if $s_i = +$,

$$\begin{aligned} & -1-\sigma \le v_i \le -1+\sigma \\ & -1-\sigma \le w_i \le -1+\sigma \\ \Rightarrow \quad & -1-\sigma \le \mu v_i + (1-\mu) w_i \le -1+\sigma. \end{aligned}$$

Thus, for all $0 \le \mu \le 1$, $\mu\{v_i\} + (1-\mu)\{w_i\} \in \mathcal{V}$. Clearly, any sequence $\{v_i\} \in \mathcal{V}$ has a subsequence which converges to an element of $\mathcal{V}$, in suitable topology. Apply the Schauder fixed point theorem, we know that there exists $\{\overline{u}_i\}_{i\in\mathbb{Z}}$, which solves (2.5), (2.6), for each $i \in \mathbb{Z}$. And it is a mosaic solution of the form $\{s_i\}$ for the SD-RDE. □

Remark 3.2. *In a finite lattice SD-RDE, we only project the global patterns on the infinite lattice onto local part of the lattice and make it satisfy the boundary condition. Thus, according to the Brouwer fixed point theorem, we can get the result.*

In the rest of this report, we omit the term "pseudo" in discussing patterns, due to Theorem 3.1.

4 Pattern Formation and Spatial Entropy

In this section, we divide it into three three subsections. Here, we discuss the spatial entropy and effect of boundary conditions in one and two dimension. First subsection, we discuss the spatial entropy in one-dimensional lattices. Second subsection, we discuss the spatial entropy in two-dimensional lattices. Last subsection, we discuss the effect of boundary conditions and give some examples.

4.1 SD-RDE on one-dimensional lattices

In this subsection, we will find out the transition matrices and then use them to obtain mosaic patterns and the spatial entropy as in [2] in one-dimensional infinite lattice. In subsection 4.3, we impose three kinds of boundary conditions to discuss its effect on pattern formation and spatial entropy for finite lattice.

Firstly, we take the following identification between the indices $\{1, 2, 3, \cdots, 9\}$ and the nine 1×2 patterns $\{++, +\times, +-, \times+, \times\times, \times-, -+, -\times, --\}$ using

$$
\begin{aligned}
&1 \longleftrightarrow ++,\ 2 \longleftrightarrow +\times,\ 3 \longleftrightarrow +-,\\
&4 \longleftrightarrow \times+,\ 5 \longleftrightarrow \times\times,\ 6 \longleftrightarrow \times-,\\
&7 \longleftrightarrow -+,\ 8 \longleftrightarrow -\times,\ 9 \longleftrightarrow --,
\end{aligned}
\tag{4.1}
$$

we consider the transition matrix M :

$$
M = M(r) \equiv \begin{pmatrix}
r_1 & r_2 & r_3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & r_4 & r_5 & r_6 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & r_7 & r_8 & r_9\\
r_{10} & r_{11} & r_{12} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & r_{13} & r_{14} & r_{15} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & r_{16} & r_{17} & r_{18}\\
r_{19} & r_{20} & r_{21} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & r_{22} & r_{23} & r_{24} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & r_{25} & r_{26} & r_{27}
\end{pmatrix}
\tag{4.2}
$$

where $r = \{r_j\}_{j=1}^{27}$, $r_j = 0$ or 1. The formation of feasible mosaic patterns related to the transition matrix can be described as follows: the (i, j)-entry of M is one if and only if the jth 1×2 pattern in (4.1) can be joined, with one site overlapped, to the right of ith 1×2 pattern in (4.1) to form a 1×3 feasible pattern. For example, in case of $b \in I_5 = [f(-\sigma), \frac{f(1+\sigma)}{1+4\sigma}]$, we have the basic patterns

$\{ \overline{\underline{+++}}, \overline{\underline{++\times}}, \overline{\underline{\times++}}, \overline{\underline{\times\times\times}}, \overline{\underline{\times--}}, \overline{\underline{--\times}}, \overline{\underline{---}} \}$,

and the corresponding transition matrix.

$$
M_1 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}.
\tag{4.3}
$$

Now, if there exists a pattern having $(i-1,\ i,\ i+1)-$entries as $\overline{\underline{+++}}$, as in Figure 3, we could continue the attaching from the right only with $\overline{\underline{+++}}$ or $\overline{\underline{++\times}}$. For the opposite use spelling checks to left sites, follow the same device.

Notably, according to [9], $M = [m_{ij}]$ is a transition matrix, if (i) $a_{ij} = 0,\ 1$, for all i and j, (ii) $\sum_{j=1}^{9} a_{ij} \geq 1$ for all i, and (iii) $\sum_{i=1}^{9} a_{ij} \geq 1$ for all j. In our case, (ii) or (iii)

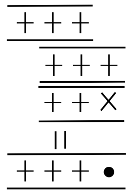

Figure 3: Attaching the two basic patterns, where $\bullet = +$ or $\times$.

may not hold. But, in infinite lattice system $\Lambda = \mathbb{Z}$, this is of little significance. Because, observe the same case in the parameter region $I_5 = [f(-\sigma), \frac{f(1+\sigma)}{1+4\sigma}]$. The corresponding transition matrix is M_1 in (4.3), which has the 3th row to be $[0,\ 0,\ \cdots,\ 0]$. It means that when $\overline{\underline{+-}}$ appears in somewhere of two adjacent sites, say $(s_i,\ s_{i+1}) = \overline{\underline{+-}}$, the attaching could not go on to the site right of s_{i+1}, and impossible to form a global pattern in the infinite lattice $\Lambda = \mathbb{Z}$. Hence, we can drop the 3th row and 3th column of the transition matrix M_1. And it does not affect the complexity of the pattern formation. Similarly, we also drop the 7th row and the 7th column of M_1. Eventually, we could apply the following theorem to study the spatial entropy of the pattern formation.

Theorem 4.1. *[9] Let A be a transition matrix on N symbols. Let $\varphi_A : \Sigma_A \rightarrow \Sigma_A$ be the associated subshift of finite type (either one or two sided). Then $h(\varphi_A) = log(\lambda_1)$ where λ_1 is the real eigenvalue of A such that $\lambda_1 \geq |\lambda_j|$ for all the other eigenvalues λ_j of A.*

Definition 4.2. *[9] We say that the system* (1.1) *and* (1.4) *exhibits spatial chaos at a choice of parameters $(\alpha,\ \beta)$, if the spatial entropy is positive there. And say that the system* (1.1) *and* (1.4) *exhibits pattern formation at such a choice of parameters, if the spatial entropy is zero.*

Definition 4.3. *A basic pattern exists in the parameter region, we call it feasible basic pattern. A basic pattern may be exist in the parameter region, then we call it "possible" basic pattern.*

For the computations of spatial entropy, we need to explore all possible basic patterns in each parameter region. In order to achieve this, further partitioning in each parameter region is necessary. In previous discussions, we have observed the feasible basic patterns for each parameter region I_i. For example, consider the parameter region I_5 again . We have divided the parameter region I_5 into three subregions I_5^1, I_5^2, and I_5^3. Take a look at Fig. 4, we may indicate the certain basic patterns in I_5^3, and then compute the spatial entropy exactly. But, in I_5^1 and I_5^2, we do the estimates of the spatial entropy. According to the results, we could describe the complexity of the pattern formation in each parameter region.

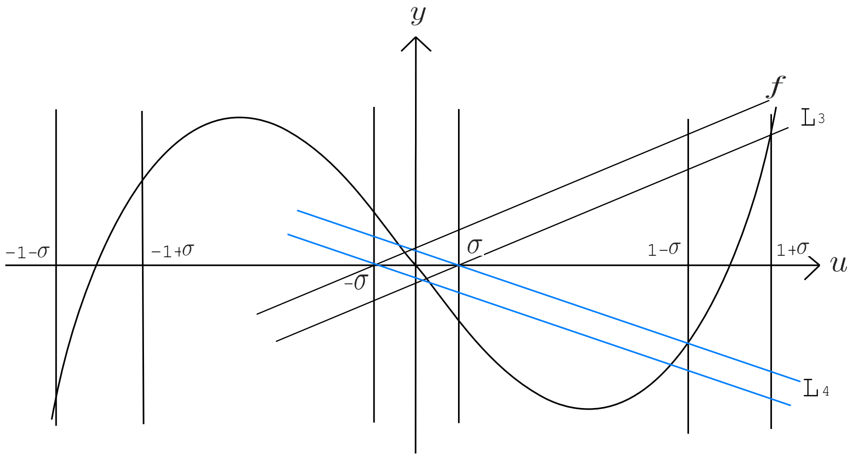

Figure 4: The intersection of (2.6) and (2.5). The slope of L_3 is m_3 and the slope of L_4 is m_4.

Remark 4.4. *Any parameter region, it must at least have basic patterns are feasible basic patterns and at most basic patterns are possible basic patterns. And the set of "possible" basic patterns is contains the set of feasible basic patterns.*

Theorem 4.5. *Consider the parameter regions I_i and I_i^j, the system (1.1) with nonlinearity in (1.4) exhibits spatial chaos in each parameter region I_i or I_i^j, for $-5 \leq i \leq 5$; and the system (1.1) with (1.4) exhibits pattern formation in parameter regions $I_{\pm 7}$ and $I_{\pm 6}^2$.*

Proof. For the proof of the theorem, it suffices to observe the upper bounds and lower bounds for the spatial entropy h in each parameter region. We list the computation results in Table 3, and the assertion is proved. □

After partitioning the parameter space, we know that the set of basic patterns in each parameter region. Here, there are some parameter regions have upper bounds and lower bounds of basic patterns. Therefore, there are two different transition matrices in each parameter region above. Refer to [2], we can get the transition matrix in each parameter region, and we use the numerical computation to estimate the eigenvalues.

