1 Introduction

The study of traveling wave and standing wave solutions for partial differential equations and lattice dynamical systems has drawn considerable attention in the past decades. For instance, the existence and stability of such solutions for lattice dynamical systems has been much studied by many authors. (see, e.g., [3], [6], [7], [11]-[14], [21], [26]-[27] and [31]-[34], and many references cited there in.) On the other hand, the study of the discrete (in time) analog of such systems has been only focused on diffusion Huxley-Nagumo (see, e.g., [1], [2].) equation. In this thesis, we study stationary traveling wave solutions of a discrete analog of one-dimensional Cellular Neural Networks(CNNs). The dynamics of one-dimensional CNNs (see, e.g., [3]-[4], [7]-[9], [15]-[16], [30] and the references cited there in.) is of the form

$$\frac{dx_i}{dt} = -\overline{k}\, x_i + \overline{z} + \overline{\alpha} f(x_{i-1}) + \overline{a} f(x_i) + \overline{\beta} f(x_{i+1}), \quad i \in \mathbb{Z}. \tag{1-1a}$$

Here f is a piecewise linear output function defined by

$$f(x) = \begin{cases} rx + 1 - r, & \text{if } x \geq 1\,, \\ x\,, & \text{if } |x| \leq 1, \\ rx - 1 + r, & \text{if } x \leq -1\,, \end{cases} \tag{1-1b}$$

where r is a nonegative constant, $\overline{k}$ is positive. The quantity $\overline{z}$ is called threshold or biased term. The constants $\overline{\alpha}$, $\overline{a}$ and $\overline{\beta}$ are the interaction weights between neighboring cells.

Discretizing equation (1-1a) by Euler method, we have the discrete-time CNNs of the form

$$x_i(t+1) = kx_i(t) + z + \alpha f(x_{i-1}(t)) + af(x_i(t)) + \beta f(x_{i+1}(t)). \tag{1-2}$$

Here $k = 1 - \triangle t\, \overline{k}$, $z = \triangle t\, \overline{z}$, $\alpha = \triangle t\, \overline{\alpha}$, $a = \triangle t\, \overline{a}$, $\beta = \triangle t\, \overline{\beta}$, and $\triangle t$ is a step size.

We will refer to the solutions of system (1-2) of the form $x_i(n) = \varphi(i+cn)$, $c \in$

$\mathbb{Z}$ being a wave speed, as stationary waves by analogy with the continuous case. Apparently, the function $\varphi(i+cn)$ must satisfy the equation

$$\varphi(i+c\,n+c) = k\,\varphi(i+cn)+z+\alpha\,f(\varphi(i-1+cn))+a\,f(\varphi(i+cn))+\beta\,f(\varphi(i+1+cn)). \tag{1-3}$$

Setting the " iteration index " $j = i + cn$, we see that (1-3) becomes

$$y_{j+c} = k\,y_j + \alpha\,f(y_{j-1}) + a\,f(y_j) + \beta\,f(y_{j+1}) + z. \tag{1-4}$$

where $y_j = \varphi(i + cn) = \varphi(j)$. For $c > 1$, equation (1-4) induces a (c+1)-dimensional map F of the form

$$T(x_1, x_2, \cdots, x_{c+1}) = (x_2, \cdots, x_{c+1}, k\,x_2 + \alpha\,f(x_1) + a\,f(x_2) + \beta\,f(x_3) + z). \tag{1-5}$$

For $c = 1$, equation (1-4) becomes

$$x_{j+1} := g(y_{j+1}) := y_{j+1} - \beta f(y_{j+1}) = k\,y_j + a\,f(y_j) + \alpha\,f(y_{j-1}) + z. \tag{1-6a}$$

or

$$x_{j-1} := f(y_{j-1}) = \frac{-k}{\alpha}\,y_j - \frac{a}{\alpha}\,x_j + \frac{1}{\alpha}\,y_{j+1} - \frac{\beta}{\alpha}\,x_{j+1} - \frac{z}{\alpha}. \tag{1-6b}$$

If we assume momentarily that g (resp. f) is invertible, then equation (1-6a) (resp., (1-6b)) can be represented by a 2-dimensional map F (resp., B) of the form

$$F(x, y) = (y\,,\ f_1(y) + f_2(x) + z). \tag{1-7a}$$

$$(\text{ resp.},\ B(x, y) = (y\,,\ g_1(y) + g_2(x) - \frac{z}{\alpha}).\ ) \tag{1-7b}$$

Here ,

$$f_1(x) = k\,g^{-1}(x) + a\,f(g^{-1}(x)), \tag{1-8a}$$

$$(\text{ resp.},\ g_1(x) = -\frac{k}{\alpha}\,f^{-1}(x) - \frac{a}{\alpha}\,x\ ), \tag{1-8b}$$

and ,

$$f_2(x) = \alpha\,f(g^{-1}(x)). \tag{1-9a}$$

$$(\text{ resp.},\ g_2(x) = \frac{1}{\alpha}\,f^{-1}(x) - \frac{\beta}{\alpha}\,x\ ). \tag{1-9b}$$

