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Cellular Neural Networks : Mosaic Patterns, Bifurcation and complexity

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摘 要

我們主要探討一個細胞類神經網路模型的馬賽克花樣，在這裡考慮的輸出函數在無窮遠的地方並不是平坦的。許多複雜的參數區域是可以被完整地描繪出來，每一個參數區域的熵是可以藉由轉換矩陣的方法算出來；我們也利用參數 z 和 β 來討論一些馬賽克花樣的分歧現象，在這裡 z 是一個偏壓項、β 是和鄰近細胞的互動比重。特別地，對於一個小的互動比重 β ，我們發現當加入偏壓項之後，許多新的複雜參數區域都會產生。然而當 β 增加到某一個範圍之後，許多上述的複雜參數區域會消失，但是又有一些新的複雜參數範圍會產生。

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Abstract

We study mosaic patterns of a one-dimensional Cellular Neural Network with an output function which is non-flat at infinity. Spatial chaotic regions are completely characterized. Moreover, each of their exact corresponding entropy is obtained via the method of transition matrices. We also study the bifurcation phenomenon of mosaic patterns with bifurcation parameters z and β. Here z is a source (or bias) term and β is the interaction weight between the neighboring cells. In particular, we find that by injecting the source term, i.e. $z \neq 0$, a lot of new chaotic patterns emerge with a smaller interaction weight β. However, as β increases to a certain range, most of previously observed chaotic patterns disappear, while other new chaotic patterns emerge.

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1 Introduction

Of concern is one-dimensional Cellular Neural Networks (CNNs) of the form

$$\frac{dx_i}{dt} = -x_i + z + \alpha f(x_{i-1}) + af(x_i) + \beta f(x_{i+1}), \quad i \in \mathbb{Z}. \tag{1.1a}$$

Here x_i denote the state of a cell C_i and $f(x)$ is a piecewise-linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r, & \text{if } x \geq 1\,, \\ x\,, & \text{if } |x| \leq 1, \\ rx - 1 + r, & \text{if } x \leq -1\,, \end{cases} \tag{1.1b}$$

where r is a positive constant. The quantity z is called a source term or a bias term. The numbers α, a and β are arranged in a vector form $[\alpha,\, a,\, \beta]$, which is called a space-invariant A-template

$$A = [\alpha,\ a,\ \beta]. \tag{1.2}$$

A is called symmetric (resp., antisymmetric) if $\alpha = \beta$ (resp., $\alpha = -\beta$).

CNNs were first proposed by Chua and Yang [1988a, 1988b]. Their main applications are in image processing and pattern recognition [Chua, 1998]. For additional background information, applications, and theory, see [Special Issue, 1995; Thiran, 1997; Chua, 1998] among others.

A basic and important class of solutions of (1.1) is the stable stationary solutions. Specifically, a stationary solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ of (1.1) satisfies the following equation

$$f(x_i) = \frac{1}{a}\{x_i - z - \alpha f(x_{i-1}) - \beta f(x_{i+1})\}, \quad i \in \mathbb{Z}. \tag{1.3}$$

Let $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ be a solution of (1.3). The associated output $\mathbf{y} = (y_i)_{i\in\mathbb{Z}} = (f(x_i))_{i\in\mathbb{Z}}$ is called a pattern. The following two types of stationary solutions are of particular interest.

Definition 1.1. *A solution $\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}}$ is called a mosaic solution if $|x_i| > 1$ for all $i \in \mathbb{Z}$. Its associated pattern $\boldsymbol{y} = (y_i)_{i\in\mathbb{Z}} = (f(x_i))_{i\in\mathbb{Z}}$ is called a mosaic pattern. If $|x_i| \neq 1$ for all $i \in \mathbb{Z}$ and there are $i,\ j \in \mathbb{Z}$ such that $|x_i| < 1$ and $|x_j| > 1$, then $\boldsymbol{x} = (x_i)_{i\in\mathbb{Z}}$ and $\boldsymbol{y} = (f(x_i))_{i\in\mathbb{Z}}$ are called, respectively, a defect solution and a defect pattern.*

To define the stability of the stationary solution, we consider the following linearized stability. Let $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \ell^2$, the linearized operator $L(\mathbf{x})$ of (1.1) at a stationary solution $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ is given by

$$(L(\mathbf{x})\xi)_i = -\xi_i + \alpha f'(x_{i-1})\xi_{i-1} + af'(x_i)\xi_i + \beta f'(x_{i+1})\xi_{i+1}. \tag{1.4}$$

Definition 1.2. *Let* $\boldsymbol{x}$= $(x_i)_{i\in\mathbb{Z}}$ *be a solution of* (1.3) *with* $|x_i| \neq 1$ *for all* $i \in \mathbb{Z}$. *The stationary solution* $\boldsymbol{x}$ *is called* (*linearized*) *stable if all eigenvalues of* $L(\boldsymbol{x})$ *have negative real parts. The solution is called unstable if there is an eigenvalue* λ *of* $L(\boldsymbol{x})$ *such that* λ *has a positive real part.*

It is well-known, see e.g., [Juang and Lin, 2000; Hsu, 2000], that for

$$\frac{1}{|a| + |\alpha| + |\beta|} > r \geq 0, \tag{1.5}$$

where r, a, α and β are defined as in (1.1), $-L(\mathbf{x})$ is a self-adjoint and positive operator. Therefore, if r is sufficiently small, all mosaic solutions of (1.1) are stable. For $r = 0$, the complexity of stable stationary solutions of (1.1) with respect to all the parameters has been completely characterized when the template A is symmetric or antisymmetric (see [Thiran et. al., 1995; Juang and Lin, 2000]). For $r > 0$, sufficiently small, a map approach was introduced to study the complexity of stable stationary solutions of (1.1) with limited success (see e.g., [Hsu, 2000; Chang and Juang, 2004]). Specifically, only the parameters region that would yield Smale horseshoe, hence, the spatial entropy of $\ln 2$, is located in those papers. That is to say, only regions that yield the full shift with 2 symbols are found. For $r = 0$ [Juang and Lin, 2000], the parameters regions corresponding to the positive entropy less than $\ln 2$ can also be found. Those are the regions that yield the subshift of finite types (see e.g., [Robinson, 1995]). It would be reasonable to expect that for $r \neq 0$, one can find such regions as well.

The purpose of this thesis is to find parameters regions yielding the subshift of finite types when the template A is symmetric. Our approach here makes use of the techniques originated in [Juang and Lin, 2000] and, later, generalized by [Cheng and Shih, 2005]. The thesis is organized as follows. In section 2, we introduce the notion of (local) basic mosaic patterns. We then identify all these basic mosaic patterns. Moreover, the solvability conditions for the existence of such patterns are also given. Section 3 is devoted to the global mosaic patterns for the symmetric template A and $z = 0$. Specifically, we find parameters regions whose corresponding positive spatial entropy is less than $\ln 2$. The exact entropy of those regions are obtained via the method of the transition matrix. The effect on the pattern formation with the presence of the bias term z and with the intensity of the interaction weight β is recorded in section 4. In particular, with the injection of a source term z ($\neq 0$), a lot of new patterns, which correspond to a certain subshifts of finite types, emerge with a smaller interaction weight β. However, as β increases to a certain range, most of previously observed chaotic patterns disappear, while other new patterns with positive entropy emerge.

2 Basic Mosaic Solutions and Patterns

As in the map approach case, we seek to find the set of solutions of (1.3) that is uniformly bounded. This is also the essence of the thesis in [Cheng and Shih, 2005]. Specifically, we consider the set of solutions $(x_i)_{i\in\mathbb{Z}}$ for which

$$|x_i| < 1+\delta \; for \; all \; i \in \mathbb{Z}, \tag{2.1a}$$

or equivalently,

$$|f(x_i)| < 1+r\delta \; for \; all \; i \in \mathbb{Z}, \tag{2.1b}$$

where $\delta > 0$ is a constant.