parameter regions		$\underline{\lambda}$	$\overline{\lambda}$	h (entropy)
$I_7 = [\frac{f(-1+\sigma)}{4\sigma}, \infty]$		1	1	0
I_6	$I_6^2 = [f(1+\sigma), \frac{f(-1+\sigma)}{4\sigma}]$	1	1	0
	$I_6^1 = [\frac{f(1+\sigma)}{1+4\sigma}, f(1+\sigma)]$	1	1.4656	$0 \leq h \leq 0.3823$
I_5	$I_5^3 = [\frac{f(-\sigma)}{1-4\sigma}, \frac{f(1+\sigma)}{1+4\sigma}]$	1.4656	1.4656	0.3823
	$I_5^2 = [\frac{f(1+\sigma)}{2}, \frac{f(-\sigma)}{1-4\sigma}]$	1.4656	1.8972	$0.3823 \leq h \leq 0.6404$
	$I_5^1 = [f(-\sigma), \frac{f(1+\sigma)}{2}]$	1.4656	2.3165	$0.3823 \leq h \leq 0.8401$
$I_4 = [\frac{f(1+\sigma)}{2+4\sigma}, f(-\sigma)]$		1.8972	2.3165	$0.6404 \leq h \leq 0.8401$
I_3	$I_3^2 = [\frac{f(1+\sigma)}{3}, \frac{f(1+\sigma)}{2+4\sigma}]$	2.3165	2.3165	0.8401
	$I_3^1 = [\frac{f(1+\sigma)}{3+4\sigma}, \frac{f(1+\sigma)}{3}]$	2.3165	2.5921	$0.8401 \leq h \leq 0.9525$
I_2	$I_2^3 = [\frac{f(-\sigma)}{2-4\sigma}, \frac{f(1+\sigma)}{3+4\sigma}]$	2.5921	2.5921	0.9525
	$I_2^2 = [\frac{f(1+\sigma)}{4}, \frac{f(-\sigma)}{2-4\sigma}]$	2.5921	2.8312	$0.9525 \leq h \leq 1.0407$
	$I_2^1 = [\frac{f(1+\sigma)}{4+4\sigma}, \frac{f(1+\sigma)}{4}]$	2.5921	3	$0.9525 \leq h \leq 1.0986$
$I_1 = [\frac{f(-\sigma)}{2}, \frac{f(1+\sigma)}{4+4\sigma}]$		2.7693	3	$1.0186 \leq h \leq 1.0986$
$I_0 = [-\frac{f(-\sigma)}{2+4\sigma}, \frac{f(-\sigma)}{2}]$		3	3	1.0986
$I_{-1} = [-\frac{f(-1+\sigma)}{4}, -\frac{f(-\sigma)}{2+4\sigma}]$		2.7693	3	$1.0186 \leq h \leq 1.0986$
I_{-2}	$I_{-2}^1 = [-\frac{f(-1+\sigma)}{4-4\sigma}, -\frac{f(-1+\sigma)}{4}]$	2.5921	3	$0.9525 \leq h \leq 1.0986$
	$I_{-2}^2 = [-\frac{f(-\sigma)}{2}, -\frac{f(-1+\sigma)}{4-4\sigma}]$	2.5921	2.8312	$0.9525 \leq h \leq 1.0407$
	$I_{-2}^3 = [-\frac{f(-1+\sigma)}{3}, -\frac{f(-\sigma)}{2}]$	2.5921	2.5921	0.9525
I_{-3}	$I_{-3}^1 = [-\frac{f(-1+\sigma)}{3-4\sigma}, -\frac{f(-1+\sigma)}{3}]$	2.3165	2.5921	$0.8401 \leq h \leq 0.9525$
	$I_{-3}^2 = [-\frac{f(-\sigma)}{1+4\sigma}, -\frac{f(-1+\sigma)}{3-4\sigma}]$	2.3165	2.3165	0.8401
$I_{-4} = [-\frac{f(-1+\sigma)}{2}, -\frac{f(-\sigma)}{1+4\sigma}]$		1.9052	2.3165	$0.6446 \leq h \leq 0.8401$
I_{-5}	$I_{-5}^1 = [-f(-\sigma), -\frac{f(-1+\sigma)}{2}]$	1.4656	2.3165	$0.3823 \leq h \leq 0.8401$
	$I_{-5}^2 = [-\frac{f(-1+\sigma)}{2-4\sigma}, -f(-\sigma)]$	1.4656	1.9052	$0.3823 \leq h \leq 0.6446$
	$I_{-5}^3 = [-f(-1+\sigma), -\frac{f(-1+\sigma)}{2-4\sigma}]$	1.4656	1.4656	0.3823
I_{-6}	$I_{-6}^1 = [-\frac{f(-1+\sigma)}{1-4\sigma}, -f(-1+\sigma)]$	1	1.4656	$0 \leq h \leq 0.3823$
	$I_{-6}^2 = [-\frac{f(-\sigma)}{4\sigma} - \frac{f(-1+\sigma)}{1-4\sigma}]$	1	1	0
$I_{-7} = [-\infty, -\frac{f(-\sigma)}{4\sigma}]$		1	1	0

Table 3: Upper bounds and lower bounds for the spatial entropy h in the case of one-dimensional lattice. $\underline{\lambda}$ is the maximal eigenvalue of the transition matrix corresponding to generate patterns from the set of the feasible basic patterns, in each parameter region; $\overline{\lambda}$ is the maximal eigenvalue of the transition matrix corresponding to generate patterns from the set of possible basic patterns, in each parameter region.

parameter regions	possible basic patterns
$\mathrm{I}_6^2 = [f(1+\sigma), \frac{f(-1+\sigma)}{4\sigma}]$	$B^+_{\{2\}}, B^\times_{\{0\}}, B^-_{\{-2\}}$
$\mathrm{I}_6^1 = [\frac{f(1+\sigma)}{1+4\sigma}, f(1+\sigma)]$	$B^+_{\{1,2\}}, B^\times_{\{0\}}, B^-_{\{-2,-1\}}$
$\mathrm{I}_5^3 = [-\frac{f(\sigma)}{1-4\sigma}, \frac{f(1+\sigma)}{1+4\sigma}]$	$B^+_{\{1,2\}}, B^\times_{\{0\}}, B^-_{\{-2,-1\}}$
$\mathrm{I}_5^2 = [\frac{f(1+\sigma)}{2}, -\frac{f(\sigma)}{1-4\sigma}]$	$B^+_{\{1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1\}}$
$\mathrm{I}_5^1 = [-f(\sigma), \frac{f(1+\sigma)}{2}]$	$B^+_{\{0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0\}}$
$\mathrm{I}_3^2 = [\frac{f(1+\sigma)}{3}, \frac{f(1+\sigma)}{2+4\sigma}]$	$B^+_{\{0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0\}}$
$\mathrm{I}_3^1 = [\frac{f(1+\sigma)}{3+4\sigma}, \frac{f(1+\sigma)}{3}]$	$B^+_{\{-1,0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0,1\}}$
$\mathrm{I}_2^3 = [-\frac{f(\sigma)}{2-4\sigma}, \frac{f(1+\sigma)}{3+4\sigma}]$	$B^+_{\{-1,0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0,1\}}$
$\mathrm{I}_2^2 = [\frac{f(1+\sigma)}{4}, -\frac{f(\sigma)}{2-4\sigma}]$	$B^+_{\{-1,0,1,2\}}, B^\times_{\{-2,-1,0,1,2\}}, B^-_{\{-2,-1,0,1\}}$
$\mathrm{I}_2^1 = [\frac{f(1+\sigma)}{4+4\sigma}, \frac{f(1+\sigma)}{4}]$	$B^+_{\{-2,-1,0,1,2\}}, B^\times_{\{-2,-1,0,1,2\}}, B^-_{\{-2,-1,0,1,2\}}$
$\mathrm{I}_{-2}^1 = [\frac{f(1-\sigma)}{4-4\sigma}, -\frac{f(-1+\sigma)}{4}]$	$B^+_{\{-2,-1,0,1,2\}}, B^\times_{\{-2,-1,0,1,2\}}, B^-_{\{-2,-1,0,1,2\}}$
$\mathrm{I}_{-2}^2 = [\frac{f(\sigma)}{2}, \frac{f(1-\sigma)}{4-4\sigma}]$	$B^+_{\{-1,0,1,2\}}, B^\times_{\{-2,-1,0,1,2\}}, B^-_{\{-2,-1,0,1\}}$
$\mathrm{I}_{-2}^3 = [\frac{f(1-\sigma)}{3}, \frac{f(\sigma)}{2}]$	$B^+_{\{-1,0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0,1\}}$
$\mathrm{I}_{-3}^1 = [\frac{f(1-\sigma)}{3-4\sigma}, \frac{f(1-\sigma)}{3}]$	$B^+_{\{-1,0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0,1\}}$
$\mathrm{I}_{-3}^2 = [\frac{f(\sigma)}{1+4\sigma}, \frac{f(1-\sigma)}{3-4\sigma}]$	$B^+_{\{0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0\}}$
$\mathrm{I}_{-5}^1 = [f(\sigma), -\frac{f(-1+\sigma)}{2}]$	$B^+_{\{0,1,2\}}, B^\times_{\{-1,0,1\}}, B^-_{\{-2,-1,0\}}$
$\mathrm{I}_{-5}^2 = [\frac{f(1-\sigma)}{2-4\sigma}, f(\sigma)]$	$B^+_{\{0,1,2\}}, B^\times_{\{0\}}, B^-_{\{-2,-1,0\}}$
$\mathrm{I}_{-5}^3 = [f(1-\sigma), \frac{f(1-\sigma)}{2-4\sigma}]$	$B^+_{\{1,2\}}, B^\times_{\{0\}}, B^-_{\{-2,-1\}}$
$\mathrm{I}_{-6}^1 = [\frac{f(1-\sigma)}{1-4\sigma}, f(1-\sigma)]$	$B^+_{\{1,2\}}, B^\times_{\{0\}}, B^-_{\{-2,-1\}}$
$\mathrm{I}_{-6}^2 = [\frac{f(\sigma)}{4\sigma}, \frac{f(1-\sigma)}{1-4\sigma}]$	$B^+_{\{2\}}, B^\times_{\{0\}}, B^-_{\{-2\}}$

Table 4: Possible existence of basic patterns in parameter regions.

4.2 SD-RDE on two-dimensional lattices

In this subsection, similar to one-dimensional case we use the same method to discuss two-dimensional SD-RDE (1.2).

