We use the same method to partition the parameter space and the ideas from [10] to estimate lower and upper bounds of the spatial entropy. In subsection 4.3, We discuss the entropy in infinite lattices and effect of the boundary conditions on spatial entropy. We use the Proposition in [11] to discuss the question "$h = h_N = h_P = h_D$?" In Section 5, we provide some numerical illustrations. Those patterns in the illustrations also appear in [3].

2 Partitioning Parameter Space and Basic Patterns

In [3], $\mathbf{u} = \{u_\mathbf{i}\}_{\mathbf{i}\in\mathbb{Z}^d}$ is called a *mosaic solution*, if $u_\mathbf{i} \in \{-1, 0, 1\}$, for all $\mathbf{i} \in \mathbb{Z}^d$, where $d = 1$ or 2. In this work, we want to extend such a notion to mosaic solutions for SD-RDE (1.1), (1.2). Their corresponding patterns have the same appearance as mosaic patterns.

Definition 2.1. *For a small fixed positive number σ, we say that a stationary solution $\mathbf{u} = \{u_\mathbf{i}\}_{\mathbf{i}\in\mathbb{Z}^d}$ of (1.1) or (1.2), with (1.4), is a mosaic solution if*

$$u_\mathbf{i} \in [-1-\sigma, -1+\sigma] \cup [-\sigma, \sigma] \cup [1-\sigma, 1+\sigma], \ \textit{for all } \mathbf{i} \in \mathbb{Z}^d,$$

where $d = 1$ or 2.

For sure that with a small $\sigma > 0$, there associate different groups of similar solutions, corresponding to different parameters of the system. However, their appearances as patterns are similar, despite of small differences among the values of σ. We introduce $\{s_\mathbf{i}\}_{\mathbf{i}\in\mathbb{Z}^d}$ to represent the corresponding pattern of $\{u_\mathbf{i}\}_{\mathbf{i}\in\mathbb{Z}^d}$ by

$$\begin{aligned} s_i &= - \text{ , if } -1-\sigma \le u_i \le -1+\sigma, \\ s_i &= \times \text{ , if } -\sigma \le u_i \le \sigma, \\ s_i &= + \text{ , if } 1-\sigma \le u_i \le 1+\sigma. \end{aligned} \tag{2.1}$$

Consider the case $d = 1$, the stationary (equilibrium) equation for (1.1) is

$$\beta(u_{i+1} + u_{i-1} - 2u_i) + \alpha f(u_i) = 0, \ i \in \mathbb{Z}^1, \tag{2.2}$$

where, α and β are two parameters. For an $i \in \mathbb{Z}^1$, let $(\eta_{i-1}, \eta_i, \eta_{i+1}) \in \mathbb{R}^3$ with

$$\eta_l \in [-1-\sigma, -1+\sigma] \cup [-\sigma, \sigma] \cup [1-\sigma, 1+\sigma], \ l = i-1, i, i+1.$$

If $(u_{i-1}, u_i, u_{i+1}) = (\eta_{i-1}, \eta_i, \eta_{i+1})$ satisfies (2.2) for this specific $i \in \mathbb{Z}^1$ and if $(\eta_{i-1}, \eta_i, \eta_{i+1})$ is represented by (s_{i-1}, s_i, s_{i+1}), $s_l =$ " $+$ ", " $\times$ ", " $-$ ", $l = i-1, i, i+1$ according to

(2.1), then the triple tuple symbol (s_{i-1}, s_i, s_{i+1}) consisting of "$+$", "$\times$", "$-$" is called a pseudo basic pattern for (2.2).

Next, we want to partition the parameter space $\mathcal{P} = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{R}\}$ to establish the existence of pseudo mosaic patterns with respect to the parameters. For convenience, we rewrite (2.2) as

$$b(u_{i+1} + u_{i-1} - 2u_i) + f(u_i) = 0, \; i \in \mathbb{Z} \tag{2.3}$$

where

$$b = \frac{\beta}{\alpha}, \; \alpha \neq 0. \tag{2.4}$$

Then we partition the parameter space $\mathcal{P}_1 = \{b : b \in \mathbb{R}\}$ into finitely many regions such that in each region, (2.2) has the same pseudo basic patterns. For any $i \in \mathbb{Z}$, (u_{i-1}, u_i, u_{i+1}) satisfies (2.2) if (u_i, y_i) satisfies