To study (1.3), we first define the following concepts.

Definition 2.1. *Given any $i \in \mathbb{Z}$, let x_{i-1} and x_{i+1} be any real numbers for which $|x_j| < 1+\delta$, $j = i-1, i+1$. If there is a unique x_i satisfying* (1.3)*, then $[x_{i-1}, x_i, x_{i+1}]$ is called a basic solution of* (1.3)*. Its corresponding output $[f(x_{i-1}), f(x_i), f(x_{i+1})]$ is called a basic pattern of* (1.3)*. If, in addition, $|x_j| > 1$, $j = i-1, i, i+1$, then $[x_{i-1}, x_i, x_{i+1}]$* (*resp., $[f(x_{i-1}), f(x_i), f(x_{i+1})]$*) *is called a basic mosaic solution* (*resp., pattern*) *of* (1.3)*. Note that the template A is space-invariant. Therefore, a basic solution pattern is independent of the spatial variable i.*

Notation 2.1. *For any mosaic pattern $\{y_i\}_{i\in\mathbb{Z}}$, we shall denote by $+$* (*resp., $-$*) *if $y_i = f(x_i) > 1$* (*resp., $y_i = f(x_i) < -1$*)*. There are only* 8 *types of basic mosaic patterns. We list as below.*

$$[+\,+\,+]_\delta, [-\,-\,-]_\delta, [+\,+\,-]_\delta, [-\,+\,+]_\delta, [+\,-\,-]_\delta, [-\,-\,+]_\delta, [-\,+\,-]_\delta \; and \; [+\,-\,+]_\delta. \tag{2.2}$$

Notation 2.2. *The parameters regions that would yield the* 8 *basic mosaic patterns are, respectively, denoted by Γ_2^+, Γ_{-2}^-, Γ_0^+, Γ_0^+, Γ_0^-, Γ_0^-, Γ_{-2}^+ and Γ_2^-.*

Remark 2.1. *Since the template A under consideration is symmetric, the parameter regions generating $[+\,+\,-]_\delta$ and $[-\,+\,+]_\delta$ are exactly the same* (*see Propositions* 2.3 *and* 4.1)*. Thus, we make no distinction for the region that would yield those two types mosaic patterns. Likewise, the same is true for $[+\,-\,-]_\delta$ and $[-\,-\,+]_\delta$.*

We next study the range of parameters a, α, β, z and r for which the existence of each of 8 basic mosaic patterns is guaranteed. For simplification, we first consider $0 < r < \frac{1}{2}$, $z = 0$ and $\alpha = \beta$. We need the following useful proposition.

Proposition 2.1. *Let* $A = (a_1, 0)$*,* $B = (b_1, 0)$*,* $C = (1, 1)$*,* $D = (1 + \delta, 1 + r\delta)$*,* $C' = (-1, -1)$ *and* $D' = (-1 - \delta, -1 - r\delta)$*. Suppose* $-1 < a_1 < b_1 < 1$ *and* $0 < r < \frac{1}{2}$*. Let* $E \in \overline{AB}$ *be arbitrarily given. The necessary and sufficient condition for any straight line* l *passing through* E *with the slope* m_E *and intercepting the open line segment* $\overline{CD}$ *(resp.,* $\overline{C'D'}$*) is that the slope* m_E *satisfies the following inequalities.*

$$m_{\overline{BD}} < m_E < m_{\overline{AC}} \tag{2.3a}$$

$$(resp., m_{\overline{AD'}} < m_E < m_{\overline{BC'}}). \tag{2.3b}$$

Here $m_{\overline{EF}}$ *means the slope of the line through* E *and* F*.*

Proof. Form Figure 2.1., we see clearly that $l \cap$ open segment $\overline{CD} \neq \varnothing$ if and only if

$$m_{\overline{ED}} < m_E < m_{\overline{EC}}. \tag{2.4}$$

Note that we need $0 < r < \frac{1}{2}$ to ensure (2.4) holds. The slopes $m_{\overline{ED}}$ and $m_{\overline{EC}}$ are increasing in E as long as E is in between A and B. Thus if m_E satisfies (2.3a), then the intersection of l and open segment $\overline{CD}$ is nonempty. On the other hand, if $m_E \geq m_{\overline{EC}}$, we see immediately that the line passing through E with such slope m_E either intersects $\overline{CD}$ at C or does not intersect $\overline{CD}$ at all, a contradiction. Similarly, if $m_E \leq m_{\overline{ED}}$, we draw the same conclusion. The proof for second assertion of the proposition is similar. $\square$

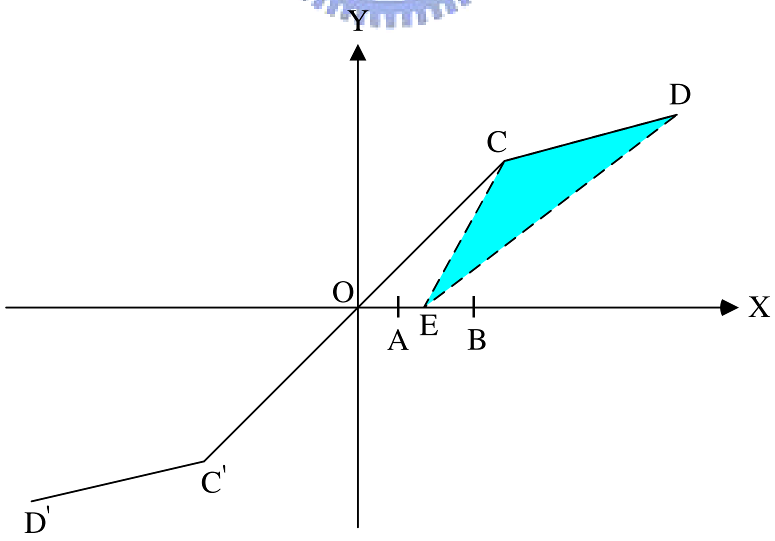

Figure 2.1: .

In the following, we describe the parameters regions, Γ_2^+ and Γ_{-2}^-, which are the same if $0 < r < \frac{1}{2}$, $\alpha = \beta$ and $z = 0$.

Proposition 2.2. *Let*

$$0 < r < \frac{1}{2},\ z = 0 \ \textit{and}\ -\frac{1}{2(1+r\delta)} < \alpha = \beta < \frac{1}{2(1+r\delta)}. \tag{2.5}$$

Then the basic mosaic patterns $[+\,+\,+]_\delta$ *and* $[-\,-\,-]_\delta$ *exist provided that* (2.6) *or* (2.7) *holds. Here* (2.6) *and* (2.7) *are given in the following.*

$$a + 2\beta > 1, \tag{2.6a}$$

$$(1 + r\delta)a + 2(1 + r\delta)\beta < 1 + \delta, \tag{2.6b}$$

$$\beta > 0, \tag{2.6c}$$

and

$$a + 2\beta(1 + r\delta) > 1, \tag{2.7a}$$

$$(1 + r\delta)a + 2\beta < 1 + \delta, \tag{2.7b}$$

$$\beta < 0. \tag{2.7c}$$

Proof. We illustrate only the case that $\beta > 0$. Let x_{i+1} and x_{i-1} be any numbers in between 1 and $1+\delta$, then $2\beta < \beta(f(x_{i-1}) + f(x_{i+1})) =: p < 2\beta(1 + r\delta)$ and equation (1.3) reduces to

$$f(x_i) = \frac{1}{a}[x_i - p]. \tag{2.8}$$

Set $A = (2\beta, 0), B = (2\beta(1 + r\delta), 0)$ and $E = (p, 0)$. It then follows from Proposition 2.1 that

$$if\ (2.6)\ holds,\ then\ (2.8)\ has\ a\ unique\ solution\ x_i\ with\ 1 < x_i < 1 + \delta. \tag{2.9}$$

Similarly, if x_{i-1} and x_{i+1} are any numbers in between $-1-\delta$ and -1. Then $2\beta(1+r\delta) < p < 2\beta$. Set $A = (2\beta(1 + r\delta), 0)$ and $B = (2\beta, 0)$, we also conclude that if (2.6) holds, than (2.8) has a unique solution x_i with $-1 - \delta < x_i < -1$. Since (2.9) holds for any $1 < x_{i-1}, x_{i+1} < 1 + \delta$ or $-1 - \delta < x_{i-1}, x_{i+1} < -1$, we conclude that $[x_{i-1},\ x_i,\ x_{i+1}]$ is indeed a local solution. □

From Proposition 2.2, we see that for fixed r, $0 < r < \frac{1}{2}$, and $\delta > 0$,

$$\Gamma_2^+ = \Gamma_{-2}^- = \{(a, \beta) : (2.6)\ \textit{holds or}\ (2.7)\ \textit{holds and}\ |\beta| < \frac{1}{2(1 + r\delta)}\} =: \Gamma_2.$$

We next study the parameters regions Γ_0^+ and Γ_0^-.