The stationary equation for Eq. (1.2) is

$$\beta^+\Delta^+u_{i,j} + \beta^\times\Delta^\times u_{i,j} + \alpha f(u_{i,j}) = 0, \text{ for } (i,j) \in \mathbb{Z}^2. \tag{4.4}$$

In Eq. (4.4), we have three parameters β^+, $\beta^\times$ and α. Similarly, we also partition the parameter regions $\mathcal{P}^{2D} = \{(\beta^+, \beta^\times, \alpha) : \beta^+, \beta^\times, \alpha \in \mathbb{Z}\}$. We rewrite (4.4) as

$$b_1\Delta^+u_{i,j} + b_2\Delta^\times u_{i,j} + f(u_{i,j}) = 0, \text{ for } (i,j) \in \mathbb{Z}^2. \tag{4.5}$$

where

$$b_1 = \frac{\beta^+}{\alpha}, b_2 = \frac{\beta^\times}{\alpha}, \alpha \neq 0. \tag{4.6}$$

Then we partition the parameter regions $\mathcal{P}_1^{2D} = \{(b_1, b_2) : b_1, b_2 \in \mathbb{R}\}$. For any $(i,j) \in \mathbb{Z}^2$, $(u_{i,j}, y)$ satisfies

$$y = f(u_{i,j}) \text{ with } f(u_{i,j}) = u_{i,j}(u_{i,j} - 1)(u_{i,j} + 1) \tag{4.7}$$

and

$$y = -b_1\Delta^+ u_{i,j} - b_2\Delta^\times u_{i,j}. \tag{4.8}$$

We rewrite (4.8):

$$\begin{aligned} y &= -b_1\Delta^+ u_{i,j} - b_2\Delta^\times u_{i,j} \\ &= 4(b_1 + b_2)u_{i,j} - b_1 g_1 - b_2 g_2. \\ &= 4(b_1 + b_2)(u_{i,j} - \frac{b_1 g_1 + b_2 g_2}{4(b_1 + b_2)}), \end{aligned}$$

where

$$\begin{aligned} g_1 &= u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}, \\ g_2 &= u_{i+1,j+1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i-1,j-1}. \end{aligned}$$

For any small $\sigma > 0$ and $u_{i,j} \in [-1-\sigma, -1+\sigma]$, $[-\sigma, \sigma]$ or $[1-\sigma, 1+\sigma]$. Then $g_i \in [-4-4\sigma, -4+\sigma]$, $[-3-4\sigma, -3+4\sigma]$, $[-2-4\sigma, -2+4\sigma]$, $[-1-4\sigma, -1+4\sigma]$, $[-4\sigma, 4\sigma]$, $[1-4\sigma, 1+4\sigma]$, $[2-4\sigma, 2+4\sigma]$, $[3-4\sigma, 3+4\sigma]$ or $[4-4\sigma, 4+\sigma]$ for $i \in \{1,2\}$ and respectively denotes $B^\bullet_{-4}$, $B^\bullet_{-3}$, $B^\bullet_{-2}$, $B^\bullet_{-1}$, $B^\bullet_0$, $B^\bullet_1$, $B^\bullet_2$, $B^\bullet_3$, $B^\bullet_4$.

We use intercept $u_{i,j} = \frac{b_1 g_1 + b_2 g_2}{4(b_1+b_2)}$ and slope $4(b_1+b_2)$ to classify the parameter regions and characterize what basic patterns will appear in each parameter region. The method is similar to one dimension case. Different part is the intercept in two dimension case have eighty-one intercepts.

Partitioning the parameter space that must divide two cases to discuss, in the first case, the parameter region of the patterns appear certainly(i.e. the feasible basic patterns), in the second case, the parameter region of the patterns maybe appear(i.e. the "possible" basic patterns). The following process is the same above one dimension case.

Theorem 4.6. *Suppose that σ is small enough and $b_1 = 0$ or $b_2 = 0$. The existence of feasible basic patterns in each parameter region in Table 5 are confirmed.*

We can use the same method to estimate the greatest lower bound of spatial entropy in each parameter regions as [10]. We only present the results are summarized in Table 6, 7 as $b_2 = 0$.

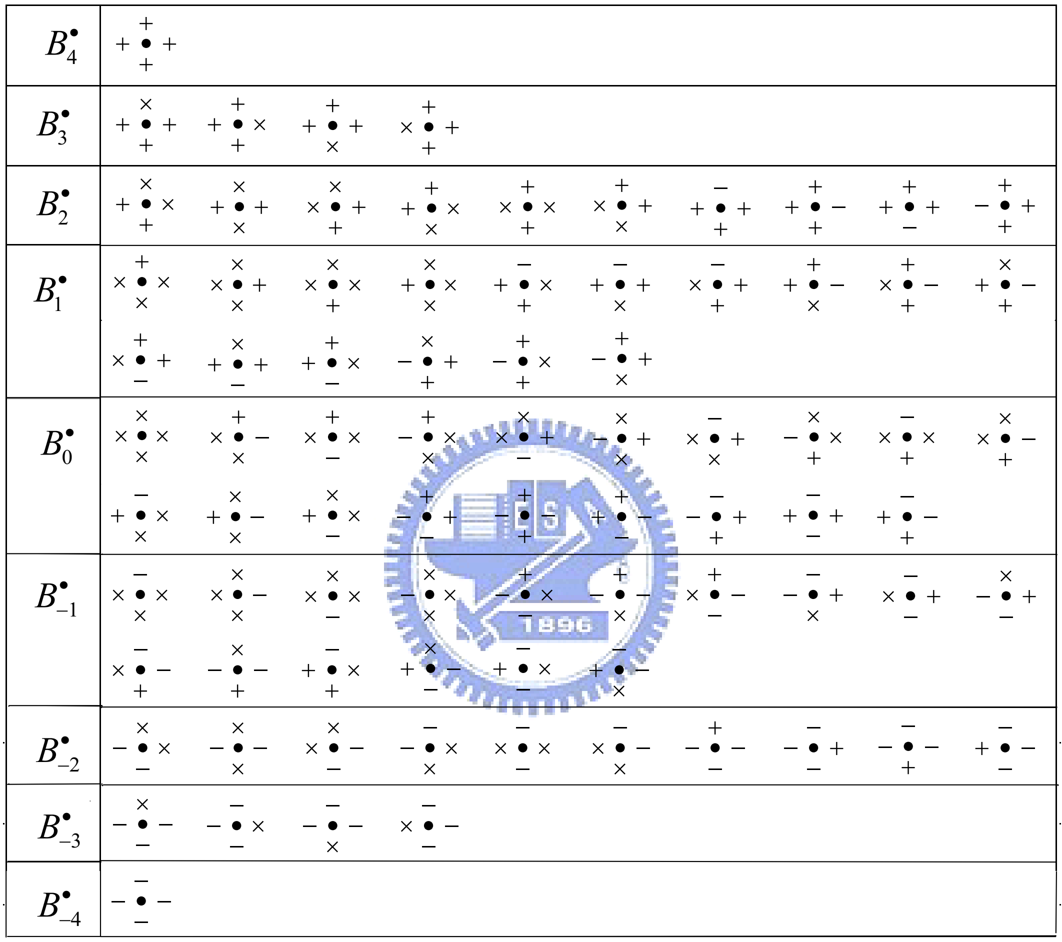

Figure 5: $g_1 \in B_i^{\bullet}$, where $i \in \{-4,-3,-2,-1,0,1,2,3,4\}$ and $\bullet =$ " $+$ ", " $\times$ " or " $-$ ".