The map F (resp., B) generates the forward (resp., backward) wave solutions of (1-2). Assuming$(1-\beta r)(1-\beta) > 0$, we see that g is invertible. After some calculations, we obtain that, for $1-\beta > 0$,

$$f_1(x) = \begin{cases} \frac{k+ar}{1-\beta r}\, x + \frac{(1-r)(k\beta+a)}{1-\beta r}\,, & \text{if } x \geq 1-\beta\,, \\ \frac{k+a}{1-\beta}\, x\,, & \text{if } |x| \leq 1-\beta, \\ \frac{k+ar}{1-\beta r}\, x - \frac{(1-r)(k\beta+a)}{1-\beta r}\,, & \text{if } x \leq -1+\beta\,, \end{cases} \tag{1-10a}$$

and,

$$f_2(x) = \begin{cases} \frac{\alpha r}{1-\beta r}\, x + \frac{\alpha\,(1-r)}{1-\beta r}\,, & \text{if } x \geq 1-\beta\,, \\ \frac{\alpha}{1-\beta}\, x\,, & \text{if } |x| \leq 1-\beta, \\ \frac{\alpha r}{1-\beta r}\, x - \frac{\alpha\,(1-r)}{1-\beta r}\,, & \text{if } x \leq -1+\beta\,, \end{cases} \tag{1-10b}$$

for $1-\beta < 0$,

$$f_1(x) = \begin{cases} \frac{k+ar}{1-\beta r}\, x - \frac{(1-r)(k\beta+a)}{1-\beta r}\,, & \text{if } x \geq -1+\beta\,, \\ \frac{k+a}{1-\beta}\, x\,, & \text{if } |x| \leq -1+\beta, \\ \frac{k+ar}{1-\beta r}\, x + \frac{(1-r)(k\beta+a)}{1-\beta r}, & \text{if } x \leq 1-\beta\,, \end{cases} \tag{1-10c}$$

and,

$$f_2(x) = \begin{cases} \frac{\alpha r}{1-\beta r}\, x + \frac{\alpha(r-1)}{1-\beta r}, & \text{if } x \geq -1+\beta\,, \\ \frac{\alpha}{1-\beta} x\,, & \text{if } |x| \leq -1+\beta, \\ \frac{\alpha r}{1-\beta r}\, x - \frac{\alpha(r-1)}{1-\beta r}\,, & \text{if } x \leq 1-\beta\,. \end{cases} \tag{1-10d}$$

Replacing x by $(1-\beta)\, x$ or $(\beta-1)\, x$ depending upon the sign of $1-\beta$, we have that

$$f_1(x) = \begin{cases} a_1\, x + a_{10} - a_1\,, & \text{if } x \geq 1\,, \\ a_{10}\, x\,, & \text{if } |x| \leq 1, \\ a_1\, x - a_{10} + a_1\,, & \text{if } x \leq -1\,, \end{cases} \tag{1-11a}$$

and,

$$f_2(x) = \begin{cases} a_2\, x + a_{20} - a_2\,, & \text{if } x \geq 1\,, \\ a_{20}\, x\,, & \text{if } |x| \leq 1, \\ a_2\, x - a_{20} + a_2\,, & \text{if } x \leq -1\,. \end{cases} \tag{1-11b}$$

Here ,

$$a_1 = \begin{cases} \frac{(1-\beta)(k+ar)}{1-\beta r}\,, & \text{if } 1-\beta > 0\,, \\ \frac{(\beta-1)(k+ar)}{1-\beta r}\,, & \text{if } 1-\beta < 0. \end{cases}$$

$$a_{10} = \begin{cases} k+a\,, & \text{if } 1-\beta > 0\,, \\ -(k+a)\,, & \text{if } 1-\beta < 0. \end{cases}$$

$$a_2 = \begin{cases} \frac{(1-\beta)\alpha r}{1-\beta r}\,, & \text{if } 1-\beta > 0\,, \\ \frac{(\beta-1)\alpha r}{1-\beta r}\,, & \text{if } 1-\beta < 0. \end{cases}$$

and ,

$$a_{20} = \begin{cases} \alpha\,, & \text{if } 1-\beta > 0\,, \\ -\alpha\,, & \text{if } 1-\beta < 0. \end{cases}$$

Assuming $r > 0$, which in turn guarantee the invertiblity of f , we have that

$$g_1(x) = \begin{cases} b_1\,x + b_{10} - b_1, & \text{if } x \geq 1\,, \\ b_{10}\,x\,, & \text{if } |x| \leq 1, \\ b_1\,x - b_{10} + b_1, & \text{if } x \leq -1\,, \end{cases} \tag{1-12a}$$

and ,

$$g_2(x) = \begin{cases} b_2\,x + b_{20} - b_2, & \text{if } x \geq 1\,, \\ b_{20}\,x\,, & \text{if } |x| \leq 1, \\ b_2\,x - b_{20} + b_2, & \text{if } x \leq -1\,, \end{cases} \tag{1-12b}$$

Here $b_1 = \frac{-(k+ra)}{r\alpha}$, $b_{10} = \frac{-(k+a)}{\alpha}$, $b_2 = \frac{1-\beta r}{r\alpha}$, $b_{20} = \frac{1-\beta}{\alpha}$.

To consider various possibilities of the graphs of $f_i(x)$ and $g_i(x)$, $i = 1,\ 2$, we need the following notions.