$$y_i = f(u_i) \text{ with } f(u_i) = u_i(u_i - 1)(u_i + 1) \tag{2.5}$$

and

$$y_i = b[2u_i - u_{i-1} - u_{i+1}]. \tag{2.6}$$

We rewrite (2.5):

$$y_i = 2b[u_i - \frac{(u_{i-1} + u_{i+1})}{2}] \tag{2.7}$$

We use intercept $u_i = \frac{u_{i-1}+u_{i+1}}{2}$ and slope $2b$ to classify the parameter regions and characterize what basic patterns will appear in each parameter region.

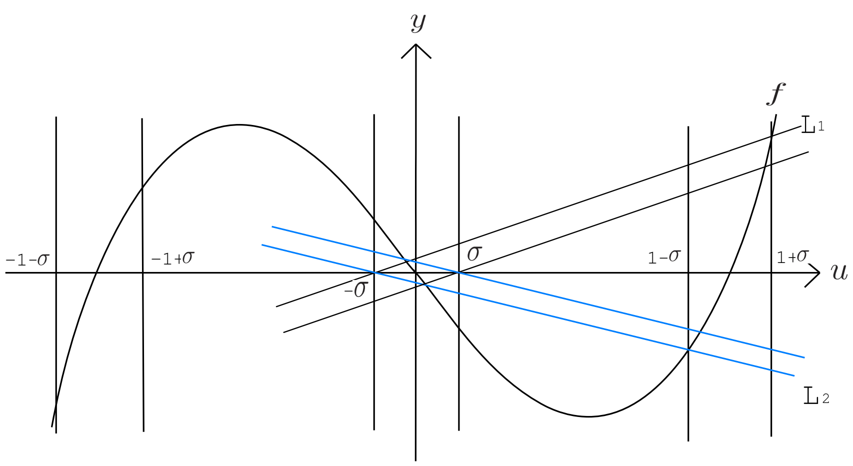

Figure 1: The intersection of (2.5) and (2.6). The slope of L_1 is m_1 and the slope of L_2 is m_2.

Let us use the following example to illustrate the idea. Given $\widetilde{u}_{i-1}, \widetilde{u}_{i+1} \in [-\sigma, \sigma]$, so $(\widetilde{u}_{i-1} + \widetilde{u}_{i+1}) \in [-2\sigma, 2\sigma]$. If there is an intersection for (2.5) and (2.6) with $u_{i-1} = \widetilde{u}_{i-1}$, $u_{i+1} = \widetilde{u}_{i+1}$ at $u_i \in [1-\sigma, 1+\sigma]$, then we have a pseudo basic pattern $(\widetilde{u}_{i-1}, u_i, \widetilde{u}_{i+1}) = \overline{\times + \times}$. In order to guarantee the existence of intersection, we need to restrict the value of b such that the graph of f is located between $L_1 : y = m_1(u+\sigma)$ and $L_2 : y = m_2(u+\sigma)$. In Fig. 1, this parameter region is $\frac{f(1-\sigma)}{2} \leq b \leq \frac{f(1+\sigma)}{2+4\sigma}$ (for the confirmative existence of the pseudo basic pattern $\overline{\times + \times}$).

Notation 2.2. *We classify the pseudo basic patterns into $B^\bullet_2$, $B^\bullet_1$, $B^\bullet_0$, $B^\bullet_{-1}$, $B^\bullet_{-2}$, where $\bullet = +, \times$ or $-$, as in Table 1, and denote the following sets:*

$$B^\bullet_{\{i_1,i_2,\cdots,i_k\}} = \bigcup_{i=i_1,\cdots,i_k} B^\bullet_i, \; i_\ell = \pm 2, \; \pm 1, \; \textit{or } 0.$$

Notation	basic patterns
$B^\bullet_2$	$+ \bullet +$
$B^\bullet_1$	$+ \bullet \times \quad \times \bullet +$
$B^\bullet_0$	$+ \bullet - \quad \times \bullet \times \quad - \bullet +$
$B^\bullet_{-1}$	$\times \bullet - \quad - \bullet \times$
$B^\bullet_{-2}$	$- \bullet -$

Table 1: Notations for collections of basic patterns, $\bullet = +, \times, -$.