parameter regions	feasible basic patterns
$\mathrm{I}_{13} = [\frac{f(1+\sigma)}{-8\sigma}, \infty]$	$B^{\times}_{\{0\}}$
$\mathrm{I}_{12} = [\frac{f(1+\sigma)}{1+8\sigma}, \frac{f(1+\sigma)}{-8\sigma}]$	$B^{+}_{\{4\}}, B^{\times}_{\{0\}}, B^{-}_{\{-4\}}$
$\mathrm{I}_{11} = [\frac{f(\sigma)}{-1}, \frac{f(1+\sigma)}{1+8\sigma}]$	$B^{+}_{\{3,4\}}, B^{\times}_{\{0\}}, B^{-}_{\{-4,-3\}}$
$\mathrm{I}_{10} = [\frac{f(1+\sigma)}{2+8\sigma}, \frac{f(\sigma)}{-1}]$	$B^{+}_{\{3,4\}}, B^{\times}_{\{-1,0,1\}}, B^{-}_{\{-4,-3\}}$
$\mathrm{I}_{9} = [\frac{f(1+\sigma)}{3+8\sigma}, \frac{f(1+\sigma)}{2+8\sigma}]$	$B^{+}_{\{2,3,4\}}, B^{\times}_{\{-1,0,1\}}, B^{-}_{\{-4,-3,-2\}}$
$\mathrm{I}_{8} = [\frac{f(\sigma)}{-2}, \frac{f(1+\sigma)}{3+8\sigma}]$	$B^{+}_{\{1,2,3,4\}}, B^{\times}_{\{-1,0,1\}}, B^{-}_{\{-4,-3,-2,-1\}}$
$\mathrm{I}_{7} = [\frac{f(1+\sigma)}{4+8\sigma}, \frac{f(\sigma)}{-2}]$	$B^{+}_{\{1,2,3,4\}}, B^{\times}_{\{-2,-1,0,1,2\}}, B^{-}_{\{-4,-3,-2,-1\}}$
$\mathrm{I}_{6} = [\frac{f(1+\sigma)}{5+8\sigma}, \frac{f(1+\sigma)}{4+8\sigma}]$	$B^{+}_{\{0,1,2,3,4\}}, B^{\times}_{\{-2,-1,0,1,2\}}, B^{-}_{\{-4,-3,-2,-1,0\}}$
$\mathrm{I}_{5} = [\frac{f(1+\sigma)}{6+8\sigma}, \frac{f(1+\sigma)}{5+8\sigma}]$	$B^{+}_{\{-1,0,1,2,3,4\}}, B^{\times}_{\{-2,-1,0,1,2\}}, B^{-}_{\{-4,-3,-2,-1,0,1\}}$
$\mathrm{I}_{4} = [\frac{f(\sigma)}{-3}, \frac{f(1+\sigma)}{6+8\sigma}]$	$B^{+}_{\{-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-2,-1,0,1,2\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2\}}$
$\mathrm{I}_{3} = [\frac{f(1+\sigma)}{7+8\sigma}, \frac{f(\sigma)}{-3}]$	$B^{+}_{\{-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-3,-2,-1,0,1,2,3\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2\}}$
$\mathrm{I}_{2} = [\frac{f(1+\sigma)}{8+8\sigma}, \frac{f(1+\sigma)}{7+8\sigma}]$	$B^{+}_{\{-3,-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-3,-2,-1,0,1,2,3\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2,3\}}$
$\mathrm{I}_{1} = [\frac{f(\sigma)}{-4}, \frac{f(1+\sigma)}{8+8\sigma}]$	$B^{+}_{\{-4,-3,-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-3,-2,-1,0,1,2,3\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2,3,4\}}$
$\mathrm{I}_{0} = [\frac{f(\sigma)}{4+8\sigma}, \frac{f(\sigma)}{-4}]$	$B^{+}_{\{-4,-3,-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-4,-3,-2,-1,0,1,2,3,4\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2,3,4\}}$
$\mathrm{I}_{-1} = [\frac{f(1-\sigma)}{8}, \frac{f(\sigma)}{4+8\sigma}]$	$B^{+}_{\{-4,-3,-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-3,-2,-1,0,1,2,3\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2,3,4\}}$
$\mathrm{I}_{-2} = [\frac{f(1-\sigma)}{7}, \frac{f(1-\sigma)}{8}]$	$B^{+}_{\{-3,-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-3,-2,-1,0,1,2,3\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2,3\}}$
$\mathrm{I}_{-3} = [\frac{f(\sigma)}{3+8\sigma}, \frac{f(1-\sigma)}{7}]$	$B^{+}_{\{-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-3,-2,-1,0,1,2,3\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2\}}$
$\mathrm{I}_{-4} = [\frac{f(1-\sigma)}{6}, \frac{f(\sigma)}{3+8\sigma}]$	$B^{+}_{\{-2,-1,0,1,2,3,4\}}, B^{\times}_{\{-2,-1,0,1,2\}}, B^{-}_{\{-4,-3,-2,-1,0,1,2\}}$
$\mathrm{I}_{-5} = [\frac{f(1-\sigma)}{5}, \frac{f(1-\sigma)}{6}]$	$B^{+}_{\{-1,0,1,2,3,4\}}, B^{\times}_{\{-2,-1,0,1,2\}}, B^{-}_{\{-4,-3,-2,-1,0\}}$
$\mathrm{I}_{-6} = [\frac{f(\sigma)}{2+8\sigma}, \frac{f(1-\sigma)}{5}]$	$B^{+}_{\{0,1,2,3,4\}}, B^{\times}_{\{-2,-1,0,1,2\}}, B^{-}_{\{-4,-3,-2,-1,0\}}$
$\mathrm{I}_{-7} = [\frac{f(1-\sigma)}{4}, \frac{f(\sigma)}{2+8\sigma}]$	$B^{+}_{\{0,1,2,3,4\}}, B^{\times}_{\{-1,0,1\}}, B^{-}_{\{-4,-3,-2,-1,0\}}$
$\mathrm{I}_{-8} = [\frac{f(1-\sigma)}{3}, \frac{f(1-\sigma)}{4}]$	$B^{+}_{\{1,2,3,4\}}, B^{\times}_{\{-1,0,1\}}, B^{-}_{\{-4,-3,-2,-1\}}$
$\mathrm{I}_{-9} = [\frac{f(\sigma)}{1+8\sigma}, \frac{f(1-\sigma)}{3}]$	$B^{+}_{\{2,3,4\}}, B^{\times}_{\{-1,0,1\}}, B^{-}_{\{-4,-3,-2\}}$
$\mathrm{I}_{-10} = [\frac{f(1-\sigma)}{2}, \frac{f(\sigma)}{1+8\sigma}]$	$B^{+}_{\{2,3,4\}}, B^{\times}_{\{0\}}, B^{-}_{\{-4,-3,-2\}}$
$\mathrm{I}_{-11} = [f(1-\sigma), \frac{f(1-\sigma)}{2}]$	$B^{+}_{\{3,4\}}, B^{\times}_{\{0\}}, B^{-}_{\{-4,-3\}}$
$\mathrm{I}_{-12} = [\frac{f(\sigma)}{8\sigma}, f(1-\sigma)]$	$B^{+}_{\{4\}}, B^{\times}_{\{0\}}, B^{-}_{\{-4\}}$
$\mathrm{I}_{-13} = [-\infty, \frac{f(\sigma)}{8\sigma}]$	$B^{+}_{\{4\}}, B^{-}_{\{-4\}}$

Table 5: The feasible basic patterns of the parameter space in $b_1 = 0$ or $b_2 = 0$.

parameter region		$\underline{h}(\mathcal{U})$	$\overline{h}(\mathcal{U})$
I_{12}	$I_{12}^2 = [f(1+\sigma), \infty]$	0	0
	$I_{12}^1 = [\frac{f(1+\sigma)}{1+8\sigma}, f(1+\sigma)]$	0	0
I_{11}	$I_{11}^3 = [\frac{f(\sigma)}{-1+8\sigma}, \frac{f(1+\sigma)}{1+8\sigma}]$	0	0
	$I_{11}^2 = [\frac{f(1+\sigma)}{2}, \frac{f(\sigma)}{-1+8\sigma}]$	0	0
	$I_{11}^1 = [\frac{f(\sigma)}{-1}, \frac{f(1+\sigma)}{2}]$	0	$\frac{ln3}{16}$
$I_{10} = [\frac{f(1+\sigma)}{2+8\sigma}, \frac{f(\sigma)}{-1}]$		0	$\frac{ln3}{16}$
I_9	$I_9^2 = [\frac{f(1+\sigma)}{3}, \frac{f(1+\sigma)}{2+8\sigma}]$	$\frac{ln3}{16}$	$\frac{ln3}{16}$
	$I_9^2 = [\frac{f(1+\sigma)}{3+8\sigma}, \frac{f(1+\sigma)}{3}]$	$\frac{ln3}{16}$	$\frac{3ln2}{16}$
I_8	$I_8^3 = [\frac{f(\sigma)}{-2+8\sigma}, \frac{f(1+\sigma)}{3+8\sigma}]$	$\frac{3ln2}{16}$	$\frac{3ln2}{16}$
	$I_8^2 = [\frac{f(1+\sigma)}{4}, \frac{f(\sigma)}{-2+8\sigma}]$	$\frac{3ln2}{16}$	$\frac{ln26}{16}$
	$I_8^1 = [\frac{f(\sigma)}{-2}, \frac{f(1+\sigma)}{4}]$	$\frac{3ln2}{16}$	$\frac{ln3}{4}$
$I_7 = [\frac{f(1+\sigma)}{4+8\sigma}, \frac{f(\sigma)}{-2}]$		$\frac{ln26}{16}$	$\frac{ln3}{4}$
I_6	$I_6^2 = [\frac{f(1+\sigma)}{5}, \frac{f(1+\sigma)}{4+8\sigma}]$	$\frac{ln3}{4}$	$\frac{ln3}{4}$
	$I_6^1 = [\frac{f(1+\sigma)}{5+8\sigma}, \frac{f(1+\sigma)}{5}]$	$\frac{ln3}{4}$	$\frac{ln3}{4}$
I_5	$I_5^3 = [\frac{f(\sigma)}{-3+8\sigma}, \frac{f(1+\sigma)}{5+8\sigma}]$	$\frac{ln3}{4}$	$\frac{ln3}{4}$
	$I_5^2 = [\frac{f(1+\sigma)}{6}, \frac{f(\sigma)}{-3+8\sigma}]$	$\frac{ln3}{4}$	$\frac{ln11}{4}$
	$I_5^1 = [\frac{f(1+\sigma)}{6+8\sigma}, \frac{f(1+\sigma)}{6}]$	$\frac{ln3}{4}$	$\frac{ln27}{4}$
$I_4 = [\frac{f(\sigma)}{-3}, \frac{f(1+\sigma)}{6+8\sigma}]$		$\frac{17}{4}$	$\frac{ln27}{4}$
I_3	$I_3^2 = [\frac{f(1+\sigma)}{7}, \frac{f(\sigma)}{-3}]$	$\frac{ln27}{4}$	$\frac{ln27}{4}$
	$I_3^1 = [\frac{f(1+\sigma)}{7+8\sigma}, \frac{f(1+\sigma)}{7}]$	$\frac{ln27}{4}$	$\frac{ln51}{4}$
I_2	$I_2^3 = [\frac{f(\sigma)}{-4+8\sigma}, \frac{f(1+\sigma)}{7+8\sigma}]$	$\frac{ln51}{4}$	$\frac{ln51}{4}$
	$I_2^2 = [\frac{f(1+\sigma)}{8}, \frac{f(\sigma)}{-4+8\sigma}]$	$\frac{51}{4}$	$\frac{ln63}{4}$
	$I_2^1 = [\frac{f(1+\sigma)}{8+8\sigma}, \frac{f(1+\sigma)}{8}]$	$\frac{ln51}{4}$	$ln3$
$I_1 = [\frac{f(\sigma)}{-4}, \frac{f(1+\sigma)}{8+8\sigma}]$		$\frac{ln61}{4}$	$ln3$
$I_0 = [0, \frac{f(\sigma)}{-4}]$		$ln3$	$ln3$

Table 6: Estimate the greatest lower bound of spatial entropy for feasible basic pattern and possible basic pattern in each parameter region, where $b_2 = 0$. This effective estimate is similar to [10].