Given a piecewise function f in the form of $f_1(x)$, then $f(x)$ is said to be of Type (I), Type (II), Type (III) and Type (IV), respectively, if $a_1 < 0$ and $a_{10} > 0$, $a_1 > 0$ and $a_{10} < 0$, a_1 , $a_{10} > 0$ and a_1 , $a_{10} < 0$. See Figure 1.1

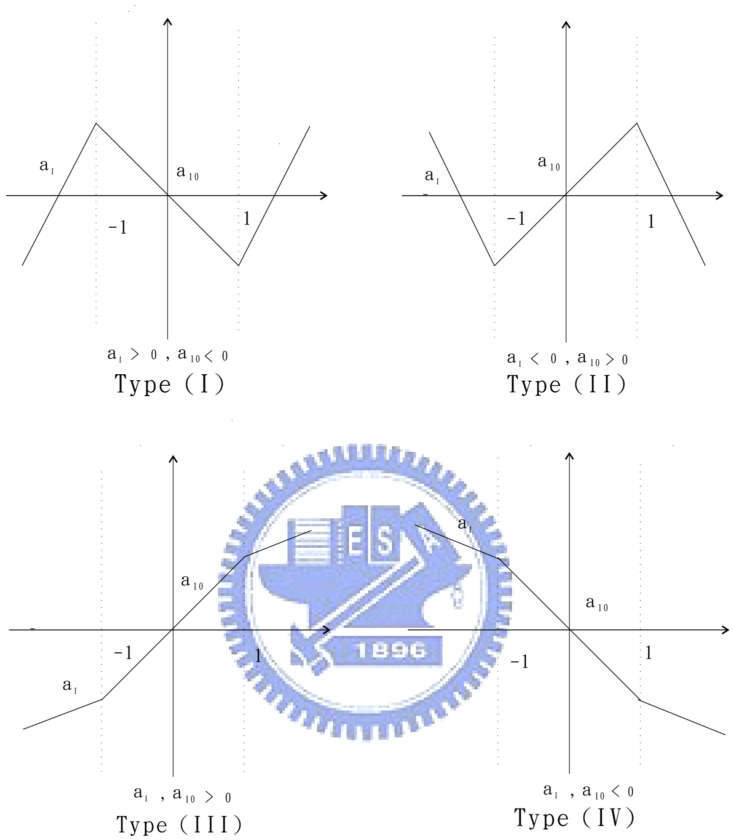

Figure 1.1:

To see $f_i(x)$ and $g_i(x)$, $i = 1\,2$, are which type of functions, we then group the parameters as follows.

(I) Dividing β into 2 parts: (i) $1 > \beta$ (ii) $1 < \beta$

(II) Dividing k into 4 parts (see figure 1.2):

$$\text{(i)}\;\; k > -ar\,, a > 0 \;\;(\text{or } k > -a\,, a < 0)\,,$$

$$\text{(ii)}\;\; k < -a\,, a > 0 \;\;(\text{or } k < -ar\,, a < 0)\,,$$

$$\text{(iii)}\;\; -a < k < -ar < 0 \;\;,$$

$$\text{(iv)}\;\; 0 < -ar < k < -a \;\;.$$

(III) Dividing α into 2 parts:(i) $\alpha > 0$ (ii) $\alpha < 0$

(IV) Dividing β into 3 parts: (i) $\beta > \frac{1}{r}$ (ii) $1 < \beta < \frac{1}{r}$ (iii) $\beta < 1$

Figure 1.2:

For applications purpose in CNNs, r is either zero or a small positive constant. Thus, $-ar > -a$ when $a > 0$. Likewise $-ar < -a$ when $a < 0$. We also note that to insure the existence of the forward map F, we must have $(1-\beta)(1-\beta\, r) > 0$.

We are now ready to give conditions on parameters for which the type of functions $f_i(x)$ and $g_i(x)$, $i = 1\,2$, are characterized.

Cases of forward map F	Function type		Conditions satisfied
	$f_1(x)$	$f_2(x)$	
$F1.$	III	III	(I-i)+(II-i)+(III-i) or (I-ii)+(II-ii)+(III-ii)
$F2.$	III	IV	(I-i)+(II-i)+(III-ii) or (I-ii)+(II-ii)+(III-i)
$F3.$	IV	III	(I-i)+(II-ii)+(III-i) or (I-ii)+(II-i)+(III-ii)
$F4.$	IV	IV	(I-i)+(II-ii)+(III-ii) or (I-ii)+(II-i)+(III-i)
$F5.$	II	III	(I-i)+(II-iii)+(III-i) or (I-ii)+(II-iv)+(III-ii)
$F6.$	II	IV	(I-i)+(II-iii)+(III-ii) or (I-ii)+(II-iv)+(III-i)
$F7.$	I	III	(I-i)+(II-iv)+(III-i) or (I-ii)+(II-iii)+(III-ii)
$F8.$	I	IV	(I-i)+(II-iv)+(III-ii) or (I-ii)+(II-iii)+(III-i)

Table 1.1:

Case of backward map B	Function type		Conditions satisfied
	$g_1(x)$	$g_2(x)$	
$B1.$	III	III	(II-iv)+(III-i)+(IV-iii) or (II-i)+(III-ii)+(IV-i)
$B2.$	III	IV	(II-iv)+(III-i)+(IV-i) or(II-i)+(III-ii)+(IV-iii)
$B3.$	IV	III	(II-i)+(III-i)+(IV-iii) or (II-iv)+(III-ii)+(IV-i)
$B4.$	IV	IV	(II-i)+(III-i)+(IV-i) or (II-iv)+(III-ii)+(IV-iii)
$B5.$	III	I	(II-iv)+(IV-ii)
$B6.$	III	II	(II-i)+(III-ii)+(IV-ii)
$B7.$	IV	I	(II-i)+(IV-ii)
$B8.$	IV	II	(II-iv)+(III-ii)+(IV-ii)
$B9.$	I	III	(II-ii)+(III-i)+(IV-iii) or (II-iii)+(III-ii)+(IV-i)
$B10.$	I	IV	(II-ii)+(III-i)+(IV-i) or (II-iii)+(III-ii)+(IV-iii)
$B11.$	II	III	(II-iii)+(III-i)+(IV-iii) or (II-ii)+(III-ii)+(IV-i)
$B12.$	II	IV	(II-iii)+(III-i)+(IV-i) or (II-ii)+(III-ii)+(IV-iii)
$B13.$	I	I	(II-ii)+(IV-ii)
$B14.$	I	II	(II-iii)+(III-ii)+(IV-ii)
$B15.$	II	I	(II-iii)+(IV-ii)
$B16.$	II	II	(II-ii)+(III-ii)+(IV-ii)

Table 1.2:

If $f_i(x)$ and $g_i(x)$, $i = 1\,,\,2$ are monotonic, such as the cases $F1$-$F4$ and $B1$-$B4$, then the forward map F and the backward map B generate no chaotic dynamics. All other cases may produce chaotic dynamics. To generate the kind of chaotic dynamics by the presence of a snap-back repeller, we need consider the noninvertible maps, such as $B5$-$B7$ and $B13$-$B16$. In the thesis, we will only consider $B15$.

In 1998, Chen et al. [6], found an error in Marotto's paper [25]. The problem is that the existence of an expanding fixed point z of a map F does not necessarily imply that F is expanding in $B_r(z)$, the ball of radius r with center at z. Subsequent efforts (see e.g., [6], [23]-[24].) in fixing the problem all have some discrepancies. One of the problems is that they only give conditions for which F is expanding "locally". In this thesis, we give a sufficient condition so that F is "globally" expanding. This, in turn, gives more satisfying definitions of a snap-back repeller. We then use those results to show the existence of chaotic backward traveling waves in a discrete time analogy of one-dimensional Cellular Neural Networks (CNNs).

We conclude this introduction section by mentioning that in Section 2, we will review the problems in the early definitions of a snap-back repeller. Moreover, we will point out what would be the more satisfying definitions of snap-back repeller. The sufficient conditions under which F is "globally" expanding are recorded in Section 2 as well. Section 3 contains the applications of those results to a discrete time analogy of CNNs.

2 Snap Back Repellers

In 1975, Li and Yorke [22] proved a celebrated result "period three implies chaos". This result plays an important role in predicting and analyzing one-dimensional chaotic systems. Motivated by Li-Yorke's work, Marotto[25] generalized such notion of chaos to higher-dimensional discrete dynamical systems. Specifically, he proved that Snap-Back repellers imply chaos in $\mathbb{R}^n$. This theorem was widely applied ever since. However, in 1998, Chen et al., [6] found that there is an error in Marotto's paper[25]. Specifically, let (**A**) and (**B**) be as follows.

(**A**): All eigenvalues of the Jacobian $DF(x)$, where $x \in B_r(z)$, $r > 0$ and z is a fixed point of F, are greater than 1 in norm.
(**B**): There exists some $s > 1$ such that any $x \neq z \in B_r(z)$, $\|F(x) - z\| > s\|x - z\|$.

A fixed point z of F satisfying (**A**) is called an expanding fixed point of F. A map F satisfying (**B**) is said to be expanding in $B_r(z)$.

The problem is that the existence of an expanding fixed point z of a high-dimensional map F in $B_r(z)$ does not necessarily guarantee that F is expanding in $B_r(z)$. Even in the case that F is linear, (**A**) does not necessarily imply (**B**). See Figure 2.1 (Fig. 1 of [23]). Consequently, for $x \in B_r(z)$, x does not necessarily lie on the local unstable manifold $W^u_{loc}(z)$. To fix such problem, Chen et al., imported a new norm different from the Euclidean norm to guarantee the map F's expansibility in the neighborhood of it's fixed point.

However, as pointed out by Li and Chen [24], and Lin, Ruan and Zhao [23] point that the incorporation of a new norm by Chen et al., into Marotto Theorem does not close the gap of the proof. This is because the new norm depends on the points $x \in B_r(z)$ and the map F. Thus, it is unclear how using such new norm can be used to prove the assertion in (**B**) when F is nonlinear. Nevertheless, Chen et al., also gave a modified definition of

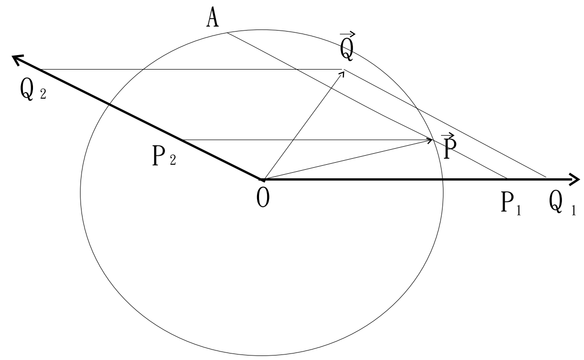

Figure 2.1:

a snap-back repeller, which refined the Marotto's Theorem in the spirit of Devaney's Theorem [10]. Their proof seems to be correct. (see also Li and Chen [24] and Lin, Ruan and Zhao [23]) We next describe their definition and results.