Proposition 2.3. *Suppose* (2.5) *holds, then the basic mosaic patterns* $[++-]_\delta$, $[-++]_\delta$, $[+--]_\delta$ *and* $[--+]_\delta$ *exist provided that* (2.10) *or* (2.11) *holds. Here* (2.10) *and* (2.11) *are given in the following.*

$$(1+r\delta)a + r\delta\beta < 1+\delta, \tag{2.10a}$$

$$a - r\delta\beta > 1, \tag{2.10b}$$

$$\beta > 0, \tag{2.10c}$$

and

$$(1+r\delta)a - r\delta\beta < 1+\delta, \tag{2.11a}$$

$$a + r\delta\beta > 1, \tag{2.11b}$$

$$\beta < 0. \tag{2.11c}$$

The parameters regions Γ_2^- and Γ_{-2}^+ are given in the following.

Proposition 2.4. *Suppose* (2.5) *holds, then the basic mosaic patterns* $[+-+]_\delta$ *and* $[-+-]_\delta$ *exist provided that* (2.12) *or* (2.13) *holds. Here* (2.12) *and* (2.13) *are given in the following.*

$$a - 2(1+r\delta)\beta > 1, \tag{2.12a}$$

$$(1+r\delta)a - 2\beta < 1+\delta, \tag{2.12b}$$

$$\beta > 0, \tag{2.12c}$$

and

$$a - 2\beta > 1, \tag{2.13a}$$

$$(1+r\delta)a - 2(1+r\delta)\beta < 1+\delta, \tag{2.13b}$$

$$\beta < 0. \tag{2.13c}$$

The proof of Propositions 2.3 and 2.4 are similar to that of Proposition 2.2, and is thus omitted.

Clearly, we have that for fixed r, $0 < r < \frac{1}{2}$, and $\delta > 0$,

$$\Gamma_0^+ = \Gamma_0^- = \{(a,\beta) : (2.10) \textit{ holds or } (2.11) \textit{ holds and } |\beta| < \frac{1}{2(1+r\delta)}\} =: \Gamma_0,$$

and

$$\Gamma_2^- = \Gamma_{-2}^+ = \{(a,\beta) : (2.12) \textit{ holds or } (2.13) \textit{ holds and } |\beta| < \frac{1}{2(1+r\delta)}\} =: \Gamma_{-2}.$$

3 Global Patterns and Their Entropy

To construct the global solutions/patterns from the local solutions/patterns, we need the following notation and proposition.

Notation 3.1. *Set* $\Gamma_i' = R^2 - \Gamma_i$, $i = 2,\ 0,\ -2$. *Let* $\Gamma_{(i_1,i_2,i_3)} = R_1 \cap R_2 \cap R_3$, *where* $i_j = 0$ *or* 1, $j = 1,2,3$, *and* $R_j = \begin{cases} \Gamma_{-2j+4}, & \text{if } i_j = 1\,, \\ \Gamma'_{-2j+4}, & \text{if } i_j = 0\,. \end{cases}$ *For instance,* $\Gamma_{(1,0,1)} = \Gamma_2 \cap \Gamma_0' \cap \Gamma_{-2}$. *The set of basic mosaic patterns whose corresponding parameters are in* $\Gamma_{(i_1,i_2,i_3)}$ *is denoted by* $B_{(i_1,i_2,i_3)}$.

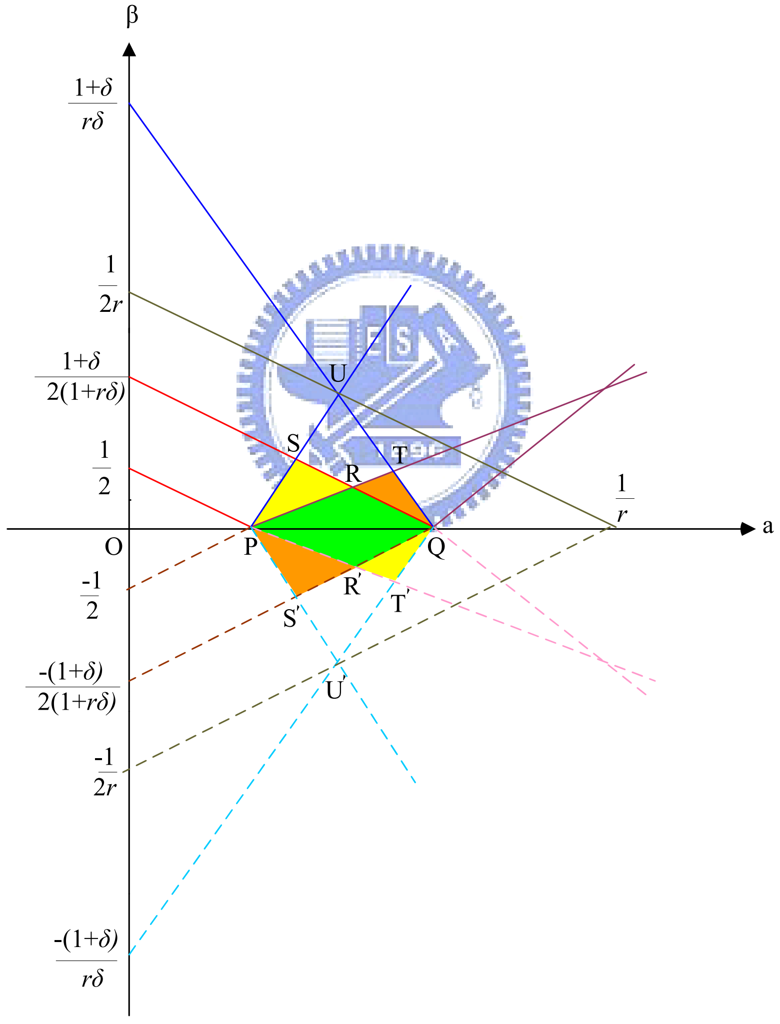

Figure 3.1: $P = (1,0)$, $Q = (\frac{1+\delta}{1+r\delta}, 0)$, $U = (\frac{2+\delta}{2+r\delta}, \frac{1-r}{r(2+r\delta)})$.

