text{ , if } -1-\sigma \le u_i \le -1+\sigma, \\ s_i &= \times \text{ , if } -\sigma \le u_i \le \sigma, \\ s_i &= + \text{ , if } 1-\sigma \le u_i \le 1+\sigma. \end{aligned} \tag{2.1}$$

Consider the case $d = 1$, the stationary (equilibrium) equation for (1.1) is

$$\beta(u_{i+1} + u_{i-1} - 2u_i) + \alpha f(u_i) = 0, \ i \in \mathbb{Z}^1, \tag{2.2}$$

where, α and β are two parameters. For an $i \in \mathbb{Z}^1$, let $(\eta_{i-1}, \eta_i, \eta_{i+1}) \in \mathbb{R}^3$ with

$$\eta_l \in [-1-\sigma, -1+\sigma] \cup [-\sigma, \sigma] \cup [1-\sigma, 1+\sigma], \ l = i-1, i, i+1.$$

If $(u_{i-1}, u_i, u_{i+1}) = (\eta_{i-1}, \eta_i, \eta_{i+1})$ satisfies (2.2) for this specific $i \in \mathbb{Z}^1$ and if $(\eta_{i-1}, \eta_i, \eta_{i+1})$ is represented by (s_{i-1}, s_i, s_{i+1}), $s_l =$ " $+$ ", " $\times$ ", " $-$ ", $l = i-1, i, i+1$ according to