Chen's Definition. $F : \mathbb{R}^N \to \mathbb{R}^N$. *Let z be a fixed point of F such that all of the eigenvalues of $DF(z)$ have absolute values larger than 1. We say that z is a snap-back repeller if there exists a point x_0 in $W^u_{loc}(z)$, the local unstable set of z, and some integer m, such that $F^m(x_0) = z$ and $detDF^m(x_0) \neq 0$.*

Chen-Hsu-Zhou Theorem. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be C^1, and z be a snap-back repeller of F. Then for each neighborhood U of z, there is an integer $m > 0$ such that F^m has a hyperbolic invariant subset in U on which F^m is topologically conjugate to the shift map on the binary symbol space $\sum_2$.*

We remark that Chen's definition of a snap-back repeller is a special case of a transverse homoclinic point. Consequently, their theorem above is a direct consequence of the results induced by the presence of a transverse homoclinic point. (see, e.q., Theorem 4.5 of [29].) As an effort to prove the existence of chaos in the sense of Marotto, Lin, Ruan and Zhao [23], proposed another modified definition of snap-back repeller to ensure chaos in the sense of Marotto. However, as pointed out by Li and Chen[24], the proof of their corresponding theorem is incorrect. What is at fault is that they used a differential mean value theorem, which generally does not exist for high dimensional vector-valued functions. Li and Chen[24] then gave an improved version of the Marotto theorem.

Li and Chen's Definition. *A fix point z of system $x_{k+1} = F(x_k), k = 0, 1, 2 \cdots$ is called a snap-back repeller if*
(i) $F(x)$ is continuously differentiable in $B_r(z)$;
(ii) all eigenvalues of $(DF(z))^T DF(z)$ are greater than 1;
(iii) there exists a point $x_0 \in B_r(z)$ with $x_0 \neq z$ such that $F^m(x_0) = z$, and $detDF^m(x_0) \neq 0$ for some positive integer m.

We next record a Lemma of Li and Chen [24], which showed the "local" expansibility of F.

Lemma 2.1. *(Lemma 5 of [24]) Suppose that z is a fixed point of system $x_{k+1} = F(x_k), k = 0, 1, 2 \cdots$ and F is continuously differentiable in some closed ball $B_r(z)$. Also, assume, that, all eigenvalues of $(DF(z))^T DF(z)$ are larger than 1. Then, there exist some $s > 1$ and $r' \in (0\,,\,r]$ such that*
(i) $\| F(x) - F(y) \| > s \| x - y \|$ for $\forall x \neq z \in B_{r'}(z)$;
(ii) all eigenvalues of $(DF(z))^T DF(z)$ exceed 1 for all $x \in B_{r'}(z)$.

Unfortunately, Li and Chen's Definition of a snap-back repeller still contains some discrepancies. Specifically, they proved, though correctly, in the lemma above that F is expanding in a small neighborhood, $B_{r'}(z)$. However, for $x_0 \in B_r(z)$ with $r \geq r'$, there is no guarantee that $F^{-k}(x_0)$ is to be in $B_{r'}(z)$, for some $k \in \mathbb{N}$. In the original paper of Marotto, the following alternative definition of a snap-back repeller was also given.

Definition 2.1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be continuous and z be a fixed point of F. We say that z is a snap-back repeller if there exists a sequence of compact sets $\{B_k\}_{k=-\infty}^{m}$ (homeomorphic to the unit ball in $\mathbb{R}^N$) which satisfy: (a) $B_k \to \{z\}$ as $k \to -\infty$; (b) $F(B_k) = B_{k+1}$; (c) F is 1-1 in B_k; (d) $B_k \cap B_m = \emptyset$ for $1 \leq k < m$; and (e) $z \in B_m^0$, the interior of B_m.*

In the original proof of Marotto's chaos, the property that F is expanding in $B_r(z)$ was used to show the existence of such sequence of compact sets. Thus, if one makes such existence of a sequence of compact sets as the definition of a snap-back repeller, then the existence of Marotto's chaos holds.

In light of the comment above, we will also define a snap-back repeller as follows

Definition 2.2. *Let $z \in \mathbb{R}$ be a fixed point of F. We say that z is a snap-back repeller if*
(i) F is expanding in $B_r(z)$, for some $r > 0$;
(ii) There exists a point $x_0 \in B_r(z)$ with $x_0 \neq z$, $F^m(x_0) = z$ and $detDF^m(x_0) \neq 0$ for some positive integer m.

For such definitions (Definition 2.1 or Definition 2.2) of snap-back repellers, the following notion of Marotto's chaos, indeed, can be achieved. Thus, from here on, when we say a point z is a snap-back repeller it means z satisfies either Definition 2.1 or Definition 2.2.