For fixed r and δ, we put Γ_i, $i = 2,\ 0,\ -2$, on the $a-\beta$ plane as in Figure 3.1.. Let

$$\bigwedge = \{(a,\beta) : |\beta| < \frac{1}{2(1+r\delta)}\}.$$

Note that

$$\Gamma_{(1,1,1)} = \Gamma_2 \cap \Gamma_0 \cap \Gamma_{-2} = Quadrilateral\ PRQR' \cap \bigwedge \neq \phi,$$

$$\Gamma_{(1,1,0)} = \Gamma_2 \cap \Gamma_0 \cap \Gamma'_{-2} = Triangular\ PSR \cup\ Triangular\ QT'R' \cap \bigwedge \neq \phi,$$

$$\Gamma_{(0,1,1)} = \Gamma'_2 \cap \Gamma_0 \cap \Gamma_{-2} = Triangular\ QTR \cup\ Triangular\ PS'R' \cap \bigwedge \neq \phi,$$

and

$$\Gamma_{(1,0,1)} = \Gamma_2 \cap \Gamma'_0 \cap \Gamma_{-2} = \phi.$$

Proposition 3.1. *Suppose* (2.5) *holds, and that* $(a,\beta) \in \Gamma_{(i_1,i_2,i_3)}$, $i_j = 0$ *or* 1, $j = 1,2,3$. *If* $[x_{i-1},\ x_i,\ x_{i+1}] := x_L$ *is a local mosaic solution of* (1.3) *for some* i, *then* x_L *can be extended to be a global solution* $\bar{x}_G = (\bar{x}_j)_{j\in Z}$, *where* $x_k = \bar{x}_k,\ k = i-1,\ i,\ i+1$, *and for all* $i \neq j$, $[\bar{x}_{j-1},\ \bar{x}_j,\ \bar{x}_{j+1}]$ *are any local solutions of* (1.3) *in* $B_{(i_1,i_2,i_3)}$.

Proof. We only illustrate the case that $(a,\beta) \in \Gamma_{(1,0,0)}$, since the others are similar. In this case either $1 < x_k < 1+\delta$ or $-1-\delta < x_k < -1$ for all $k = i-1,\ i,\ i+1$. Now suppose the former holds, then we assign x_{i+3} to be any number in between 1 and $1+\delta$. Since $(a,\beta) \in \Gamma_{(1,0,0)}$, x_{i+2} can be uniquely determined and its value lies between 1 and $1+\delta$. By proceeding similarly, we get to a global solution $\bar{x}_G$ as claimed. $\square$

From here on, by a mosaic pattern, we mean that the pattern consists of only + or − sign. That is to say we make distinction on only the signs of $f(x_i)$. Using Proposition 3.1, we see immediately that if $(a,\beta) \in \Gamma_{(1,0,0)}$, then the only mosaic patterns are of the following two types

$$\ldots\ldots + + + + + + + + + + + + + \ldots\ldots$$

$$\ldots\ldots - - - - - - - - - - - - - \ldots\ldots$$

Similarly, if $(a,\beta) \in \Gamma_{(0,1,0)}$ (resp., $\Gamma_{(0,0,1)}$), then the mosaic pattern produced is unique up to the translation.

$$\ldots\ldots + + - - + + - - + + - - \ldots\ldots$$

$$(resp., \ldots\ldots + - + - + - + - + - + - \ldots\ldots)$$

Theorem 3.1. *Suppose* (2.5) *holds. Then the following are true.*
(i) If $(a,\beta) \in \Gamma_{(1,1,1)}$, *then any mosaic pattern* $(*_i)_{i\in\mathbb{Z}}, *_i = +$ *or* $-$, *is a pattern for* (1.3). *(ii) If* $(a,\beta) \in \Gamma_{(1,1,0)}$, *then any mosaic pattern* $(*_i)_{i\in\mathbb{Z}}, *_i = +$ *or* $-$, *satisfying the rules that any* $+$ *is adjacent to at least one* $+$, *any* $-$ *is adjacent to at least one* $-$, *is a pattern for* (1.3). *(iii) If* $(a,\beta) \in \Gamma_{(0,1,1)}$, *then any mosaic pattern* $(*)_{i\in\mathbb{Z}}$, $* = +$ *or* $-$, *satisfying the rules that any* $+$ *is adjacent to at least one* $-$, *any* $-$ *is adjacent to at least one* $+$, *is a pattern for* (1.3).

Proof. We illustrate only *(ii)*. The other cases are similar. If $(a,\beta) \in \Gamma_{(1,1,0)}$, then its corresponding basic mosaic patterns are

$$[+++]_\delta, [---]_\delta, [++-]_\delta, [-++]_\delta, [+--]_\delta, [--+]_\delta =: B_{(1,1,0)}. \tag{3.1}$$

In view of (3.1) and Proposition 3.1, we see, immediately, that the assertion in (ii) holds true. □

We next study the complexity of the patterns for given choices of parameters .

Definition 3.1. *Let* $\mu_\Gamma = \{(*_i)_{i\in\mathbb{Z}} : *_i = +$ *or* $-\}$ *be a set of stable mosaic patterns of* (1.3) *for given choices of parameters in* Γ. *The spatial entropy* $h(\mu_\Gamma)$ *is defined as the limit*

$$h(\mu_\Gamma) = \lim_{n\to\infty} \frac{\ln \sharp(\mu_\Gamma^n)}{n}. \tag{3.2}$$

Here $\sharp(\mu_\Gamma^n)$ = *the cardinality of the set* $\mu_\Gamma^n = \{(*_i)_{i=1}^n : *_i = +$ *or* $-, (*_i)_{i\in\mathbb{Z}} \in \mu_\Gamma\}$

Note that μ_Γ is a translation invariant set and the limit in (3.2) is well-defined (see e.g., [Chow et. al., 1996]).

Definition 3.2. *We say the system* (1.1) *or* (1.3) *exhibits spatial chaos for given choices of parameters in* Γ, *in case that spatial entropy* $h(\mu_\Gamma)$ *is positive. We say that the system* (1.1) *or* (1.3) *exhibits pattern formation for given choices of parameters in* Γ *in case the spatial entropy* $h(\mu_\Gamma)$ *is zero.*

We next recall a well-known result (see e.g., Robinson, 1995).

Theorem 3.2. *Suppose there is a one-to-one and onto correspondence between the set* μ_Γ *and the sequence space* Σ_A. *Here* A *is a matrix of dimension* $n \times n$ *whose elements are* 0 *and* 1, *and that* $\Sigma_A = \{(s_i) : (A)_{s_i,s_{i+1}} = 1$ *for all* $i\}$. *Then* $h(\mu_\Gamma) = \ln\lambda$, *where* λ *is the maximal eigenvalue of* A.

Theorem 3.3. *Suppose* (2.5) *holds. If* $(a, \beta) \in \Gamma_{(i_1,i_2,i_3)}$, $i_j \in \{0,1\}$, $j = 1,\ 2,\ 3$, *then system* (1.1) *exhibits spatial chaos if and only if* $i_2 = 1$ *and* $i_1 + i_3 \geq 1$. *Moreover,* $h(\mu_{\Gamma_{(1,1,1)}}) = \ln 2$ *and* $h(\mu_{\Gamma_{(1,1,0)}}) = h(\mu_{\Gamma_{(0,1,1)}}) = \ln \frac{1+\sqrt{5}}{2}$. *Consequently,* $\Gamma_{(1,1,1)}$, $\Gamma_{(1,1,0)}$ *and* $\Gamma_{(0,1,1)}$ *are the only chaotic parameters regions.*

Proof. We first show that the mosaic patterns produced from $\Gamma_{(1,1,1)}$, $\Gamma_{(1,1,0)}$ and $\Gamma_{(0,1,1)}$ are all stable. Note that the stability condition (1.5) reduces to

$$|a| + 2|\beta| < \frac{1}{r}. \tag{3.3}$$

If $0 < r < \frac{1}{2}$, then Q, see Figure 3.1., is to the left of the a-intercept of the line $a+2\beta = \frac{1}{r}$. Moreover, a direct computation could yield that the point U, see Figure 3.1., lies on the line $a + 2\beta = \frac{1}{r}$. Similarly, the point U' lies on the line $a - 2\beta = \frac{1}{r}$. Thus, the mosaic patterns under consideration are all stable. We illustrate only the cases that $(a,\beta) \in \Gamma_{(1,1,0)}$, and $(a,\beta) \in \Gamma_{(0,1,1)}$. We assign 4 symbols $++, +-, -+$ and $--$ to be $1, 2, 3$ and 4 , respectively. We define i_l and i_r, respectively, to be the left (resp., right) side of the symbol corresponding to i. For instance, let $2 = + -$, then $2_l = +$ and $2_r = -$. We construct a 4×4 transition matrix $A = (a_{i,j})$ as follows.

