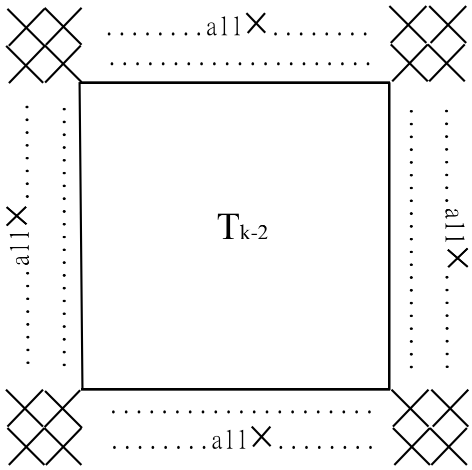

Figure 14: In I_3^1, the pattern T_{k-2} is surrounded by two layers "×".

5 Numerical Illustrations

In this section, we use basic pattern formation to create the same patterns as [3]. We observe the basic patterns (3×3 herein) as needed to obtain the designated patterns and look for parameters with which these basic patterns are feasible for (1.4). We then choose these parameters for (1.4) and use numerical computations (Newton's method) to compute the corresponding solutions of (1.4). Our theory can thus be justified.

Recall the two-dimensional reaction diffusion equation (1.2) with (1.4):

$$\frac{du_{i,j}}{dt} = \beta^+ \Delta^+ u_{i,j} + \beta^\times \Delta^\times u_{i,j} + \alpha f(u_{i,j}), \text{ where } (i,j) \in \mathbb{Z}^2, \tag{5.1}$$

where

$$f(\xi) = \xi^3 - \xi.$$

Patterns in Color: The value of $u_{i,j}$ is to colored as in Fig. 15, 16.

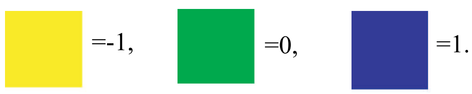

Figure 15: Patterns in color.

Example 5.1. *Checkerboard with horizontal interface.*
We need the following 3×3 basic patterns in Fig. 18 to generate the following 7×7 checkerboard with horizontal interface in Fig. 17, through attaching process.

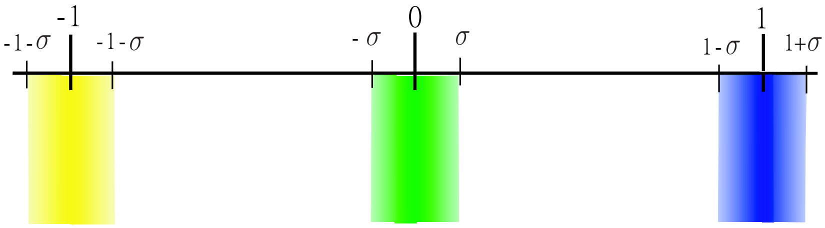

Figure 16: Patterns in color.

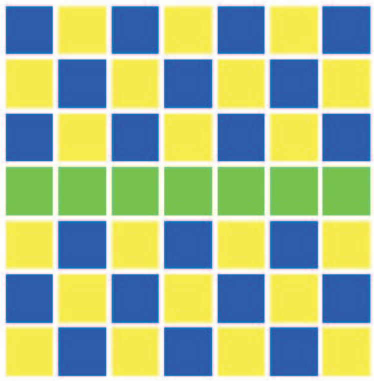

Figure 17: Checkerboard with horizontal interface.

We use previous intersection concept to find the parameter region. Thus, for the basic patterns in Fig. 18 to exist, suppose $b_1 > 0$ and $b_2 > 0$, we derive four conditions:

$$(5.2)\quad \begin{cases} \frac{f(1-\sigma)}{\frac{6b_1+3b_2}{b_1+b_2}} \leq b_1 + b_2 \leq \frac{f(1+\sigma)}{\frac{6b_1+3b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{7b_1+2b_2}{b_1+b_2}} \leq b_1 + b_2 \leq \frac{f(1+\sigma)}{\frac{7b_1+2b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{8b_1}{b_1+b_2}} \leq b_1 + b_2 \leq \frac{f(1+\sigma)}{\frac{8b_1}{b_1+b_2}+8\sigma} \\ \frac{f(\sigma)}{8\sigma} \leq b_1 + b_2. \end{cases}$$