Theorem 2.1. *(Marotto's chaos) Suppose $F : \mathbb{R}^N \to \mathbb{R}^N$, and z is a snap-back repeller, defined as in Definition 2.1 or Definition 2.2. Then the map F is chaotic in the sense of Li-Yorke:*
(i) There is a positive integer N such that for each integer $p \geq N$, F has a point of period p .
(ii) There is a "scrambled set" of F, i.e., an uncountable set S containing no periodic points of F such that

(b_1) $F(S) \subset S$,

(b_2) for every $X_S, Y_S \in S$ with $X_S \neq Y_S$,

$$\lim_{k\to\infty} sup \parallel F^k(X_S) - F^k(Y_S) \parallel > 0\,.$$

(b_3) for every $X_S \in S$ and any periodic point Y_{per} of F,

$$\lim_{k\to\infty} sup \parallel F^k(X_S) - F^k(Y_{per}) \parallel > 0\,,$$

(iii) There is an uncountable subset S_0 of S such that for every $X_{S_0}, Y_{S_0} \in S_0$:

$$\lim_{k\to\infty} sup \parallel F^k(X_{S_0}) - F^k(Y_{S_0}) \parallel = 0\,.$$

In the following, we will give sufficient conditions for which the "global" expansibility of a map can be obtained. Thus, the verification of the existence of a snap-back repeller should be made more friendly.

Theorem 2.2. *(i) Let $F = (f_1, f_2, \cdots, f_n)$ be a smooth vector-valued function from $\mathbb{R}^N \to \mathbb{R}^N$, and z be a fixed point of F. Suppose $DF(z)$ is a normal matrix. Let α and β be defined as*

$$\alpha = \min_{1\le i\le n} \mid \lambda_i \mid,$$
$$\beta = \max_{1\le i\le n} \max_{x\in B_r(z)} \max_{1\le j\le n} \mid \beta_{i,j}(x) \mid,$$

where $\lambda_i, i = 1, \cdots, n$, are eigenvalues of $F(x)$ and $\beta_{i,j}(x)$, $j = 1, 2, \cdots, n$, are eigenvalues of Hessian matrices $H_{f_i}(x) = (\partial_k\partial_l f_i(x))_{k\times l}$ and $B_r(z)$ is a closed ball with center at z and radius $r > 0$. If $\alpha - \frac{\beta}{2}r > 1$, then F is expanding in $B_r(z)$.
(ii) Let F be linear all eigenvalues of $DF(0)$ have absolute values larger than 1, then F is expanding in $\mathbb{R}^N$ with respect to a certain operator matrix norm.

Proof. (i) For $y \in B_r(z)$, we have, see e.g., 3.3.11 of [28] in p.80, that

$$F(y) - z = DF(z)(y-z) + \int_0^1 (1-t)F''(z+t(y-z))(y-z)(y-z)dt. \quad (2\text{-}1)$$

We next estimate the first term on the right hand side of (3-1),

$$\parallel DF(z)(y-z) \parallel = \parallel T^{-1}\Lambda T(y-z) \parallel = \parallel \Lambda T(y-z) \parallel \ge \alpha \parallel y - z \parallel, \quad (2\text{-}2)$$

where T is a unitary matrix and

$$\Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

Since Hessian matrices $H_{f_i}(x)$ is symmetric, for all x , $y \in B_r(z)$, we have

$$| \, y^T \, H_{f_i}(x) \, y \, | \leq \beta \; \| \, y \, \|^2 \leq \beta r \, \| \, y \, \| \tag{2-3}$$

Using (2-2) and (2-3) and the fact that
$[F''(x) \, h \, k]^T = (k^T \, H_{f_1(x)} \, h, k^T \, H_{f_2(x)} \, h, \cdots, k^T \, H_{f_n(x)} \, h)$, we see that

$$\| \, F(y) - z \, \| \geq (\alpha - \frac{\beta}{2} r) \, \| \, y - z \, \| \, .$$

Thus, F is expanding in $B_r(z)$.
(ii) Let $\lambda = \min_{1 \leq i \leq n} \{ \, | \, \lambda_i \, | : \lambda_i$ is an eigenvalue of $DF(0) \, \}$. It follows from, see e.g., p.12 of [19], that for every $\epsilon > 0$ there is an operator matrix norm, denoted by $\| \cdot \|_\epsilon$, for which

$$\| \, (DF(0))^{-1} \, \|_\epsilon \leq \frac{1}{\lambda} + \epsilon =: \lambda_\epsilon$$

Choose ϵ sufficiently small so that $\lambda_\epsilon < 1$. Consequently, $\| \, DF(0) \, \|_\epsilon \geq \frac{1}{\lambda_\epsilon} > 1$. Thus, F is expanding in $\mathbb{R}^N$ with respect to $\| \cdot \|_\epsilon$. We just completed the proof of the theorem. □

Remark 2.1. *The second assertion of the theorem was first appeared in [6].*

3 Chaotic Backward Map

In this section, we consider the backward map B with $1 < \beta < \frac{1}{r}$, $-\frac{k}{r} > a > -k > 0$, and $\alpha > 0$. (see $B15$.) Under the circumstances,

$$b_1 < 0,\ b_{10} > 0,\ b_2 > 0,\ \textit{and}\ b_{20} < 0. \tag{3-1}$$

We denote by $\Omega_1, \Omega_0, \Omega_{-1}, \Omega_{1,-1}$ and $\Omega_{-1,1}$ the regions $\Omega_1 = \{(x,y) : x, y \geq 1\}$, $\Omega_0 = \{(x,y) : -1 \leq x, y \leq 1\}$, $\Omega_{-1} = \{(x,y) : x, y \leq -1\}$, $\Omega_{1,-1} = \{(x,y) : x \geq 1, y \leq -1\}$, and $\Omega_{-1,1} = \{(x,y) : x \leq -1, y \geq 1\}$, respectively.