$$\textit{Set } a_{i,j} = \begin{cases} 1, & \text{if } i_r = j_l \text{ and } [i_l, i_r, j_r] \text{ is a basic mosaic patterns in } B_{(1,1,0)}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Thus the transition matrix with the choice of parameters in $\Gamma_{(1,1,0)}$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} =: A_{(1,1,0)}.$$

Now, the set of $\mu_{\Gamma_{(1,1,0)}}$ has a one-to-one and onto correspondence with the sequence space $\Sigma_{A_{(1,1,0)}}$. Here $\Sigma_{A_{(1,1,0)}} = \{(s_i) : s_i \in \{1,2,3,4\}, (A_{(1,1,0)})_{s_i,s_{i+1}} = 1 \textit{ for all } i\}$. Clearly, the characteristic polynomial for $A_{(1,1,0)}$ is $\lambda^4 - 2\lambda^3 + \lambda^2 - 1 = 0$ or equivalently $(\lambda^2 - \lambda + 1)(\lambda^2 - \lambda - 1) = 0$. It then follows from Theorem 3.2 that $h(\mu_{\Gamma_{(1,1,0)}}) = \ln \frac{1+\sqrt{5}}{2}$. If $(a,\beta) \in \Gamma_{(0,1,1)}$, we will define the corresponding transition matrix $A_{(1,1,0)}$ as

Notation	Parameters' regions	Corresponding patterns
Γ_2^+	$a+z>1-2\beta$, $(1+r\delta)a+z<1+\delta-2(1+r\delta)\beta$, $\beta>0$. or $a+z>1-2(1+r\delta)\beta$, $(1+r\delta)a+z<1+\delta-2\beta$, $\beta<0$	$[+++]_\delta$
Γ_{-2}^-	replacing z by $-z$ in the equations right above.	$[---]_\delta$,
Γ_0^+	$a+z>1+r\delta\beta$, $(1+r\delta)a+z<1+\delta-r\delta\beta$, $\beta>0$. or $a+z>1-r\delta\beta$, $(1+r\delta)a+z<1+\delta+r\delta\beta$, $\beta<0$	$[++-]_\delta, [-++]_\delta$
Γ_0^-	replacing z by $-z$ in the equations right above.	$[+--]_\delta, [--+]_\delta$
Γ_{-2}^+	$a+z>1+2(1+r\delta)\beta$, $(1+r\delta)a+z<1+\delta+2\beta$, $\beta>0$. or $a+z>1+2\beta$, $(1+r\delta)a+z<1+\delta+2(1+r\delta)\beta$, $\beta<0$	$[-+-]_\delta$
Γ_2^-	replacing z by $-z$ in the equations right above.	$[+-+]_\delta$

Table 4.1: .

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} =: A_{(0,1,1)},$$

the characteristic polynomial of $A_{(0,1,1)}$ is $(\lambda^2+\lambda+1)(\lambda^2-\lambda-1)=0$. Thus $h(\mu_{\Gamma_{(0,1,1)}}) = \ln\frac{1+\sqrt{5}}{2}$. $\square$

4 The Effect of the Source Term on Patterns

In this section, we first consider the effect of the source term z on patterns. With the presence of the source term $z \neq 0$, the regions Γ_2^+ and Γ_{-2}^- are no longer identical. Same can be said to the two pairs of regions Γ_0^+ and Γ_0^-, and Γ_2^- and Γ_{-2}^+. Therefore, some new patterns emerge as z moves away from zero.

Proposition 4.1. *Suppose*

$$-1+2|\beta|(1+r\delta) < z < 1-2|\beta|(1+r\delta) \text{ and } 0<r<\frac{1}{2}. \tag{4.1}$$

Then the Table 4.1. *holds true.*

The first two inequalities imply that $|\beta| < \frac{1}{2(1+r\delta)} =: \beta_4$

For fixed $0<r<\frac{1}{2}$ and $\delta>0$ to draw parameters regions in $z-a$ space, we need the following notations.

Notation 4.1. *Denote by* $z = 1-2\beta(1+r\delta)$, $a+z = 1-2\beta$, $(1+r\delta)a+z = 1+\delta-2(1+r\delta)\beta$, $a+z = 1+r\delta\beta$, $(1+r\delta)a+z = 1+\delta-r\delta\beta$, $a+z = 1+2(1+r\delta)\beta$ *and* $(1+r\delta)a+z = 1+\delta+2\beta$ *by* l_0, l_1, l_4, l_2, l_5, l_3 *and* l_6, *respectively. Replacing* z *and* $-z$ *in those equations above, we shall denote the corresponding equations by* r_0, r_1, r_4, r_2, r_5, r_3 *and* r_6, *respectively.*

Notation 4.2. *(i) We shall denote the intersection of the lines* l_i *and* r_j, $i, j = 1, 2, ..6$ *by* $A_{i,j}$. *(ii) We shall denote by the quadrilateral* $A_{i,j}A_{i,k}A_{l,k}A_{l,j} = (l_i, r_k, l_l, r_j) = (i, k, l, j)$. *Here the a-coordinate of* $A_{i,j}$ *is greater than those of* $A_{i,k}$, $A_{l,k}$ *and* $A_{l,j}$. *Note that such tuple is well-defined.*

Let $0 < r < \frac{1}{2}$ and $\delta > 0$ be fixed and $0 < \beta < \frac{(1-r)\delta}{2(1+r\delta)(2+r\delta)} =: \beta_1$. Putting r_i and l_i, $i = 0, 1, 2, ...6$, on $z - a$ plane, we have the Figure 4.1..

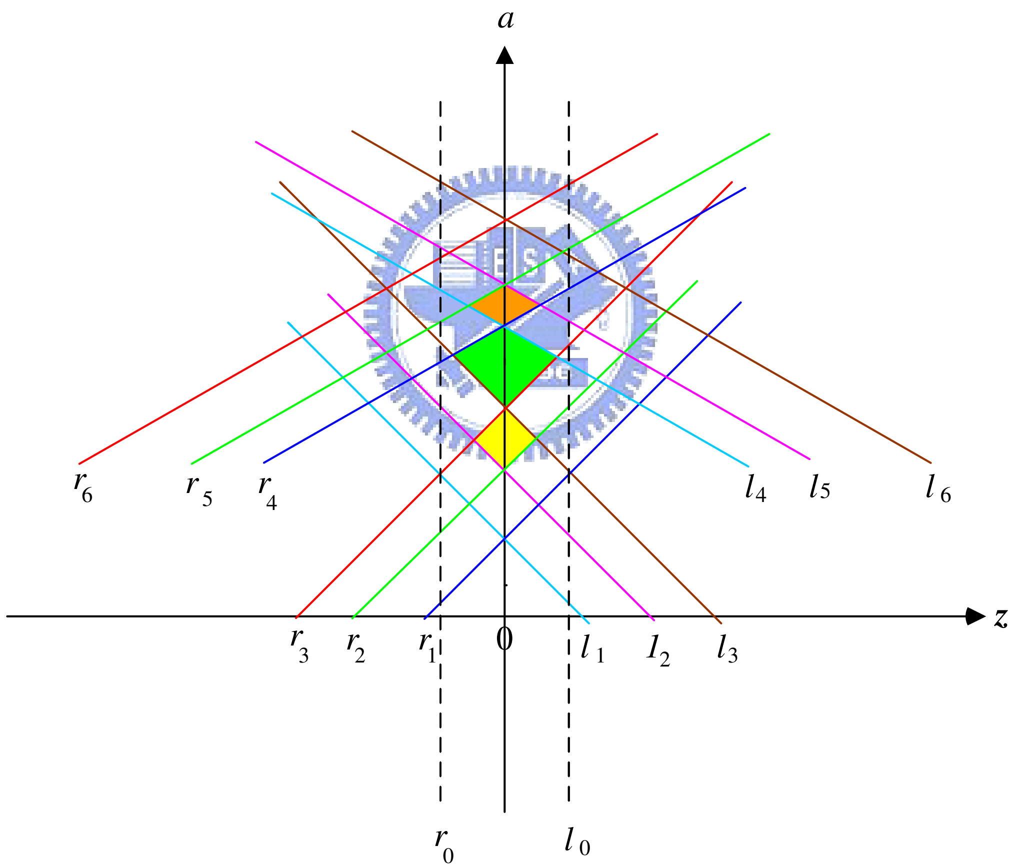

Figure 4.1: Orange region: $(5, 4, 4, 5)$, green region: $(4, 3, 3, 4)$ and yellow region: $(3, 2, 2, 3)$.