Then (5.2) *becomes the following conditions:*

$$\begin{cases} b_1 > 0 \\ b_2 > 0 \\ (6+8\sigma)b_1 + (3+8\sigma)b_2 \leq f(1+\sigma) \\ (7+8\sigma)b_1 + (2+8\sigma)b_2 \leq f(1+\sigma) \\ (8+8\sigma)b_1 + (8\sigma)b_2 \leq f(1+\sigma). \end{cases}$$

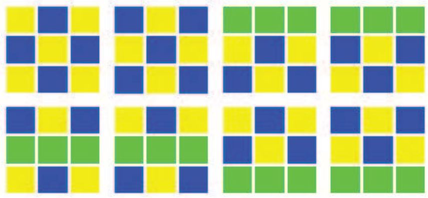

Figure 18: Basic patterns of Fig. 17.

1.0036	-1.0039	1.0039	-1.0039	1.0039	-1.0039	1.0036
-1.0039	1.004	-1.004	1.004	-1.004	1.004	-1.0039
1.0036	-1.0039	1.0039	-1.0039	1.0039	-1.0039	1.0036
3.434e-22	8.590e-22	6.866e-22	8.590e-22	6.866e-22	8.590e-22	3.434e-22
-1.0036	1.0039	-1.0039	1.0039	-1.0039	1.0039	-1.0036
1.0039	-1.004	1.004	-1.004	1.004	-1.004	1.0039
-1.0036	1.0039	-1.0039	1.0039	-1.0039	1.0039	-1.0036

Figure 19: The values $\{\overline{u}_{i,j}\}_{1\leq i,j\leq 7}$ in Example 5.1.

As an illustration, we choose the parameters $\sigma = 0.01$, $b_1 = 0.001$, $b_2 = 0.0004$. *With these chosen parameters, we compute* (1.2), (1.3) *and* (1.4) *with Dirichlet boundary condition* D_0 (*i.e.* $u_{\mathbf{b}} = \hat{u}_{\mathbf{b}} := \{0, i \in \mathbf{b}\}$), *numerically to find the solutions* $\{\overline{u}_{i,j}\}_{1\leq i,j\leq 7}$. *The values* $\{\overline{u}_{i,j}\}_{1\leq i,j\leq 7}$ *are listed in Fig. 19, its associated pattern in colors is exactly the one in Fig. 17.*

Example 5.2. *Vertical and horizontal stripes with vertical interface.*
We need the following 3×3 *basic patterns in Fig. 21 to generate the following* 8×8 *vertical and horizontal stripes with vertical interface in Fig. 20, through attaching process.*

We can attach the pattern Fig. 20. Thus, we use previous intersection concept to find the parameter region. Thus, for the basic patterns in Fig. 21 to exist, suppose $b_1 > 0$ *and* $b_2 > 0$, *we derive six conditions:*

Figure 20: Vertical and horizontal stripes with vertical interface.

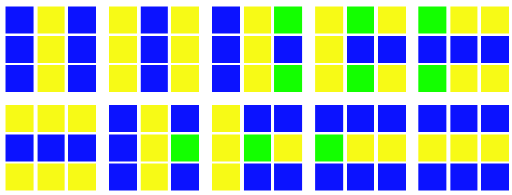

Figure 21: Basic patterns of Fig. 20.

$$(5.3)\quad \begin{cases} \frac{f(1-\sigma)}{\frac{b_1+8b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{b_1+8b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{3b_1+5b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{3b_1+5b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{3b_1+6b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{3b_1+6b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{4b_1+5b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{4b_1+5b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{4b_1+6b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{4b_1+6b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{4b_1+7b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{4b_1+7b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{5b_1+5b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{5b_1+5b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{5b_1+6b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{5b_1+6b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{5b_1+8b_2}{b_1+b_2}} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{5b_1+8b_2}{b_1+b_2}+8\sigma} \\ \frac{f(1-\sigma)}{\frac{b_1}{b_1+b_2}+8\sigma} \le b_1+b_2 & \le \frac{f(1+\sigma)}{\frac{-b_1}{b_1+b_2}} \\ \frac{f(\sigma)}{8\sigma} \le b_1+b_2. & \end{cases}$$