Using previous observation we have the following result.

Theorem 2.3. *Suppose that σ is a fixed number with $0 < \sigma < \frac{1}{11}$. Consider the equations* (1.1), (1.4). *The pseudo basic patterns exist in respective parameter regions as in Table 2.*

In the next section we shall obtain some global patterns in respective parameter regions by employing *Schauder fixed point theorem.* In order to apply the theorem, we need to construct a convex compact subset of a Banach space. In addition to the idea, we wander if there still exist other "possible" pseudo basic patterns. So, it is worthy to find out all possible pseudo basic patterns in all parameter regions. In the view point, we need to consider all the possibility of the intersection of the graph of $f = u(u-1)(u+1)$ with L_1, L_2.

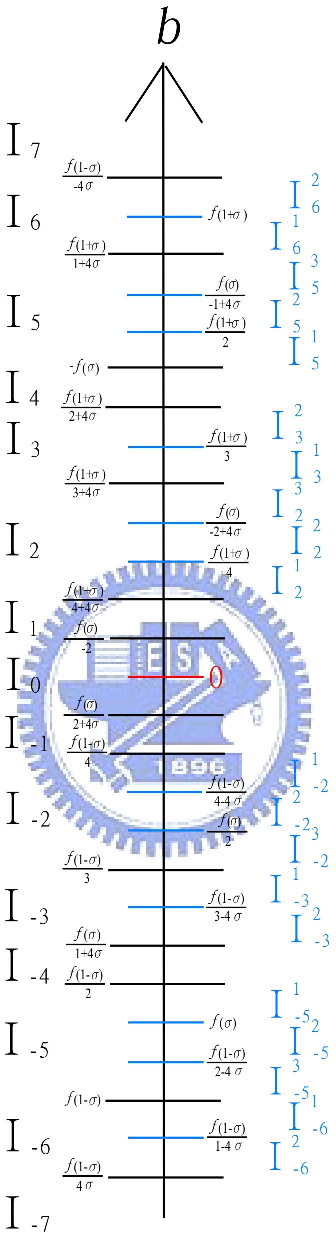

Figure 2: Partitioning parameter space for one-dimensional SD-RDE.