Notation 4.3. *Set* $\Lambda_1 = \Gamma_2^+$, $\Lambda_2 = \Gamma_{-2}^-$, $\Lambda_3 = \Gamma_0^+$, $\Lambda_4 = \Gamma_0^-$, $\Lambda_5 = \Gamma_{-2}^+$, $\Lambda_6 = \Gamma_2^-$, *and for* $i_j, j = 1, 2, ...6, \in \{0, 1\}$, *we define* $\Gamma_{(i_1,i_2,i_3,i_4,i_5,i_6)} = \bigcap_{j=1}^{6} R_j \cap \bigwedge_z$, *where*

$$R_j = \begin{cases} \Lambda_j, & \text{if } i_j = 1\,, \\ R^2 - \Lambda_j, & \text{if } i_j = 0\,, \end{cases} \quad \text{and } \bigwedge_z = \{(a, \beta) : |z| < 1 - 2\beta(1 + r\delta)\}.$$

With $z \neq 0$ and a small $\beta > 0$, we see, in the following, that a lot more chaotic parameters regions emerge. The case for $\beta < 0$ is similar and is, thus, omitted.

Theorem 4.1. *Assume that* (4.1) *holds and* r *is sufficiently small. Then the following hold:*
(i) Suppose $0 < \beta < min\{\frac{2(1-r)\delta}{(2+r\delta)(4+5r\delta)}, \frac{2}{6+5r\delta}\} =: min\{\beta_0, \hat{\beta}_0\}$ *and* $0 < \delta < \frac{2}{1-2r}$. *Then all parameters regions in Table* 4.2. *are nonempty and all assertions in Table* 4.2. *hold true.*
(ii) Suppose $min\{\beta_0, \hat{\beta}_0\} < \beta < \beta_1$ *and* $0 < \delta < \frac{2}{1-2r}$. *Then the last two parameters regions* $\Gamma_{(0,1,0,1,1,1)}$ *and* $\Gamma_{(1,0,1,0,1,1)}$ *in Table* 4.2. *are empty, and all other regions are nonempty.*
(iii) Suppose $0 < \beta < min\{\beta_0, \hat{\beta}_0\}$ *and* $\frac{2}{1-2r} < \delta$. *Then the last two parameters regions* $\Gamma_{(0,1,1,1,1,0)}$ *and* $\Gamma_{(1,0,1,1,0,1)}$ *in Table* 4.2. *are empty, and all other regions are nonempty.*
(iv) Suppose $min\{\beta_0, \hat{\beta}_0\} < \beta < \beta_4$ *and* $\frac{2}{1-2r} < \delta$. *Then the last four parameters regions* $\Gamma_{(0,1,0,1,1,1)}$, $\Gamma_{(1,0,1,0,1,1)}$, $\Gamma_{(0,1,1,1,1,0)}$ *and* $\Gamma_{(1,0,1,1,0,1)}$ *in Table* 4.2. *are empty, and all other regions are nonempty.*

Proof. We illustrate only (i). To see the non-emptiness of the parameters regions in Table 4.2., we first check that the z-coordinates of both $A_{4,3}$ and $A_{5,4}$ are smaller than $z = 1 - 2\beta(1 + r\delta)$. A direction computation would yield so provided that $0 < \delta < \frac{2}{1-2r}$ and $0 < \beta < \hat{\beta}_0$. We then need to verify that the intersection A of r_3 and r_4 lies above l_5. We see, via direct computations, that only if $0 < \beta < \beta_0$, then A lies above l_5. Note also that if r is sufficiently small, the stability condition (1.5) is satisfied. The verification of the other assertions in the theorem is then similar to the above and those in Theorem 3.1 and is thus omitted. □

Remark 4.1. *(i) If* $0 < \delta < \frac{2}{1-2r}$ *and* $0 < r < \frac{1}{2}$, *then* $\beta_4 > \beta_1$.
(ii) Note that $2 > \lambda_1 > \lambda_2 > \frac{1+\sqrt{5}}{2}$. *Thus, Table* 4.2. *is arranged in the following way : the higher row the parameters region is placed the more complex its corresponding patterns are.*

(iii) It is clear that the chaotic patterns produced from the regions $\Gamma_{(1,1,1,1,1,1)}$ *and* $\Gamma_{(1,1,1)}$ *are the same. Similarly, the pairs* $\Gamma_{(0,0,1,1,1,1)}$, $\Gamma_{(0,1,1)}$ *and* $\Gamma_{(1,1,1,1,0,0)}$, $\Gamma_{(1,1,0)}$ *generate the*

Parameters region	Exact location in Figure 4.1.	Basic mosaic patterns contained	Spatial Entropy
$\Gamma_{(1,1,1,1,1,1)}$	$(4,3,3,4)\cap\bigwedge_z$	$[+++]_\delta,[---]_\delta,[++-]_\delta,$ $[-++]_\delta,[+--]_\delta,[--+]_\delta,$ $[-+-]_\delta,[+-+]_\delta.$	$\ln 2$
$\Gamma_{(0,1,1,1,1,1)}$	$(5,3,4,4)\cap\bigwedge_z$	$[---]_\delta,[++-]_\delta,[-++]_\delta,$ $[+--]_\delta,[--+]_\delta,$ $[-+-]_\delta,[+-+]_\delta.$	$\ln\lambda_1$
$\Gamma_{(1,0,1,1,1,1)}$	$(4,4,3,5)\cap\bigwedge_z$	$[+++]_\delta,[++-]_\delta,[-++]_\delta,$ $[+--]_\delta,[--+]_\delta,$ $[-+-]_\delta,[+-+]_\delta.$	$\ln\lambda_1$
$\Gamma_{(1,1,1,1,0,1)}$	$(3,3,2,4)\cap\bigwedge_z$	$[+++]_\delta,[---]_\delta,[++-]_\delta,$ $[-++]_\delta,[+--]_\delta,$ $[--+]_\delta,[+-+_\delta].$	$\ln\lambda_2$
$\Gamma_{(1,1,1,1,1,0)}$	$(4,2,3,3)\cap\bigwedge_z$	$[+++]_\delta,[---]_\delta,[++-]_\delta,$ $[-++]_\delta,[+--]_\delta,$ $[--+]_\delta,[-+-]_\delta.$	$\ln\lambda_2$
$\Gamma_{(0,0,1,1,1,1)}$	$(5,4,4,5)\cap\bigwedge_z$	$[++-]_\delta,[-++]_\delta,[+--]_\delta,$ $[--+]_\delta,[-+-]_\delta,[+-+]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
$\Gamma_{(1,1,1,1,0,0)}$	$(3,2,2,3)\cap\bigwedge_z$	$[+++]_\delta,[---]_\delta,[++-]_\delta,$ $[-++]_\delta,[+--]_\delta,[--+]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
$\Gamma_{(0,1,1,1,1,0)}$	$(5,2,4,3)\cap\bigwedge_z$	$[---]_\delta,[++-]_\delta,[-++]_\delta,$ $[+--]_\delta,[--+]_\delta,[-+-]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
$\Gamma_{(1,0,1,1,0,1)}$	$(3,4,2,5)\cap\bigwedge_z$	$[+++]_\delta,[++-]_\delta,[-++]_\delta,$ $[+--]_\delta,[--+]_\delta,[+-+]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
$\Gamma_{(0,1,0,1,1,1)}$	$(6,3,5,4)\cap\bigwedge_z$	$[---]_\delta,[+--]_\delta,[--+]_\delta,$ $[-+-]_\delta,[+-+]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
$\Gamma_{(1,0,1,0,1,1)}$	$(4,5,3,6)\cap\bigwedge_z$	$[+++]_\delta,[++-]_\delta,[-++]_\delta,$ $[-+-]_\delta,[+-+]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
Here λ_1 and λ_2 are the maximal roots of $(\lambda^3-\lambda^2-\lambda-1)=0$ and $(\lambda^3-2\lambda^2+\lambda-1)=0$, respectively.			

