parameter regions	pseudo basic patterns
$\mathrm{I}_7 = [\frac{f(-1+\sigma)}{4\sigma}, \infty]$	$B^{\times}_{\{0\}}$
$\mathrm{I}_6 = [\frac{f(1+\sigma)}{1+4\sigma}, \frac{f(-1+\sigma)}{4\sigma}]$	$B^{+}_{\{2\}}, B^{\times}_{\{0\}}, B^{-}_{\{-2\}}$
$\mathrm{I}_5 = [f(-\sigma), \frac{f(1+\sigma)}{1+4\sigma}]$	$B^{+}_{\{2,1\}}, B^{\times}_{\{0\}}, B^{-}_{\{-1,-2\}}$
$\mathrm{I}_4 = [\frac{f(1+\sigma)}{2+4\sigma}, f(-\sigma)]$	$B^{+}_{\{2,1\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{-1,-2\}}$
$\mathrm{I}_3 = [\frac{f(1+\sigma)}{3+4\sigma}, \frac{f(1+\sigma)}{2+4\sigma}]$	$B^{+}_{\{2,1,0\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{0,-1,-2\}}$
$\mathrm{I}_2 = [\frac{f(1+\sigma)}{4+4\sigma}, \frac{f(1+\sigma)}{3+4\sigma}]$	$B^{+}_{\{2,1,0,-1\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{1,0,-1,-2\}}$
$\mathrm{I}_1 = [\frac{f(-\sigma)}{2}, \frac{f(1+\sigma)}{4+4\sigma}]$	$B^{+}_{\{2,1,0,-1,-2\}}, B^{\times}_{\{2,1,0,-1,-2\}}, B^{-}_{\{1,0,-1\}}$
$\mathrm{I}_0 = [-\frac{f(-\sigma)}{2+4\sigma}, \frac{f(-\sigma)}{2}]$	$B^{+}_{\{2,1,0,-1,-2\}}, B^{\times}_{\{2,1,0,-1,-2\}}, B^{-}_{\{2,1,0,-1,-2\}}$
$\mathrm{I}_{-1} = [-\frac{f(-1+\sigma)}{4}, -\frac{f(-\sigma)}{2+4\sigma}]$	$B^{+}_{\{2,1,0,-1,-2\}}, B^{\times}_{\{2,1,0,-1,-2\}}, B^{-}_{\{1,0,-1\}}$
$\mathrm{I}_{-2} = [-\frac{f(-1+\sigma)}{3}, -\frac{f(-1+\sigma)}{4}]$	$B^{+}_{\{2,1,0,-1\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{1,0,-1,-2\}}$
$\mathrm{I}_{-3} = [-\frac{f(-\sigma)}{1+4\sigma}, -\frac{f(-1+\sigma)}{3}]$	$B^{+}_{\{2,1,0\}}, B^{\times}_{\{1,0,-1\}}, B^{-}_{\{0,-1,-2\}}$
$\mathrm{I}_{-4} = [-\frac{f(-1+\sigma)}{2}, -\frac{f(-\sigma)}{1+4\sigma}]$	$B^{+}_{\{2,1,0\}}, B^{\times}_{\{0\}}, B^{-}_{\{0,-1,-2\}}$
$\mathrm{I}_{-5} = [-f(-1+\sigma), -\frac{f(-1+\sigma)}{2}]$	$B^{+}_{\{2,1\}}, B^{\times}_{\{0\}}, B^{-}_{\{-1,-2\}}$
$\mathrm{I}_{-6} = [-\frac{f(-\sigma)}{4\sigma}, -f(-1+\sigma)]$	$B^{+}_{\{2\}}, B^{\times}_{\{0\}}, B^{-}_{\{-2\}}$
$\mathrm{I}_{-7} = [-\infty, -\frac{f(-\sigma)}{4\sigma}]$	$B^{+}_{\{2\}}, B^{-}_{\{-2\}}$