Lemma 3.1. *Suppose (3-1) holds. Let $b_1 + b_2 > 1$, (resp., $b_1 + b_2 < 1$) and $-1+b_{10}+b_{20} < c < 1-b_{10}-b_{20}$(resp., $1-b_{10}-b_{20} < c < -1+b_{10}+b_{20}$), then the map B has exactly three fixed points $(\overline{x}_1, \overline{x}_1)$, $(\overline{x}_0, \overline{x}_0)$ and $(\overline{x}_{-1}, \overline{x}_{-1})$, in Ω_1, Ω_0, Ω_{-1}, respectively. Here*

$$\overline{x}_1 = \frac{b_{10} + b_{20} - b_1 - b_2 + c}{1 - b_1 - b_2},\ \overline{x}_0 = \frac{c}{1 - b_{10} - b_{20}}\ ,\ \overline{x}_{-1} = \frac{-b_{10} - b_{20} + b_1 + b_2 + c}{1 - b_1 - b_2}. \tag{3-2}$$

Lemma 3.2. *Suppose the first set of assumptions in Lemma 3.1 holds. Then $(\overline{x}_1, \overline{x}_1)$ and $(\overline{x}_{-1}, \overline{x}_{-1})$ are repelling fixed points.*

Proof. It is obvious that $DB((\overline{x}_{\pm 1}, \overline{x}_{\pm 1})) = \begin{bmatrix} 0 & 1 \\ b_2 & b_1 \end{bmatrix}$. The eigenvalue of $DB((\overline{x}_{\pm 1}, \overline{x}_{\pm 1}))$

are $\frac{b_1 + \sqrt{b_1^2 + 4b_2}}{2}$ and $\frac{b_1 - \sqrt{b_1^2 + 4b_2}}{2}$. Moreover, $\frac{b_1 + \sqrt{b_1^2 + 4b_2}}{2} > 1$ provided that $b_1 + b_2 > 1$. We thus complete the proof of lemma. □

We are next to find a point $p = (x_0, y_0)$ for which $B(P) \in \Omega_{1,-1}$, $B^2(P) \in \Omega_{-1,1}$, $B^3(P) = (\overline{x}_1, \overline{x}_1)$. To this end, we first compute a pre-image $q = (q_1, q_2)$ of $(\overline{x}_1, \overline{x}_1)$ for which q lies in $\Omega_{-1,1}$. Clearly, $q_2 = \overline{x}_1$ and q_1 must satisfy equation $g_1(\overline{x}_1) + g_2(q_1) + c = \overline{x}_1$, or equivalently,

$$b_2 q_1 = (1 - b_1)\overline{x}_1 - b_{10} + b_{20} + b_1 - b_2 - c = b_2 \overline{x}_1 + 2b_{20} - 2b_2.$$

Thus,

$$q_1 = \overline{x}_1 + 2\frac{b_{20}}{b_2} - 2, \ \ and \ q_2 = \overline{x}_1. \tag{3-3}$$

Now, $p = (x_0, y_0)$ must satisfy the following equations

$$g_1(y_0) + g_2(x_0) + c = q_1, \tag{3-4a}$$

$$g_1(q_1) + g_2(y_0) + c = q_2. \tag{3-4b}$$

From (3-4b), we see that

$$b_2 y_0 = (1 - b_1)\overline{x}_1 - 2\tfrac{b_1 b_{20}}{b_2} + b_1 + b_{10} - b_{20} + b_2 - c = b_2\overline{x}_1 - 2\tfrac{b_1 b_{20}}{b_2} + 2b_{10}.$$

So,

$$y_0 = \overline{x}_1 - 2\frac{b_1 b_{20}}{b_2^2} + 2\frac{b_{10}}{b_2} := \overline{x}_1 + d_1. \tag{3-5a}$$

Substituting (3-5a) into (3-4a), we obtain that

$$x_0 = \overline{x}_1 + 2\frac{b_{20}}{b_2^2} - \frac{2}{b_2} - 2\frac{b_1 b_{10}}{b_2^2} + 2\frac{b_1^2 b_{20}}{b_2^3} := \overline{x}_1 + d_2. \tag{3-5b}$$

We need to show that there is a nonempty set of parameters for which

$$x_0\,,\, y_0 > 1, \tag{3-6a}$$

and

$$g_1(y_0) + g_2(x_0) + c = x_1 + 2\frac{b_{20}}{b_2} - 2 < -1. \tag{3-6b}$$

Propsition 3.1. *Let* $b_{10} = -b_{20} = q > 0$ *and* $b_2 = -p\, b_1 > 0$, *where* $p > 0$. *Suppose* $b_1(1-p) > 1$, $-1 < c < 1$, $b_1 \leq -3$, $q \geq 2p \geq 12$. *Then* $x_0 \geq \overline{x}_1$ *and* $y_0 \geq \overline{x}_1$. *Consequently, (3-6) holds.*

Proof. Clearly, y_0, given as in (3-5a), is greater than $\overline{x}_1$.
Now,

$$x_0 = \frac{c + (p-1)\, b_1}{1 + (p-1)\, b_1} - \frac{2q}{p^2\, b_1}\,[1 + \frac{1}{b_1} - \frac{1}{p}] + \frac{2}{p\, b_1}$$

$$\geq \frac{c + (p-1)\, b_1}{1 + (p-1)\, b_1} - \frac{q}{p^2\, b_1} + \frac{2}{p\, b_1} \geq \frac{c + (p-1)\, b_1}{1 + (p-1)\, b_1} = \overline{x}_1 \geq 1.$$