parameter region		$\underline{h}(\mathcal{U})$	$\overline{h}(\mathcal{U})$
$I_0 = [\frac{f(\sigma)}{4+8\sigma}, 0]$		$ln3$	$ln3$
$I_{-1} = [\frac{f(1-\sigma)}{8}, \frac{f(\sigma)}{4+8\sigma}]$		$\frac{ln61}{4}$	$ln3$
I_{-2}	$I^1_{-2} = [\frac{f(1-\sigma)}{8-8\sigma}, \frac{f(1-\sigma)}{8}]$	$\frac{ln51}{4}$	$ln3$
	$I^2_{-2} = [\frac{f(\sigma)}{4}, \frac{f(1-\sigma)}{8-8\sigma}]$	$\frac{ln51}{4}$	$\frac{ln63}{4}$
	$I^3_{-2} = [\frac{f(1-\sigma)}{7}, \frac{f(\sigma)}{4}]$	$\frac{ln51}{4}$	$\frac{ln51}{4}$
I_{-3}	$I^1_{-3} = [-\frac{f(1-\sigma)}{7-8\sigma}, \frac{f(1-\sigma)}{7}]$	$\frac{ln27}{4}$	$\frac{ln51}{4}$
	$I^2_{-3} = [\frac{f(\sigma)}{3+8\sigma}, \frac{f(1-\sigma)}{7-8\sigma}]$	$\frac{ln27}{4}$	$\frac{ln27}{4}$
$I^2_{-4} = [\frac{f(1-\sigma)}{6}, \frac{f(\sigma)}{3+8\sigma}]$		$\frac{ln17}{4}$	$\frac{ln27}{4}$
I_{-5}	$I^1_{-5} = [\frac{f(1-\sigma)}{6-8\sigma}, \frac{f(1-\sigma)}{6}]$	$\frac{ln3}{4}$	$\frac{ln27}{4}$
	$I^2_{-5} = [\frac{f(\sigma)}{3}, \frac{f(1-\sigma)}{6-8\sigma}]$	$\frac{ln3}{4}$	$\frac{ln11}{4}$
	$I^3_{-5} = [\frac{f(1-\sigma)}{5}, \frac{f(\sigma)}{3}]$	$\frac{ln3}{4}$	$\frac{ln3}{4}$
I_{-6}	$I^1_{-6} = [\frac{f(1-\sigma)}{5-8\sigma}, \frac{f(1-\sigma)}{5}]$	$\frac{ln3}{4}$	$\frac{ln3}{4}$
	$I^2_{-6} = [\frac{f(\sigma)}{2+8\sigma}, \frac{f(1-\sigma)}{5-8\sigma}]$	$\frac{ln3}{4}$	$\frac{ln3}{4}$
$I_{-7} = [\frac{f(1-\sigma)}{4}, \frac{f(\sigma)}{2+8\sigma}]$		$\frac{ln2}{4}$	$\frac{ln3}{4}$
I_8	$I^1_{-8} = [\frac{f(\sigma)}{2}, \frac{f(1-\sigma)}{4}]$	$\frac{3ln2}{16}$	$\frac{ln3}{4}$
	$I^2_{-8} = [\frac{f(1-\sigma)}{4+8\sigma}, \frac{f(\sigma)}{2}]$	$\frac{3ln2}{16}$	$\frac{ln2}{4}$
	$I^3_{-8} = [\frac{f(1-\sigma)}{3}, \frac{f(1-\sigma)}{4-8\sigma}]$	$\frac{3ln2}{16}$	$\frac{3ln2}{16}$
I_{-9}	$I^1_{-9} = [\frac{f(1-\sigma)}{3-8\sigma}, \frac{f(1-\sigma)}{3}]$	$\frac{ln3}{16}$	$\frac{3ln2}{16}$
	$I^2_{-9} = [\frac{f(\sigma)}{1+8\sigma}, \frac{f(1-\sigma)}{3+8\sigma}]$	$\frac{ln3}{16}$	$\frac{ln3}{16}$
$I_{-10} = [\frac{f(1-\sigma)}{2}, \frac{f(\sigma)}{1+8\sigma}]$		?	$\frac{ln3}{16}$
I_{-11}	$I^1_{-11} = [f(\sigma), \frac{f(1-\sigma)}{2}]$	0	$\frac{ln3}{16}$
	$I^2_{-11} = [\frac{f(1-\sigma)}{2-8\sigma}, f(\sigma)]$	0	?
	$I^3_{-11} = [f(\sigma), \frac{f(1-\sigma)}{2-8\sigma}]$	0	0
I_{12}	$I^1_{-12} = [\frac{f(1-\sigma)}{1-8\sigma}, f(\sigma)]$	0	0
	$I^2_{-12} = [-\infty, \frac{f(1-\sigma)}{1-8\sigma}]$	0	0

Table 7: Estimate the greatest lower bound of spatial entropy for feasible basic pattern and possible basic pattern in each parameter region, where $b_2 = 0$. This effective estimate is similar to [10].

4.3 Effect of boundary conditions

Let $\Lambda \subset \mathbb{Z}^d$, where $d=1$ or 2. For $d=1$, consider Λ and the finite lattice T_k:

$$T_k = \{(i) \in \mathbb{Z}^1 : 1 \leq i \leq k\},$$

where k is a 1−tuple of positive integer. For $d=2$, consider Λ and the finite lattice $T_{\mathbf{k}}$:

$$T_{\mathbf{k}} = \{(i_1, i_2) \in \mathbb{Z}^2 : 1 \leq i_l \leq k, l = 1, 2\},$$

where $\mathbf{k} = (k_1, k_2)$ is a 2−tuple of positive integers.

Let $\mathcal{A}$ be a finite set of elements (symbols) which are used to represent the patterns at each site on the lattice. Let $\mathcal{A}^{\mathbb{Z}^d} = \{y | y : \mathbb{Z}^d \longrightarrow \mathcal{A}\}$. There is a natural projection

$$\pi_{\mathbf{k}} : \mathcal{A}^{\mathbb{Z}^d} \longrightarrow \mathcal{A}^{T_{\mathbf{k}}},$$

by any $y \in \mathcal{A}^{\mathbb{Z}^d}$ on finite lattice $T_{\mathbf{k}}$. Let $\underline{\mathcal{U}}$ be a translational invariant subset of the global stationary solutions $\mathcal{A}^{\mathbb{Z}^d}$ in (SD-RDE)$_\infty$(generated by feasible basic patterns), $\mathcal{U}$ be a translational invariant subset of the stationary solutions $\mathcal{A}^{\mathbb{Z}^d}$ in (SD-RDE)$_\infty$ and $\overline{\mathcal{U}}$ be a translational invariant subset of the possible stationary solutions $\mathcal{A}^{\mathbb{Z}^d}$ in (SD-RDE)$_\infty$(generated by possible basic patterns), which represent two classes of patterns in (SD-RDE)$_\infty$. Set

$$\Gamma_{\mathbf{k}}(\overline{\mathcal{U}}) := card(\pi_{\mathbf{k}}(\overline{\mathcal{U}})) := \underline{\Gamma}_{\mathbf{k}}^{\infty},$$
$$\Gamma_{\mathbf{k}}(\mathcal{U}) := card(\pi_{\mathbf{k}}(\mathcal{U})) := \Gamma_{\mathbf{k}}^{\infty},$$
$$\Gamma_{\mathbf{k}}(\underline{\mathcal{U}}) := card(\pi_{k}(\underline{\mathcal{U}})) := \overline{\Gamma}_{\mathbf{k}}^{\infty},$$

where $\underline{\Gamma}_{\mathbf{k}}$ is the number of the distinct feasible basic patterns on $T_{\mathbf{k}}$ projected from $\underline{\mathcal{U}}$, $\Gamma_{\mathbf{k}}$ is the number of the distinct patterns on $T_{\mathbf{k}}$ projected from $\mathcal{U}$ and $\overline{\Gamma}_{\mathbf{k}}$ is the number of the distinct possible basic patterns on $T_{\mathbf{k}}$ projected from $\overline{\mathcal{U}}$.

The spatial entropy in one dimension is defined as

$$h(\underline{\mathcal{U}}) := \lim_{k \to \infty} \frac{1}{k} ln\Gamma_k(\underline{\mathcal{U}}) := \underline{h},$$
$$h(\mathcal{U}) := \lim_{k \to \infty} \frac{1}{k} ln\Gamma_k(\mathcal{U}) := h,$$
$$h(\overline{\mathcal{U}}) := \lim_{k \to \infty} \frac{1}{k} ln\Gamma_k(\overline{\mathcal{U}}) := \overline{h};$$

and the spatial entropy in two dimension is defined as

$$h(\underline{\mathcal{U}}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1k_2} ln\Gamma_{\mathbf{k}}(\underline{\mathcal{U}}) := \underline{h},$$
$$h(\mathcal{U}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1k_2} ln\Gamma_{\mathbf{k}}(\mathcal{U}) := h,$$
$$h(\overline{\mathcal{U}}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1k_2} ln\Gamma_{\mathbf{k}}(\overline{\mathcal{U}}) := \overline{h},$$

where $h(\underline{\mathcal{U}})$ is the entropy of the feasible basic patterns, $h(\mathcal{U})$ is the entropy of all patterns and $h(\overline{\mathcal{U}})$ is the entropy of the possible basic patterns.

Next, we discuss the effect of boundary conditions and spatial entropy similar to [11]. The following are three types of boundary conditions for SD-RDE on $T_{\mathbf{k}}$:

(1) $(\text{SD-RDE})_{\mathbf{k}} - N$, SD-RDE with Neumann boundary condition on $T_{\mathbf{k}}$.
For $d = 1$ and $0 \le i \le k+1$,

$$u_0 = u_1,\ u_k = u_{k+1}.$$

For $d = 2$, $0 \le i \le k_1 + 1$ and $0 \le j \le k_2 + 1$

$$u_{k_1+1,j} = u_{k_1,j},\ u_{0,j} = u_{1,j},$$
$$u_{i,k_2+1} = u_{i,k_2},\ \ u_{i,0} = u_{i,1}.$$

(2) $(\text{SD-RDE})_{\mathbf{k}} - P$, SD-RDE with Periodic boundary condition on $T_{\mathbf{k}}$.
For $d = 1$ and $0 \le i \le k+1$,

$$u_0 = u_{k-1},\ u_1 = u_k,\ u_2 = u_{k+1}.$$

For $d = 2$, $0 \le i \le k_1 + 1$ and $0 \le j \le k_2 + 1$

$$u_{1,j} = u_{k_1,j},\ u_{0,j} = u_{k_1-1,j},\ u_{2,j} = u_{k_1+1,j},$$
$$u_{i,1} = u_{i,k_2},\ \ u_{i,0} = u_{i,k_2-1},\ u_{i,2} = u_{i,k_2+1}.$$

(3) $(\text{SD-RDE})_{\mathbf{k}} - D$, SD-RDE with Dirichlet boundary condition on $T_{\mathbf{k}}$.The Dirichlet boundary conditions means that $u_{\mathbf{b}} = \hat{u}_{\mathbf{b}} := \{\hat{u}_i, i \in \mathbf{b}\}$, where $\mathbf{b}$ is boundary sites in one or two dimension.