Table 2: Existence of pseudo basic patterns in parameter regions

3 From Basic Patterns to Global Patterns

In the previous section, we define twenty-seven pseudo basic patterns. In this section, we discuss one-dimensional lattice. For any two pseudo basic patterns $\mathbf{s}_1 = \overline{\mathrm{w}_1\mathrm{m}_1\mathrm{e}_1}$, $\mathbf{s}_2 = \overline{\mathrm{w}_2\mathrm{m}_2\mathrm{e}_2}$, where $\mathrm{w}_l, \mathrm{m}_l, \mathrm{e}_l = +, \times, -, l = 1, 2$, we say that the pseudo basic pattern $\mathbf{s}_2$ can be attached to the right of pseudo basic pattern $\mathbf{s}_1$, if $\mathrm{w}_2 = \mathrm{m}_1$ and $\mathrm{m}_2 = \mathrm{e}_1$. Using the method of attaching compatible pseudo basic patterns, we obtain pseudo global mosaic patterns. Since $\mathrm{w}_l, \mathrm{m}_l, \mathrm{e}_l, l = 1, 2$, correspond to some values in three possible intervals, each of length 2σ, $\mathrm{w}_2 = \mathrm{m}_1$ and $\mathrm{m}_2 = \mathrm{e}_1$ may not imply their corresponding values are identical. We shall claim the existence of global mosaic patterns in the following content.

In the case $\Lambda = \mathbb{Z}^d$, $d = 1$ or 2, consider the phase space

$$\mathcal{X} = \{u = \{u_{\mathbf{i}}\} : \mathbf{i} \in \mathbb{Z}^d, \|u_{\mathbf{i}}\| < \infty\},$$

where $u_{\mathbf{i}} \in \mathbb{R}$ and the norm $\|\cdot\|$ could be $\|\cdot\|_{\ell^\infty}$, the ℓ^∞ norm , or $\|\cdot\|_{\ell^2_q}$, the ℓ^2_q norm. The ℓ^2_q norm is defined as follows:

$$\|u\|_{\ell^2_q} = (\sum_{\mathbf{i}\in\mathbb{Z}^d} q^{-|\mathbf{i}|}|u_{\mathbf{i}}|^2)^{\frac{1}{2}},$$

where $q > 0$ is a fixed number.

Theorem 3.1. *Assume that σ is a small positive constant with $\sigma < \frac{1}{11}$. If, for* (1.1) *and* (1.2) *with certain parameters, $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d}$ is a pseudo global mosaic pattern, then there exists a global mosaic solution $\{u_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^d}$, with $u_{\mathbf{i}} \in [-1-\sigma, -1+\sigma]$ if $s_{\mathbf{i}} = -$, and $u_{\mathbf{i}} \in [-\sigma, \sigma]$ if $s_{\mathbf{i}} = \times$, and $u_{\mathbf{i}} \in [1-\sigma, 1+\sigma]$, if $s_{\mathbf{i}} = +$.*

Proof. We take the case of one-dimensional lattice as an illustration. We shall claim the existence of global mosaic solution $\{u_i\}$ by using the Schauder fixed point theorem [6]. Assume that there exists a pseudo global mosaic pattern $\{s_i\}_{i\in\mathbb{Z}^1}$, then for given $\{\widetilde{u}_i\}$, with $\widetilde{u}_i \in [-1-\sigma, -1+\sigma]$ if $s_i = -$, and $\widetilde{u}_i \in [-\sigma, \sigma]$ if $s_i = \times$, and $\widetilde{u}_i \in [1-\sigma, 1+\sigma]$, if $s_i = +$, there is an intersection (u_i, y_i) at designated location for the graphs of $y_i = b[2u_i - \widetilde{u}_{i-1} - \widetilde{u}_{i+1}]$ and $y_i = f(u_i)$, for all $i \in \mathbb{Z}$, according to our formulations. Such a u_i, for each i, lies in $[-1-\sigma, -1+\sigma]$ if $s_i = -$, $[-\sigma, \sigma]$ if $s_i = \times$, and $[1-\sigma, 1+\sigma]$, if $s_i = +$. Set

$$\begin{aligned} \mathcal{V} = \{\{v_i\} : -1-\sigma \le v_i \le -1+\sigma, \text{ if } s_i &= -, \\ -\sigma \le v_i \le \sigma, \text{ if } s_i &= \times, \\ 1-\sigma \le v_i \le 1+\sigma, \text{ if } s_i &= +\}. \end{aligned} \tag{3.1}$$