parameter regions	pseudo basic patterns
$\mathrm{I}_7 = [\frac{f(-1+\sigma)}{4\sigma}, \infty]$	$B^{\times}_{\{0\}}$
$\mathrm{I}_6 = [\frac{f(1+\sigma)}{1+4\sigma}, \frac{f(-1+\sigma)}{4\sigma}]$	$B^{+}_{\{2\}}, B^{\times}_{\{0\}}, B^{-}_{\{-2\}}$
$\mathrm{I}_5 = [f(-\sigma), \frac{f(1+\sigma)}{1+4\sigma}]$	$B^{+}_{\{2,1\}}, B^{\times}_{\{0\}}, B^{-}_{\{-1,-2\}}$
$\mathrm{I}_4 = [\frac{f(1+\sigma)}{2+4\sigma}, f(-\sigma)]$	$B^{+}_{\{2,1\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{-1,-2\}}$
$\mathrm{I}_3 = [\frac{f(1+\sigma)}{3+4\sigma}, \frac{f(1+\sigma)}{2+4\sigma}]$	$B^{+}_{\{2,1,0\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{0,-1,-2\}}$
$\mathrm{I}_2 = [\frac{f(1+\sigma)}{4+4\sigma}, \frac{f(1+\sigma)}{3+4\sigma}]$	$B^{+}_{\{2,1,0,-1\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{1,0,-1,-2\}}$
$\mathrm{I}_1 = [\frac{f(-\sigma)}{2}, \frac{f(1+\sigma)}{4+4\sigma}]$	$B^{+}_{\{2,1,0,-1,-2\}}, B^{\times}_{\{2,1,0,-1,-2\}}, B^{-}_{\{1,0,-1\}}$
$\mathrm{I}_0 = [-\frac{f(-\sigma)}{2+4\sigma}, \frac{f(-\sigma)}{2}]$	$B^{+}_{\{2,1,0,-1,-2\}}, B^{\times}_{\{2,1,0,-1,-2\}}, B^{-}_{\{2,1,0,-1,-2\}}$
$\mathrm{I}_{-1} = [-\frac{f(-1+\sigma)}{4}, -\frac{f(-\sigma)}{2+4\sigma}]$	$B^{+}_{\{2,1,0,-1,-2\}}, B^{\times}_{\{2,1,0,-1,-2\}}, B^{-}_{\{1,0,-1\}}$
$\mathrm{I}_{-2} = [-\frac{f(-1+\sigma)}{3}, -\frac{f(-1+\sigma)}{4}]$	$B^{+}_{\{2,1,0,-1\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{1,0,-1,-2\}}$
$\mathrm{I}_{-3} = [-\frac{f(-\sigma)}{1+4\sigma}, -\frac{f(-1+\sigma)}{3}]$	$B^{+}_{\{2,1,0\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{0,-1,-2\}}$
$\mathrm{I}_{-4} = [-\frac{f(-1+\sigma)}{2}, -\frac{f(-\sigma)}{1+4\sigma}]$	$B^{+}_{\{2,1,0\}}, B^{\times}_{\{0\}}, B^{-}_{\{0,-1,-2\}}$
$\mathrm{I}_{-5} = [-f(-1+\sigma), -\frac{f(-1+\sigma)}{2}]$	$B^{+}_{\{2,1\}}, B^{\times}_{\{0\}}, B^{-}_{\{-1,-2\}}$
$\mathrm{I}_{-6} = [-\frac{f(-\sigma)}{4\sigma}, -f(-1+\sigma)]$	$B^{+}_{\{2\}}, B^{\times}_{\{0\}}, B^{-}_{\{-2\}}$
$\mathrm{I}_{-7} = [-\infty, -\frac{f(-\sigma)}{4\sigma}]$	$B^{+}_{\{2\}}, B^{-}_{\{-2\}}$

Table 2: Existence of pseudo basic patterns in parameter regions

3 From Basic Patterns to Global Patterns

In the previous section, we define twenty-seven pseudo basic patterns. In this section, we discuss one-dimensional lattice. For any two pseudo basic patterns $\mathbf{s}_1 = \overline{\mathrm{w}_1\mathrm{m}_1\mathrm{e}_1}$, $\mathbf{s}_2 = \overline{\mathrm{w}_2\mathrm{m}_2\mathrm{e}_2}$, where $\mathrm{w}_l, \mathrm{m}_l, \mathrm{e}_l = +, \times, -, l = 1, 2$, we say that the pseudo basic pattern $\mathbf{s}_2$ can be attached to the right of pseudo basic pattern $\mathbf{s}_1$, if $\mathrm{w}_2 = \mathrm{m}_1$ and $\mathrm{m}_2 = \mathrm{e}_1$. Using the method of attaching compatible pseudo basic patterns, we obtain pseudo global mosaic patterns. Since $\mathrm{w}_l, \mathrm{m}_l, \mathrm{e}_l, l = 1, 2$, correspond to some values in three possible intervals, each of length 2σ, $\mathrm{w}_2 = \mathrm{m}_1$ and $\mathrm{m}_2 = \mathrm{e}_1$ may not imply their corresponding values are identical. We shall claim the existence of global mosaic patterns in the following content.

In the case $\Lambda = \mathbb{Z}^d$, $d = 1$ or 2, consider the phase space

$$\mathcal{X} = \{u = \{u_{\mathbf{i}}\} : \mathbf{i} \in \mathbb{Z}^d, \|u_{\mathbf{i}}\| < \infty\},$$

where $u_{\mathbf{i}} \in \mathbb{R}$ and the norm $\|\cdot\|$ could be $\|\cdot\|_{\ell^\infty}$, the ℓ^∞ norm , or $\|\cdot\|_{\ell^2_q}$, the ℓ^2_q norm. The ℓ^2_q norm is defined as follows:

$$\|u\|_{\ell^2_q} = (\sum_{\mathbf{i}\in\mathbb{Z}^d} q^{-|\mathbf{i}|}|u_{\mathbf{i}}|^2)^{\frac{1}{2}},$$