Then (5.3) *becomes the following conditions:*

$$\begin{cases} b_1 > 0 \\ b_2 > 0 \\ (5+8\sigma)b_1 + (8+8\sigma)b_2 \le f(1+\sigma). \end{cases}$$

As an illustration, we choose the parameters $\sigma = 0.01$, $b_1 = 0.001$, $b_2 = 0.001$. *With these chosen parameters, we compute* (1.2), (1.3) *and* (1.4) *with Dirichlet boundary condition* D_0*, numerically to find the solutions* $\{\overline{u}_{i,j}\}_{1\le i,j\le 8}$. *The values* $\{\overline{u}_{i,j}\}_{1\le i,j\le 8}$ *are listed in Fig. 22, its associated pattern in colors is exactly the one in Fig. 20.*

1.0045	-1.0055	1.0055	-1.005	1.005	1.004	1.0045	1.0045
1.0045	-1.006	1.006	-1.0055	-9.801e-07	-1.0065	-1.006	-1.0055
1.0045	-1.006	1.006	-1.005	1.006	1.005	1.006	1.0055
1.0045	-1.006	1.006	-1.0055	7.23e-10	-1.0065	-1.006	-1.0055
1.0045	-1.006	1.006	-1.005	1.006	1.005	1.006	1.0055
1.0045	-1.006	1.006	-1.0055	1.214e-09	-1.0065	-1.006	-1.0055
1.0045	-1.006	1.006	-1.005	1.006	1.005	1.006	1.0055
1.0045	-1.0055	-1.0055	-1.005	-0.0009999	-1.005	-1.0045	-1.0045

Figure 22: The values $\{\overline{u}_{i,j}\}_{1\le i,j\le 8}$ in Example 5.2.

Using the method similar to above examples, we present more examples in the following.

Example 5.3. *Checkerboard with thin horizontal interface.*
The following 3×3 *basic patterns in Fig. 24 can attach the pattern checkerboard with thin horizontal interface in Fig. 23 with these chosen parameters* $\sigma = 0.01$, $b_1 = 0.001$ *and* $b_2 = 0.0004$ *which satisfy the following, conditions we compute* (1.2), (1.3) *and* (1.4) *with Dirichlet boundary condition* D_0*, numerically to find the solutions* $\{\overline{u}_{i,j}\}_{1\le i,j\le 7}$. *Its patterns in colors are exactly the one in Fig. 23.*

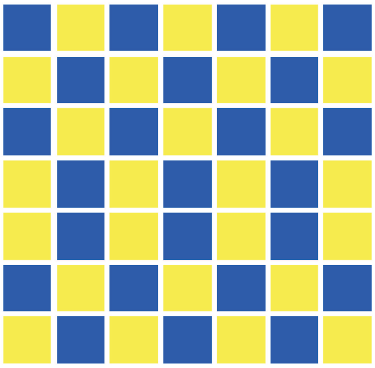

Figure 23: Checkerboard with thin horizontal interface.

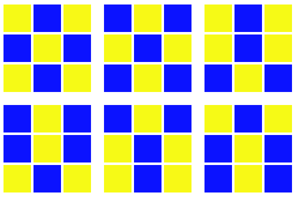

Figure 24: Basic patterns of Fig. 23.

$$\begin{cases} b_1 > 0 \\ b_2 > 0 \\ (6+8\sigma)b_1 + (4+8\sigma)b_2 \leq f(1+\sigma) \\ (7+8\sigma)b_1 + (2+8\sigma)b_2 \leq f(1+\sigma) \\ (8+8\sigma)b_1 + (8\sigma)b_2 \leq f(1+\sigma). \end{cases}$$