Table 4.2: .

exact patterns. Thus, with the presence of the bias term $z \neq 0$, some new chaotic patterns would emerge. Specifically, the patterns whose parameters regions are from $\Gamma_{(0,1,1,1,1,1)}$, $\Gamma_{(1,0,1,1,1,1)}$, $\Gamma_{(1,1,1,1,0,1)}$, $\Gamma_{(1,1,1,1,1,0)}$, $\Gamma_{(0,1,1,1,1,0)}$, $\Gamma_{(1,0,1,1,0,1)}$, $\Gamma_{(0,1,0,1,1,1)}$, and $\Gamma_{(1,0,1,0,1,1)}$ are new and chaotic.

(iv) Note that in Figure 4.1., we have $0 < \beta < \beta_1$. Such condition is to ensure that the β-intercept of l_3 is smaller than that of l_4. We also remark that β_1 is the β coordinate of R in Figure 3.1.. Therefore when $\bar{\beta}$ ($< \beta_1$) is fixed, we see in Figure 3.1. that the line $\beta = \bar{\beta}$ passes through $\Gamma_{(1,1,0)}$, $\Gamma_{(1,1,1)}$ and $\Gamma_{(0,1,1)}$, which corresponds to the line $z = 0$ in Figure 4.1. going through $\Gamma_{(1,1,1,1,0,0)}$, $\Gamma_{(1,1,1,1,1,1)}$ and $\Gamma_{(0,0,1,1,1,1)}$.

(v) In the case that $\frac{(1-r)\delta}{(2+r\delta)(2+3r\delta)} < \beta < \beta_0$, $(6,3,5,4)$ reduces to a triangular $A_{5,4}A_{5,3}A$. Here A is the intersection of lines r_3 and r_4. Likewise, $(4,5,3,6)$ reduces to a triangular too.

(vi) In the case that $\beta_0 < \beta < \beta_1$, $(5,2,4,3)$ and $(3,4,2,5)$ both reduce to a triangular. Moreover, $(6,3,5,4)$ and $(4,5,3,6)$ disappear.

For $\beta_1 < \beta < min\{\frac{(1-r)\delta}{(1+r\delta)(2+r\delta)}, \frac{(1-r)\delta}{2(1+r\delta)^2+r\delta}, \beta_4\} =: min\{\beta_2, \beta_3, \beta_4\}$, we have Figure 4.2. and Table 4.3.

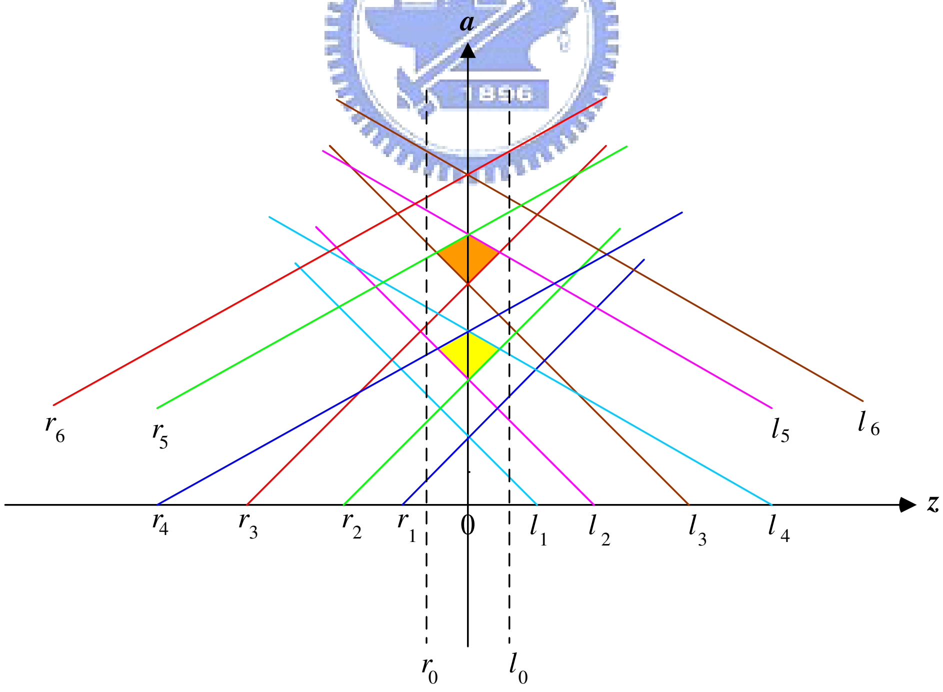

Figure 4.2: Orange region: $(5,3,3,5)$ and yellow region: $(4,2,2,4)$.

Parameters region	Exact location in Figure 4.2.	Basic mosaic patterns contained	Spatial Entropy
$\Gamma_{(0,0,1,1,1,1)}$	$(5,3,3,5)\cap\bigwedge_z$	$[++-]_\delta, [-++]_\delta, [+--]_\delta,$ $[--+]_\delta, [-+-]_\delta, [+-+]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
$\Gamma_{(1,1,1,1,0,0)}$	$(4,2,2,4)\cap\bigwedge_z$	$[+++]_\delta, [---]_\delta, [++-]_\delta,$ $[-++]_\delta, [+--]_\delta, [--+]_\delta.$	$\ln\frac{1+\sqrt{5}}{2}$
$\Gamma_{(0,1,1,1,0,0)}$	$(3,2,4,4)\cap\bigwedge_z$	$[---]_\delta, [++-]_\delta, [-++]_\delta,$ $[+--]_\delta, [--+]_\delta.$	$\ln\lambda_3$
$\Gamma_{(1,0,1,1,0,0)}$	$(4,4,2,3)\cap\bigwedge_z$	$[+++]_\delta, [++-]_\delta, [-++]_\delta,$ $[+--]_\delta, [--+_\delta].$	$\ln\lambda_3$
$\Gamma_{(0,0,1,1,0,1)}$	$(3,3,4,5)\cap\bigwedge_z$	$[++-]_\delta, [-++]_\delta, [+--]_\delta,$ $[--+]_\delta, [+-+]_\delta.$	$\ln\lambda_4$
$\Gamma_{(0,0,1,1,1,0)}$	$(5,4,3,3)\cap\bigwedge_z$	$[++-]_\delta, [-++]_\delta, [+--]_\delta,$ $[--+]_\delta, [-+-]_\delta.$	$\ln\lambda_4$
Here λ_3 and λ_4 are the maximal roots of $\lambda^4-\lambda^3-1=0$ and $\lambda^4-\lambda-1=0$, respectively. Clearly, $\frac{1+\sqrt{5}}{2}>\lambda_3>\lambda_4>1$.			