Denote by $\underline{\mathcal{U}}^B_{\mathbf{k}}$ a class of the patterns obtained from attaching all feasible basic patterns for (SD-RDE)$_{\mathbf{k}}$, $\mathcal{U}^B_{\mathbf{k}}$ a class of the patterns obtained from attaching all patterns for (SD-RDE)$_{\mathbf{k}}$ and $\overline{\mathcal{U}}^B_{\mathbf{k}}$ is a class of the all possible basic patterns for (SD-RDE)$_{\mathbf{k}}$, where $B = N$, P or D. And we set $\Gamma(\underline{\mathcal{U}}^B_{\mathbf{k}}) = \underline{\Gamma}^B_{\mathbf{k}}$ is the number of patterns of $\underline{\mathcal{U}}^B_{\mathbf{k}}$, $\Gamma(\mathcal{U}^B_{\mathbf{k}}) = \Gamma^B_{\mathbf{k}}$ is the number of patterns of $\mathcal{U}^B_{\mathbf{k}}$, $\Gamma^B_{\mathbf{k}}(\overline{\mathcal{U}}^B_{\mathbf{k}}) = \overline{\Gamma}^B_{\mathbf{k}}$ is the number of patterns of $\overline{\mathcal{U}}^B_{\mathbf{k}}$.

And we define $h(\underline{\mathcal{U}}^B) := \underline{h}_B$ as the entropy on $T_{\mathbf{k}}$ with boundary condition B generated from all feasible basic patterns, $h(\mathcal{U}^B) := h_B$ is entropy on $T_{\mathbf{k}}$ with boundary condition B generated from all basic patterns, $h(\overline{\mathcal{U}}^B) := \overline{h}_B$ is entropy on $T_{\mathbf{k}}$ with boundary condition B generated from all possible basic patterns.

Proposition 4.7. *[11] (i) Fix $s \in \mathbb{N}$, for all $\mathbf{k} > s$(means $k_d > s$, for all d), $\Gamma^B_{\mathbf{k}} \geq \Gamma^\infty_{\mathbf{k-s}}$ and (ii) $\Gamma^B_{\mathbf{k}} \leq p^c \cdot \Gamma^\infty_{\mathbf{k-s}}$ for some $p > 0$ and $c = c(\mathbf{k})$ with $lim_{\mathbf{k}\to\infty}(c/k_1k_2\cdots k_d) = 0$, then $h = h_B$, where $B = N$ or P or D.*

Proposition 4.8. *Assume two conditions: (i) Fix $s \in \mathbb{N}$, for all $\mathbf{k} > s$,$\underline{\Gamma}^B_{\mathbf{k}} \geq \overline{\Gamma}^\infty_{\mathbf{k}-s}$ and (ii) $\overline{\Gamma}^B_{\mathbf{k}} \leq p^c \cdot \underline{\Gamma}^\infty_{\mathbf{k}-s}$ for some $p > 0$ and $c = c(\mathbf{k})$ with $lim_{\mathbf{k}\to\infty}(c/k_1k_2\cdots k_d) = 0$, then $h = h_B = \underline{h} = \underline{h}_B = \overline{h} = \overline{h}_B$, where $B = N$ or P or D.*

Proof. Because $\underline{\Gamma}^B_{\mathbf{k}} \geq \overline{\Gamma}^B_{\mathbf{k}}$, then we get $\underline{\Gamma}^B_{\mathbf{k}} \geq \underline{\Gamma}^B_{\mathbf{k}}$. And we also get that $\underline{\Gamma}^B_{\mathbf{k}} \leq p^c \cdot \underline{\Gamma}^\infty_{\mathbf{k}-s}$ for some $p > 0$ and $c = c(\mathbf{k})$ with $lim_{\mathbf{k}\to\infty}(c/\mathbf{k}) = 0$. According to proposition 4.7, then we get $\underline{h}_B = \underline{h}$. Similarly, because $\overline{\Gamma}^B_{\mathbf{k}} \leq p^c \cdot \underline{\Gamma}^\infty_{\mathbf{k}-s}$, then we get $\overline{\Gamma}^B_{\mathbf{k}} \leq p^c \cdot \overline{\Gamma}^\infty_{\mathbf{k-s}}$. And because $\overline{\Gamma}^B_{\mathbf{k}} \geq \underline{\Gamma}^B_{\mathbf{k}}$, then we get $\overline{\Gamma}^B_{\mathbf{k}} \geq \overline{\Gamma}^B_{\mathbf{k}}$. According to proposition 4.7, then we get $\overline{h}_B = \overline{h}$.

We prove the two-dimensional case, according to the condition (i), we get

$$\begin{aligned}
\underline{h}_B = h(\underline{\mathcal{U}}^B) &= \lim_{\mathbf{k}\to\infty} \frac{1}{k_1k_2} \ln \underline{\Gamma}^B_{\mathbf{k}} \\
&\geq \lim_{\mathbf{k}\to\infty} \frac{1}{k_1k_2} \ln \overline{\Gamma}^\infty_{\mathbf{k-s}} \\
&= \lim_{\mathbf{k}\to\infty} \frac{(k_1-2s_1)(k_2-2s_2)}{k_1k_2} \frac{\ln \overline{\Gamma}^B_{\mathbf{k}}}{(k_1-2s_1)(k_2-2s_2)} \\
&= h(\overline{\mathcal{U}}) \\
&= \overline{h}.
\end{aligned}$$

According to the condition(ii), we get

$$
\begin{aligned}
\overline{h}_B = h(\overline{\mathcal{U}}^B) &= \lim_{\mathbf{k}\to\infty} \frac{1}{k_1k_2} ln\overline{\Gamma}^B_{\mathbf{k}} \\
&\leq \lim_{\mathbf{k}\to\infty} \frac{1}{k_1k_2} \ln(P^c \cdot \underline{\Gamma}^{\infty}_{\mathbf{k-2s}}) \\
&= \lim_{k\to\infty} \frac{(k_1-2s_1)(k_2-2s_2)}{k_1k_2} \frac{c\ln P + \ln \underline{\Gamma}^B_{\mathbf{k}}}{(k_1-2s_1)(k_2-2s_2)} \\
&= h(\underline{\mathcal{U}}) \\
&= \underline{h}.
\end{aligned}
$$

Thus, $h = h_B = \underline{h} = \underline{h}_B = \overline{h} = \overline{h}_B$, where $B = N$, P or D. □

We want to know whether if "$h = h_N = h_P = h_D$?" in each one-dimensional parameter region. So we must check the conditions of Proposition 4.7 or Proposition 4.8.

It is easy to check that the parameter in all regions satisfy the condition(ii) of Proposition 4.7 $\Gamma^B_{\mathbf{k}} \leq p^c \cdot \Gamma^{\infty}_{\mathbf{k-s}}$ for some $p > 0$ and $c = c(\mathbf{k})$ with $\lim_{\mathbf{k}\to\infty}(c/k_1k_2\cdots k_d) = 0$. Next, in each parameter region, we shall check whether if the parameters within satisfy the condition(i) of Proposition 4.7, i.e. fix $s \in \mathbb{N}$, for all $k > s, \Gamma^B_{\mathbf{k}} \geq \Gamma^{\infty}_{\mathbf{k-s}}$, where $B = N$, P or D.

After the analysis, we come to a conclusion as follow. When $B = N$, we assume $s \geq 2$, then all parameter regions satisfy that for all $k > s, \Gamma^B_k \geq \Gamma^{\infty}_{k-s}$. When $B = P$, assume $s \geq 3$, then all parameter regions satisfy that for all $k > s, \Gamma^B_k \geq \Gamma^{\infty}_{k-s}$. When $B = D_1$, i.e. $u_{\mathbf{b}} = \hat{u}_{\mathbf{b}} := \{1, i \in \mathbf{b}\}$, assume $s \geq 4$, then almost all parameter regions satisfy that for all $k > s, \Gamma^B_k \geq \Gamma^{\infty}_{k-s}$, beside the spatial entropy is zero in some parameter regions. Thus, according to Proposition 4.7 implies $h = h_N = h_P = h_{D_1}$ in one dimension.

Theorem 4.9. *In one dimension, $h = h_N = h_P = h_{D_1}$. Namely the effect of boundary conditions does not change the number of spatial entropy.*

The following we present several examples to explain how to check the Proposition 4.7 and Proposition 4.8.

Example 4.10. *In this example, we illustrate that the parameters in I^1_3 satisfy the conditions of Proposition 4.7 and justify the equality $h = h_N = h_P = h_{D_1}$. In Table 2 and Table 4.1, the parameter region I^1_3 has the feasible basic patterns: $B^+_{\{0,1,2\}}$, $B^{\times}_{\{-1,0,1\}}$ and $B^-_{\{-2,-1,0\}}$ and the possible basic patterns: $B^+_{\{-1,0,1,2\}}$, $B^{\times}_{\{-1,0,1\}}$ and $B^-_{\{-2,-1,0,1\}}$. The boundary of basic patterns on T_{k-s} may be:*

$+ + + \cdots$	$\cdots + + +$
$\times + + \cdots$	$\cdots \times + +$
$+ + \times \cdots$	$\cdots + + \times$
$\times + \times \cdots$	$\cdots \times + \times$
$- + + \cdots$	$\cdots - + +$
$+ + - \cdots$	$\cdots + + -$
$\times \times + \cdots$	$\cdots \times \times +$
$+ \times \times \cdots$	$\cdots + \times \times$
$\times \times \times \cdots$	$\cdots \times \times \times$
$- \times + \cdots$	$\cdots - \times +$
$+ \times - \cdots$	$\cdots + \times -$
$\times \times - \cdots$	$\cdots - \times \times$
$- \times \times \cdots$	$\cdots \times \times -$
$\times - \times \cdots$	$\cdots \times - \times$
$- - + \cdots$	$\cdots - - +$
$+ - - \cdots$	$\cdots + - -$
$- - \times \cdots$	$\cdots - - \times$
$\times - - \cdots$	$\cdots \times - -$
$- - - \cdots$	$\cdots - - -$

Figure 6: Left and right boundary of basic patterns on T_{k-s} that satisfy the Neumann boundary condition.

$+ + +$, $\times + +$, $+ + \times$, $\times + \times$, $+ + -$, $- + +$, $- + \times$, $\times + -$,
$\times \times +$, $+ \times \times$, $\times \times \times$, $+ \times -$, $- \times +$, $\times \times -$, $- \times \times$, $\times - +$,
$+ - \times$, $\times - \times$, $+ - -$, $- - +$, $\times - -$, $- - \times$, $- - -$.