Example 5.4. *Checkerboard with diagonal interface.*
The following 3×3 *basic patterns in Fig. 26 can attach the pattern checkerboard with diagonal interface in Fig. 25 with these chosen parameters* $\sigma = 0.01$, $b_1 = 0.001$ *and* $b_2 = 0.0009$ *which satisfy the following, conditions we compute* (1.2), (1.3) *and* (1.4) *with Dirichlet boundary condition* D_0, *numerically to find the solutions* $\{\overline{u}_{i,j}\}_{1 \leq i,j \leq 7}$. *Its associated pattern in colors is exactly the one in Fig. 25.*

$$\begin{cases} b_1 > 0 \\ b_2 > 0 \\ (5+8\sigma)b_1 + (4+8\sigma)b_2 \leq f(1+\sigma) \\ (7+8\sigma)b_1 + (3+8\sigma)b_2 \leq f(1+\sigma) \\ (8+8\sigma)b_1 + (1+8\sigma)b_2 \leq f(1+\sigma). \end{cases}$$

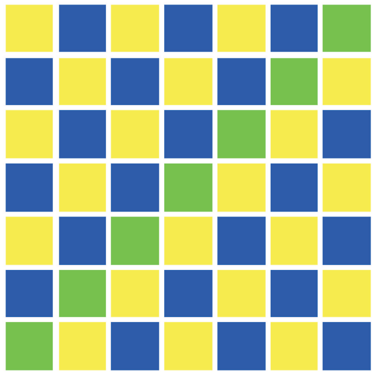

Figure 25: Checkerboard with diagonal interface.

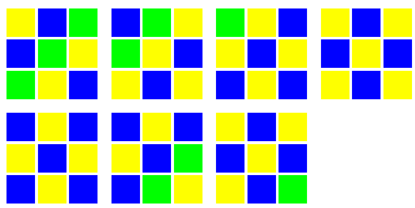

Figure 26: Basic patterns of Fig. 25.

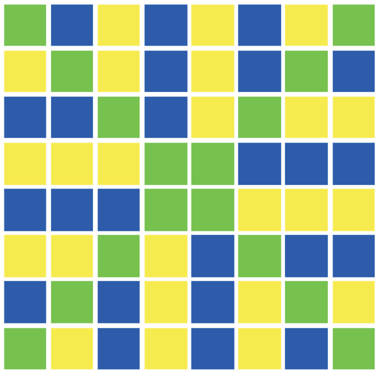

Figure 27: Quad junction.

Example 5.5. *Quad junction.*
The following 3×3 *basic patterns in Fig. 28 can attach the pattern quad junction in Fig. 27 with these chosen parameters* $\sigma = 0.01$, $b_1 = 0.001$ *and* $b_2 = 0.001$ *which satisfy the following conditions , we compute* (1.2), (1.3) *and* (1.4) *with Dirichlet boundary condition* D_0, *numerically to find the solutions* $\{\overline{u}_{i,j}\}_{1 \le i,j \le 8}$. *Its associated pattern in colors is exactly the one in Fig. 27.*

$$\begin{cases} b_1 > 0 \\ b_2 > 0 \\ (4+8\sigma)b_1 + (8+8\sigma)b_2 \le f(1+\sigma) \\ (5+8\sigma)b_1 + (6+8\sigma)b_2 \le f(1+\sigma). \end{cases}$$

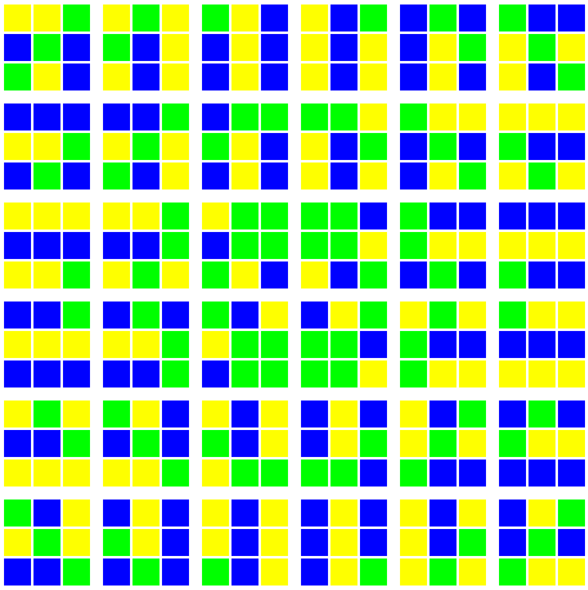

Figure 28: Basic patterns of Fig. 27.