Table 4.3: .

Theorem 4.2. *Let* (4.1) *hold,* $0<\delta<\frac{2}{1-2r}$ *and* r *be sufficiently small. In the case that* $\beta_1<\beta<min\{\beta_2,\beta_3,\beta_4\}$, *the parameters regions in Table* 4.3. *are nonempty, and all assertions in Table* 4.3. *hold true.*

Remark 4.2. *(i) If* $\beta_1<\beta<min\{\beta_2,\beta_3,\beta_4\}$, *then the a-intercept of* l_3 *is greater than that of* l_4. *We also note that* β_2 *and* β_3 *are the* β*-coordinates of* S *and* T, *respectively. So when* $\beta_1<\beta<min\{\beta_2,\beta_3,\beta_4\}$, *we see from Figure* 3.1. *that* $\Gamma_{(1,1,1)}$ *disappears. Thus, not surprisingly, most of regions in Figure* 4.1. *are destroyed; however, there are some new chaotic parameters regions as opposed to the case that* $0<\beta<\beta_1$ *appear. Specifically, the parameters regions with indexes containing three zeros newly emerge.*

(ii) For $\beta>min\{\beta_2,\beta_3,\beta_4\}$, *most of chaotic regions are destroyed and yield no new chaotic regions. We thus skip the discussion of the case.*

We conclude the thesis with the following remarks.

(i) The antisymmetric template for (1.1) can be similarly done. Moreover, the generalization of the work to two-dimensional CNNs with output function (1.1) and with the symmetric and antisymmetric templates is also straightforward.

(ii) It is of considerable interests to study the defect patterns for (1.1).

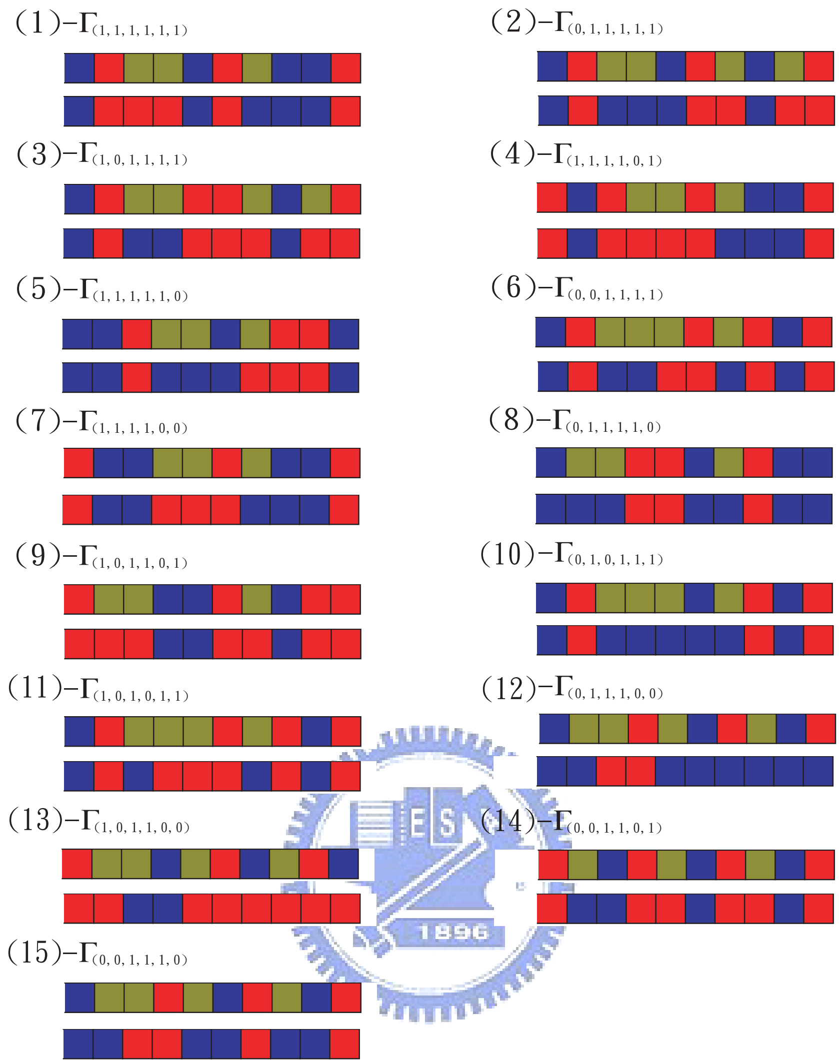

Figure 4.3:

(iii) Figure 4.3. is a collection of a computer simulation with sets of parameters chosen from the parameters regions in Tables 4.2. and 4.3.. Specifically, we set $r = 0.25$ and $\delta = 2$ for all cases. The first eleven cases in Figure 4.3. correspond to the first eleven parameters regions in Table 4.2.. The last four cases in Figure 4.3 correspond to the last four parameters regions in Table 4.3.. Each collection in Figure 4.3. contains two arrays of colors. The first array is the initial outputs. The second array represents the final outputs. If the state x_j of a cell C_j is such that $|x_j| < 1$, then we color it green. If the state x_j of a cell C_j is less than -1 (greater than 1, respectively), then we color it blue (red, respectively). Moreover, the final outputs in each of the collection consist of all basic mosaic patterns allowed in their corresponding parameters region. For instance, the final outputs in (1) consist of all 8 basic mosaic patterns. Likewise, in $(6) - \Gamma_{(0,0,1,1,1,1)}$ and $(12) - \Gamma_{(0,1,1,1,0,0)}$, their corresponding outputs contain 6 and 5 basic mosaic patterns listed in Table 4.2. and 4.3., respectively.

References

[1] Afraimovich, V. S. and Hsu, S. B. [2003] *Lecture on Chaotic Dynamical Systems*, American Mathemtional Society, International Press.

[2] Chang, H. M. and Juang, J. [2004] "Piecewise two-dimensional maps and applications to cellular neural networks," *Int. J. Bifurcation and Chaos*, Vol.14, No.7, 2223-2228.

[3] Chow, S. N., Mallet-Paret, J. and Van Vleck, E. S. [1996a] "Pattern formation and spatial chaos in spatially discrete evolution equatrion," *Rand. Comput. Dyn.* **4**(2 and 3), 109-178

[4] Cheng, C. Y. and Shih, C. W. [2005] *"Pattern formations and spatial entropy for spatially discrete reaction diffusion equations"*, to appear in *Physica D.*

[5] Chua, L. O. and Yang, L. [1988a] "Cellular neural networks : Theory," *IEEE Trans. Circuits Syst.* **35**, 1257-1272.

[6] Chua, L. O. and Yang, L. [1988b] "Cellular neural networks : Applications," *IEEE Trans. Circuits Syst.* **35**, 1273-1290.

[7] Chua, L. O. [1998] CNN : *A Paradigm for Complexity*, World Scientific Series on Nonlinear Science, Series A, Vol. 31 (World Scientific, Singapore).

[8] Hsu, C. H. [2000] "Smale horseshoe of cellular neural networks," *Int. J. Bifurcation and Chaos*, Vol.10, No.9, 2119-2127.

[9] Juang, J. and Lin, S. S. [2000] "Cellular neural networks : Mosaic pattern and spatial chaos," *SIAM J. Appl. Math.* **60**, 891-915.

[10] Robinson, C. [1995] *Dynamical Systems : Stability, Symbolic Dynamics and Chaos*, CRC Press, Boca Raton, FL.

[11] Special Issue, [1995] "Nonlinear waves, patterns and spatio-temporal chaos in dynamic arrays," *IEEE Trans. Circuits Syst.* I **42**(10).

[12] Thiran, P. [1997] *Dynamics of Self-Organization of Locally Coupled Neural Networks*, Presses Polytechniques et Universitaires Romandes, Lausanne.