The graphs of the straight lines $y_i = b(2u_i - \widetilde{u}_{i-1} + \widetilde{u}_{i+1})$ vary smoothly in $\widetilde{u}_{i-1}$, $\widetilde{u}_{i+1}$. Since there is always an intersection between the lines and and the graph of $y_i = f(u_i)$, for each i, the intersection point varies smoothly in $\widetilde{u}_{i-1}$, $\widetilde{u}_{i+1}$.

We claim that $\mathcal{V}$ is a convex, compact subset of the Banach space $\mathcal{X}$. For all $0 \le \mu \le 1$, $\{v_i\}, \{w_i\} \in \mathcal{V}$,
if $s_i = +$,

$$\begin{aligned} &1-\sigma \le v_i \le 1+\sigma \\ &1-\sigma \le w_i \le 1+\sigma \\ \Rightarrow\ &1-\sigma \le \mu v_i + (1-\mu)w_i \le 1+\sigma, \end{aligned}$$

if $s_i = \times$,

$$\begin{aligned} &-\sigma \le v_i \le \sigma \\ &-\sigma \le w_i \le \sigma \\ \Rightarrow\ &-\sigma \le \mu v_i + (1-\mu)w_i \le \sigma, \end{aligned}$$

if $s_i = +$,

$$\begin{aligned} & -1-\sigma \le v_i \le -1+\sigma \\ & -1-\sigma \le w_i \le -1+\sigma \\ \Rightarrow \quad & -1-\sigma \le \mu v_i + (1-\mu) w_i \le -1+\sigma. \end{aligned}$$

Thus, for all $0 \le \mu \le 1$, $\mu\{v_i\} + (1-\mu)\{w_i\} \in \mathcal{V}$. Clearly, any sequence $\{v_i\} \in \mathcal{V}$ has a subsequence which converges to an element of $\mathcal{V}$, in suitable topology. Apply the Schauder fixed point theorem, we know that there exists $\{\overline{u}_i\}_{i\in\mathbb{Z}}$, which solves (2.5), (2.6), for each $i \in \mathbb{Z}$. And it is a mosaic solution of the form $\{s_i\}$ for the SD-RDE. □

Remark 3.2. *In a finite lattice SD-RDE, we only project the global patterns on the infinite lattice onto local part of the lattice and make it satisfy the boundary condition. Thus, according to the Brouwer fixed point theorem, we can get the result.*

In the rest of this report, we omit the term "pseudo" in discussing patterns, due to Theorem 3.1.

4 Pattern Formation and Spatial Entropy

In this section, we divide it into three three subsections. Here, we discuss the spatial entropy and effect of boundary conditions in one and two dimension. First subsection, we discuss the spatial entropy in one-dimensional lattices. Second subsection, we discuss the spatial entropy in two-dimensional lattices. Last subsection, we discuss the effect of boundary conditions and give some examples.

4.1 SD-RDE on one-dimensional lattices

In this subsection, we will find out the transition matrices and then use them to obtain mosaic patterns and the spatial entropy as in [2] in one-dimensional infinite lattice. In subsection 4.3, we impose three kinds of boundary conditions to discuss its effect on pattern formation and spatial entropy for finite lattice.

Firstly, we take the following identification between the indices $\{1, 2, 3, \cdots, 9\}$ and the nine 1×2 patterns $\{++, +\times, +-, \times+, \times\times, \times-, -+, -\times, --\}$ using