It is easy to check the condition (ii) in I_3^1. We shall check the other condition (i), i.e. for all Γ_k^B, $\underline{\Gamma}_k^B \leq \Gamma_k^B \leq \overline{\Gamma}_k^B$. We divide the discussions into four cases.

(i) $B^+_{\{0,1,2\}}$, $B^\times_{\{-1,0,1\}}$ *and* $B^-_{\{-2,-1,0\}}$.

(ii) $B^+_{\{-1,0,1,2\}}$, $B^\times_{\{-1,0,1\}}$ *and* $B^-_{\{-2,-1,0\}}$.

(iii) $B^+_{\{0,1,2\}}$, $B^\times_{\{-1,0,1\}}$ *and* $B^-_{\{-2,-1,0,1\}}$.

(iv) $B^+_{\{-1,0,1,2\}}$, $B^\times_{\{-1,0,1\}}$ *and* $B^-_{\{-2,-1,0,1\}}$.

We use the method is to attach the least basic patterns that satisfies the boundary conditions of T_k.

(i) $B^+_{\{0,1,2\}}$, $B^\times_{\{-1,0,1\}}$ *and* $B^-_{\{-2,-1,0\}}$.

(a) In Fig. 6, we attach some basic pattern to satisfy the Neumann boundary condition of T_k. When we choose any $s \geq 0$, it satisfies the condition (i).

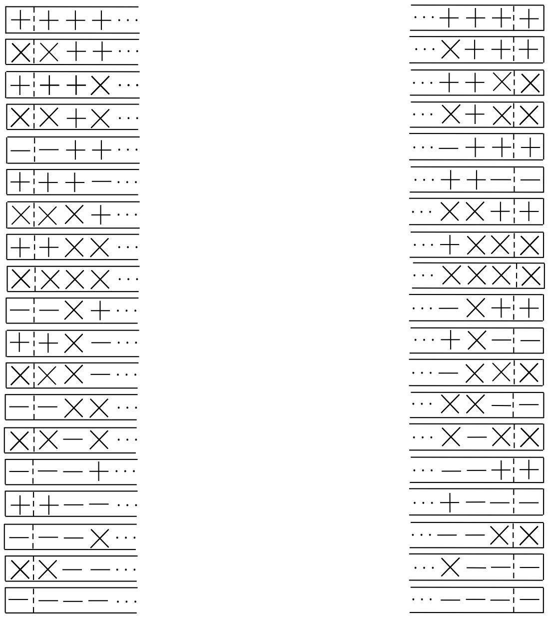

Figure 7: Left and right boundary of basic patterns on T_{k-s} that satisfy the periodic and Dirichlet boundary conditions.

(b) *In Fig. 7, we attach some basic pattern to satisfy the periodic boundary condition of T_k. When we choose any $s \geq 1$, it satisfies the condition (i).*

(c) *In Fig. 7, we attach some basic pattern to satisfy the periodic boundary condition of T_k, i.e. D_1. When we choose any $s \geq 1$, it satisfies the condition (i).*

Thus, we choose a $s \geq 1$, this case (i) satisfies the condition (i).

(ii) $B^{+}_{\{-1,0,1,2\}}$, $B^{\times}_{\{-1,0,1\}}$ *and* $B^{-}_{\{-2,-1,0\}}$

(a) *In Fig. 8, we attach some basic pattern to satisfy the Neumann boundary condition of T_k. When we choose any $s \geq 0$, it satisfies the condition (i).*

(b) *In Fig. 9, we attach some basic pattern to satisfy the periodic boundary condition of T_k. When we choose any $s \geq 1$, it satisfies the condition (i).*

(c) *In Fig. 9, we attach some basic pattern to satisfy the Dirichlet boundary condition of T_k, i.e. D_1. When we choose any $s \geq 1$, it satisfies the condition (i).*

Thus, we choose a $s \geq 1$, this case (ii) satisfies the condition (i).

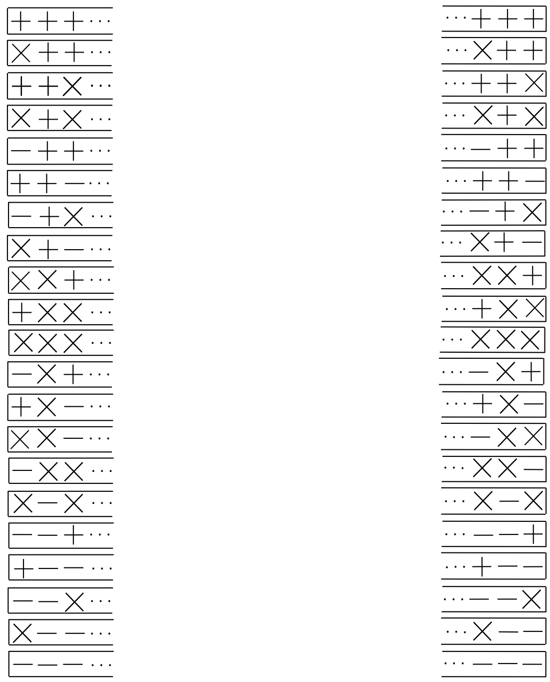

Figure 8: Left and right boundary of basic patterns on T_{k-s} that satisfy the Neumann boundary condition.

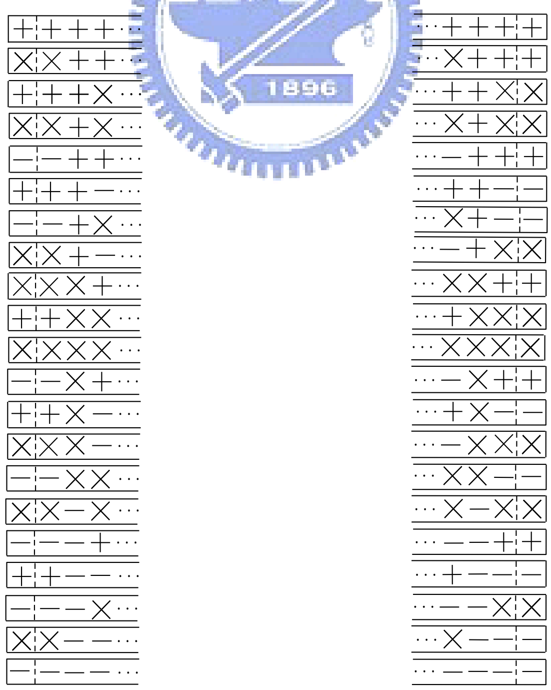

Figure 9: Left and right boundary of basic patterns on T_{k-s} that satisfy the periodic and Dirichlet boundary conditions.

The case (iii) and the case (iv) are similar, and we choose any $s \geq 1$ to satisfy the condition (i). According to the Proposition 4.7, then $h = h_N = h_P = h_{D_1}$.

Example 4.11. *In this example, we illustrate that the parameters in I_5^2 satisfy the conditions of Proposition 4.7 and justify the equality $h = h_N = h_P = h_{D_1}$. In Table 2 and Table 4.1, the parameter region I_5^2 has the feasible basic patterns: $B^+_{\{1,2\}}$, $B^\times_{\{0\}}$ and $B^-_{\{-2,-1\}}$ and the possible basic patterns: $B^+_{\{1,2\}}$, $B^\times_{\{-1,0,1\}}$ and $B^-_{\{-2,-1\}}$. The boundary of basic patterns on T_{k-s} may be:*

$\overline{\underline{\underline{+++}}}$, $\overline{\underline{\underline{\times++}}}$, $\overline{\underline{\underline{++\times}}}$, $\overline{\underline{\underline{+\times\times}}}$, $\overline{\underline{\underline{\times\times+}}}$, $\overline{\underline{\times\times\times}}$, $\overline{\underline{+\times-}}$, $\overline{\underline{-\times+}}$,
$\underline{-\times\times}$, $\underline{\times\times-}$, $\underline{--\times}$, $\underline{\times--}$, $\underline{---}$.

It is easy to check the condition (ii) in I_5^2. We shall the other condition (i), i.e. for all Γ_k^B, $\underline{\Gamma}_k^B \leq \Gamma_k^B \leq \overline{\Gamma}_k^B$. We divide the discussions into four cases.

(i) $B^+_{\{1,2\}}$, $B^\times_{\{0\}}$ *and* $B^-_{\{-2,-1\}}$.

(ii) $B^+_{\{1,2\}}$, $B^\times_{\{-1,0\}}$ *and* $B^-_{\{-2,-1\}}$.

(iii) $B^+_{\{1,2\}}$, $B^\times_{\{0,1\}}$ *and* $B^-_{\{-2,-1\}}$.

(iv) $B^+_{\{1,2\}}$, $B^\times_{\{-1,0,1\}}$ *and* $B^-_{\{-2,-1\}}$.

We use the method is to add the least lattices that satisfies the boundary conditions of T_k.

(i) $B^+_{\{1,2\}}$, $B^\times_{\{0\}}$ *and* $B^-_{\{-2,-1\}}$.

(a) In Fig. 10, we attach some basic pattern to satisfy the Neumann boundary condition of T_k. When we choose any $s \geq 1$, it satisfies the condition (i).

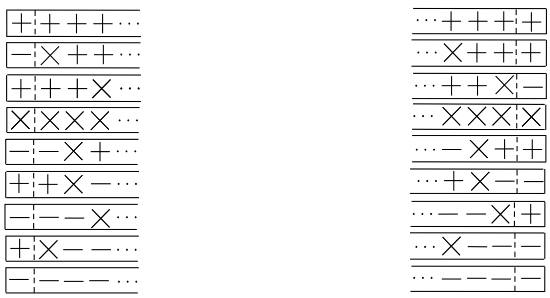

Figure 10: Left and right boundary of basic patterns on T_{k-s} that satisfy the Neumann boundary condition.

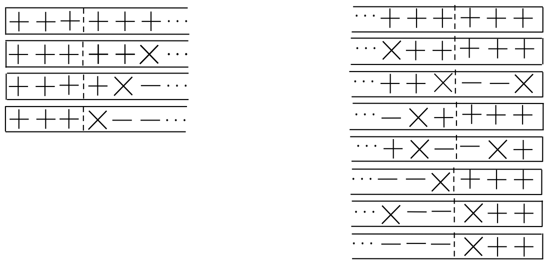

Figure 11: Left and right boundary of basic patterns on T_{k-s} that satisfy the periodic and periodic boundary condition.