To complete the proof of the proposition, we see, via (3-6b), that

$$
\begin{aligned}
&\overline{x}_1 + 2\frac{b_{20}}{b_2} - 1 = \frac{c-1}{1+(p-1)\,b_1} + \frac{2q}{p\,b_1} \\
&= \frac{1}{p\,b_1(1+(p-1)\,b_1)}\,[p\,b_1(c-1) + 2\,q(1+(p-1)\,b_1)] \\
&\leq \frac{1}{b_1(1+(p-1)\,b_1)}\,[-2\,b_1 + 4\,(1+5\,b_1)] \\
&= \frac{1}{b_1(1+(p-1)\,b_1)}\,[18\,b_1 + 4] < 0.
\end{aligned}
$$

□

We next show that there are parameter values for which B has a snap-back repeller.

Theorem 3.1. *Let* $b_{10} = -b_{20} = q > 0$ *and* $b_2 = -p\,b_1 > 0$*, where* $p > 0$*. Suppose,*

$$
-1 < c < 1\,,\; q \geq 2\,p \geq 12 \text{ and } -b_1 \text{ is sufficiently large.} \tag{3-7}
$$

Then B *has a snap-back repeller.*

Proof. Let

$$
r := x_1 - 1 = \frac{c+(p-1)\,b_1}{1+(p-1)\,b_1} = \frac{c-1}{1+(p-1)\,b_1}. \tag{3-8}
$$

In view of Definition 2.1 and Proposition 3.1, to complete the proof of the theorem we only need to show that $B^{-k}(x_0, y_0)$, where B is the backward map and (x_0, y_0) is given as in (3-5), lies in $B_r(h)$ for all $k \in \mathbb{N}$. Here r is given in (3-8) and $h = (\overline{x}_1, \overline{x}_1)$ is the fixed point of B in Ω_1. To this end, we make a change of variables $x' = x - \overline{x}_1$ and $y' = y - \overline{x}_1$ on B in the region Ω_1. The resulting map B then has the form

$$
B(x'\,,\,y') = (y'\,,\,b_2\,x' + b_1\,y'). \tag{3-9}
$$

In the new coordinate systems, (x_0, y_0) becomes (x'_0, y'_0), where

$$(x'_0, y'_0) := (x_0 - \overline{x}_1, y_0 - \overline{x}_1) = (d_2\,,\, d_1) = (\frac{-2q}{p^2 b_1^2} + \frac{2}{p\,b_1} - \frac{2\,q}{p^2\,b_1} + \frac{2\,q}{p^3\,b_1},\ \frac{2\,q}{p^2\,b_1} - \frac{2\,q}{p\,b_1}); \tag{3-10}$$

where d_1 and d_2 are given as in (3.5). Note also that $d_1\,,\, d_2 > 0$. Let the pre-image (x'_0, y'_0), located in Ω, be denoted by (x'_{-1}, y'_{-1}). We then denote, inductively, by the pre-image of (x'_{-i}, y'_{-i}), located in Ω, (x'_{-i-1}, y'_{-i-1}), for any $i \in \mathbb{N}$. Using (3-9) and (3-10), we see immediately that

$$x^{'}_{-i-1} = -\frac{b_1}{b_2}\,x^{'}_{-i} + \frac{1}{b_2}\,y^{'}_{-i} = \frac{x^{'}_{-i}}{p} - \frac{y^{'}_{-i}}{p\,b_1}, \tag{3-10a}$$

$$y^{'}_{-i-1} = x^{'}_{-i}. \tag{3-10b}$$

Let $-1 < c < 1$ and $q \geq 2\,p \geq 12$. By making $-b_1$ sufficiently large, we see that

$$r > x^{'}_0 = d_1 > 0. \tag{3-11a}$$

and

$$r > y^{'}_0 = d_2 > 0. \tag{3-11b}$$

Using (3.10) and (3.11), we may prove inductively that

$$0 < x^{'}_{-k}\,,\, y^{'}_{-k} < r.$$

Thus $B^{-k}(x_0\,,\, y_0) \in B_r(h)$ with respect to supnorm for all $k \in \mathbb{N}$. It then easy to see that h is a snap-back repeller satisfying Definition 2.1. $\square$

Remark 3.1. *Let $\|\cdot\|_\epsilon$ be an operator matrix norm given as in the proof of **Theorem 2.2**-(ii). We still have that $B^{-k}(x_{0\,,\,y_0}) \in B_r(h)$ with respect to $\|\cdot\|_\epsilon$ for all $k \in \mathbb{N}$. Thus, h is a snap-back repeller satisfying Definition 2.2.*

Theorem 3.2. *Let $b_{10} = -b_{20} = q > 0$ and $b_2 = -p\,b_1 > 0$, where $p > 0$. Suppose (3-7) holds. Then the system (1-2) exists backward traveling waves of a chaotic profile.*

We conclude this thesis with following remarks.

(1) Let $\triangle t$ be small, Let r and $\overline{\alpha}$, as given in (1-1 a-b), be such that $r > 0$ is small and $\overline{\alpha}$ is negative. Then (3-7) are satisfied under some mild compatibility conditions on other parameters.

(2) It is worthwhile to find some other sufficient conditions for which F is "globally" expanding.

(3) It is also of interest to study the chaotic dynamics of the backward map B for the other combinations of g_1 and g_2, as well as those of the forward map F.