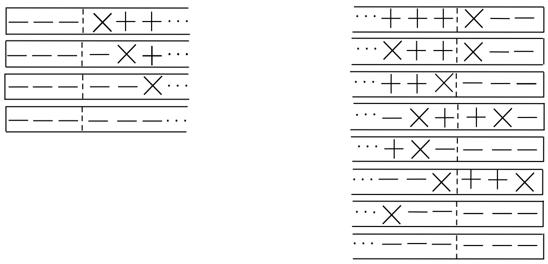

Figure 12: Left and right boundary of basic patterns on T_{k-s} that satisfy the periodic boundary condition.

(b) *In Fig. 11 and 12, we add some lattices to satisfy the periodic boundary condition of T_k. We divide two parts of the boundary patterns to discuss. When we choose any $s \geq 3$, it satisfies the condition (i).*

One part is Fig. 11.

The other part is Figure 12.

The pattern T_k $\overline{\underline{\times\times\times\cdots\times\times\times}}$ is also satisfying the condition (i).

(c) *In Fig. 13 ,we attach some basic pattern to satisfy the Dirichlet boundary condition of T_k, i.e. D_1. When we choose any $s \geq 3$, it satisfies the condition (i).*

Beside the only one pattern $\overline{\underline{\times\times\times\cdots\times\times\times}}$ can not satisfying the Dirichlet boundary condition.

Thus, we choose any $s \geq 3$, this case (i) satisfies the condition (i).

The case (ii), (iii) and (iv) are similar, and we choose any $s \geq 3$ to satisfy the condition (i). According to the Proposition 4.7, then $h = h_N = h_P = h_{D_1}$.

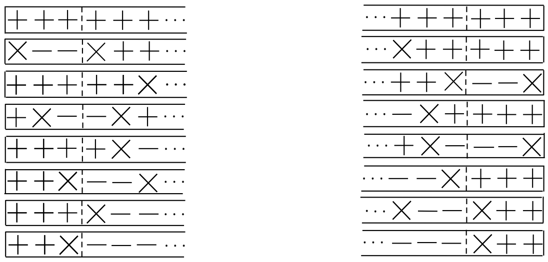

Figure 13: Left and right boundary of basic patterns on T_{k-s} that satisfy the Dirichlet boundary condition.

Example 4.12. *In this example, we check the conditions of Proposition 4.8 to get the result "$h = h_N = h_P = h_{D_1}$". In Table 2 and Table 4.1, the parameter region I_3^2, the number of feasible basic patterns is equal to the number of possible basic patterns. The set of basic patterns is $B^+_{\{0,1,2\}}$, $B^\times_{\{-1,0,1\}}$ and $B^-_{\{-2,-1,0\}}$. In Example 4.10, we know that choose any $s \geq 1$, this region satisfies the conditions of Proposition 4.8. Then $h = h_N = h_P = h_{D_1}$.*

Note: If the number of pseudo basic patterns is least for each case, then we need to take s.

Now we discuss that the parameter regions of $b_1 = 0$ or $b_2 = 0$, the other parameter regions are use the same method to discuss. Similarly, we are only check the conditions: (i) $\Gamma^B_{\mathbf{k}} \geq \Gamma^\infty_{\mathbf{k-s}}$ and (ii) $\Gamma^B_{\mathbf{k}} \leq p^c \cdot \Gamma^\infty_{\mathbf{k-s}}$ for some $p > 0$ and $c = c(\mathbf{k})$ with $\lim_{\mathbf{k}\to\infty}(c/\mathbf{k}) = 0$. The second condition is easy to check, so we only check the condition(ii) $\Gamma^B_{\mathbf{k}} \geq \Gamma^\infty_{\mathbf{k-s}}$.

We want to know whether if h is equal to h_B in two dimension, where $B = N$, P or D? Consider the case of two-dimensional lattice. We shall only discuss the parameter regions with $b_2 = 0$. Similar to one-dimensional case, it is easy to check that all parameter regions satisfy the condition (ii) of Proposition 4.7. Thus, we are only to check each parameter region satisfies the condition (i) of Proposition 4.7. In each parameter region and different boundary condition, in [11], if we can modify patterns on the layers of the lattice near the boundary, for each pattern, then satisfy the boundary condition, then we can get "$h = h_N = h_P = h_D$". Fortunately, all parameter regions of $b_2 = 0$ satisfy the assumption, so we can get the result.

Example 4.13. *Assume $b_2 = 0$. In I_3^1, this parameter region has the feasible basic patterns: $B^+_{\{-2,-1,0,1,2,3,4\}}$, $B^\times_{\{-3,-2,-1,0,1,2,3\}}$ and $B^-_{\{-4,-3,-2,-1,0,1,2\}}$ and the possible basic patterns: $B^+_{\{-3,-2,-1,0,1,2,3,4\}}$, $B^\times_{\{-3,-2,-1,0,1,2,3\}}$ and*

$B^-_{\{-4,-3,-2,-1,0,1,2,3\}}$. *According to condition(i) of Proposition 4.7(i.e. for all* $\Gamma^B_{\mathbf{k}}$, $\underline{\Gamma}^B_{\mathbf{k}} \leq \Gamma^B_{\mathbf{k}} \leq \overline{\Gamma}^B_{\mathbf{k}}$ *satisfies the conditions (i)), we need to check the following four cases satisfies the condition (i).*

(i) $B^+_{\{-2,-1,0,1,2,3,4\}}$, $B^\times_{\{-3,-2,-1,0,1,2,3\}}$ *and* $B^-_{\{-4,-3,-2,-1,0,1,2\}}$.

(ii) $B^+_{\{-3,-2,-1,0,1,2,3,4\}}$, $B^\times_{\{-3,-2,-1,0,1,2,3\}}$ *and* $B^-_{\{-4,-3,-2,-1,0,1,2\}}$.

(iii) $B^+_{\{-2,-1,0,1,2,3,4\}}$, $B^\times_{\{-3,-2,-1,0,1,2,3\}}$ *and* $B^-_{\{-4,-3,-2,-1,0,1,2,3\}}$.

(iv) $B^+_{\{-3,-2,-1,0,1,2,3,4\}}$, $B^\times_{\{-3,-2,-1,0,1,2,3\}}$ *and* $B^-_{\{-4,-3,-2,-1,0,1,2,3\}}$.

Assume the pattern $T_{\mathbf{k}-2}$ *is surrounding by two layers "×" in Fig. 14.*

(i) If a "+" in the $(\mathbf{k}-2)$*th layer is surrounded by three "−" on* $T_{\mathbf{k}-2}$*, we change this "+" to "−". And if a "−" in the* $(\mathbf{k}-2)$*th layer is surrounded by three "+" on* $T_{\mathbf{k}-2}$*, we change this "−" to "+". After these steps, pattern* $T_{\mathbf{k}}$ *satisfy those boundary conditions. Thus, we know that satisfies the condition (ii). According to propsition 4.7 we get that* $h = h_B$*, where* $B = N$, P *or* D_1.

(ii) If a "−" in the $(\mathbf{k}-2)$*th layer is surrounded by three "+" on* $T_{\mathbf{k}-2}$*, we change this "−" to "+". After this step, pattern* $T_{\mathbf{k}}$ *satisfy those boundary conditions. Thus, we know that satisfies the condition (ii). According to propsition1 we get that* $h = h_B$*, where* $B = N$, P *or* D_1.

(iii) If a "+" in the $(\mathbf{k}-2)$*th layer is surrounded by three "−" on* $T_{\mathbf{k}-2}$*, we change this "+" to "−". After this step, pattern* $T_{\mathbf{k}}$ *satisfy those boundary conditions. Thus, we know that satisfies the condition (ii). According to propsition1 we get that* $h = h_B$*, where* $B = N$, P *or* D_1.

(iv) We do not change the $(k-2)$*th layer, then only add* $\mathbf{k}$*th layer to satisfy those boundary conditions. According to propsition1 we get that* $h = h_B$*. Then we get* $h = h_B$*, where* $B = N$, P *or* D_1.

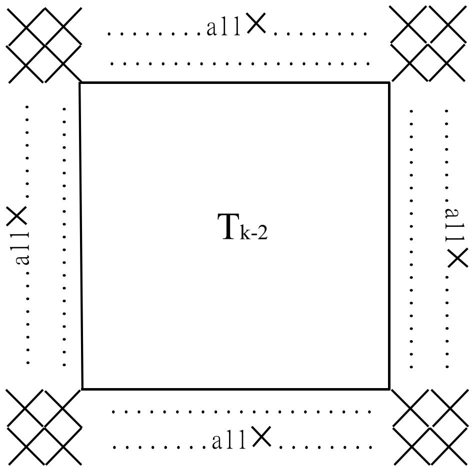

Figure 14: In I_3^1, the pattern T_{k-2} is surrounded by two layers "×".

5 Numerical Illustrations

In this section, we use basic pattern formation to create the same patterns as [3]. We observe the basic patterns (3 × 3 herein) as needed to obtain the designated patterns and look for parameters with which these basic patterns are feasible for (1.4). We then choose these parameters for (1.4) and use numerical computations (Newton's method) to compute the corresponding solutions of (1.4). Our theory can thus be justified.

Recall the two-dimensional reaction diffusion equation (1.2) with (1.4):

$$\frac{du_{i,j}}{dt} = \beta^{+}\Delta^{+}u_{i,j} + \beta^{\times}\Delta^{\times}u_{i,j} + \alpha f(u_{i,j}), \text{ where } (i,j) \in \mathbb{Z}^2, \tag{5.1}$$

where

$$f(\xi) = \xi^3 - \xi.$$

Patterns in Color: The value of $u_{i,j}$ is to colored as in Fig. 15, 16.

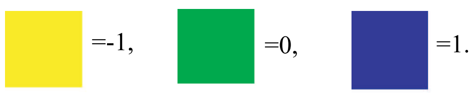

Figure 15: Patterns in color.

Example 5.1. *Checkerboard with horizontal interface.*
We need the following 3 × 3 *basic patterns in Fig. 18 to generate the following* 7 × 7 *checkerboard with horizontal interface in Fig. 17, through attaching process.*