# 國 立 交 通 大 學

## 應 用 數 學 研 究 所

## 碩 士 論 文

三邊圖裝填的研究

Packing Graphs with Graphs of Size Three

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中華民國九十三年六月

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## 摘要 (中文)

在 1994 年, Chartrand 等人，提出下列兩個猜測：（1）一個 2 連通的圖，只要點數大於等於 4 而且邊數爲 3 的倍數，則此圖爲 $P_4$ 可分割；（2）對於任一個邊數爲 3 的倍數且最小度數大於等於 2 的圖, 都存在一個邊數爲 3 的圖 $H$，使得 $G$ 爲 $H$ 可分割。

我們在這篇論文中首先證明了猜測（2）, 然後，我們對於指定的 3 邊圖 $H$, 就完全多部圖，三正則圖和超立方體分別研究他們的分割。最後，我們在研究配對分割方面得到一些結果，並且猜測當 $q(G) = k\Delta(G)$ 及 $\Delta(G) \geq 2k-1$ 成立時 $G$ 爲第一類圖。

# Packing Graphs with Graphs of Size Three

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**Abstract**

In 1994, Chartrand et al. conjectured : (1) If $G$ is a 2-connected graph of order $p \geq 4$ and size $q(G) \equiv 0(\text{mod } 3)$, then $G$ is $P_4$-decomposable; (2) If $G$ is a graph of size $q(G) \equiv 0(\text{mod } 3)$ and $\delta(G) \geq 2$, then $G$ is $H$-decomposable for some graph $H$ of size 3. In the thesis, we first prove the second conjecture. Then, we study the $H$-decompositions of $G$ with fixed $H$ of size 3, where $G$ is a complete multipartite graph, a cubic graph or a hypercube. Finally, we obtain some results on $M_k$-decomposability of a graph $G$. Subsequently, we conjecture that a graph $G$ is of Class 1 provided $q(G) = k\Delta(G)$ and $\Delta(G) \geq 2k - 1$.

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# Contents

# Chapter 1

# Introduction

A *graph* $G$ is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of *vertices*, a set $E(G)$ of *edges*, and an *incidence function* $\psi_G$. For convenience, $G$ is also denoted by $(V, E)$ or $(V(G), E(G))$. Two vertices which are incident with a common edge are *adjacent*. An edge with identical ends is called a *loop*, and an edge with distinct ends is a *link*.

A graph is finite if both its vertex set and edge set are finite. A graph is simple if it has no loop and no two of its edges join the same pair of vertices. The *order* and *size* are the numbers of vertices and edges in graph $G$ respectively.

Two graph, $G$ and $H$ are said to be isomorphic (written $G = H$) if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$.

A simple graph on $n$ vertices in which each pair of distinct vertices are joined by an edge is called a *complete graph* of order $n$, denoted by $K_n$. A *bipartite graph* is a graph whose vertex set can be partitioned into two subsets $X$ and $Y$, so that each edge has one end in $X$ and one end in $Y$. A *complete bipartite graph* $K_{m,n}$ is a simple bipartite graph with bipartition $(X, Y)$ such that $|X| = m$, $|Y| = n$ and each vertex of $X$ is joined to each vertex of $Y$. An *r-partite graph* is whose vertex set can be partitioned into $r$ subsets such that no edge has both ends in any subset, a *complete r-bipartite graph* is a simple graph such that two vertices are adjacent if and only if they are not in the same subset.

A graph $H$ is a *subgraph* of $G$ (written $H \subseteq G$) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and $\psi_H$ is the restriction of $\psi_G$ to $E(H)$. The *degree* of a vertex $v$ in $G$ (written $d(v)$) is the number of edges of $G$ incident with $v$, each loop counting as two edges. A graph $G$ is *k-regular* if $d(v) = k$ for all $v \in V$.

A $(v_0, v_k)$*-path* in $G$ is a finite non-null sequence $P = v_0e_1v_1e_2v_2 \cdots e_kv_k$ where $v_0, v_1, \cdots v_k$ are distinct, a path $P$ in which $v_0 = v_k$ is a *cycle*; a path of length $n$ is denoted by $P_{n+1}$; a cycle of length $n$ is denoted by $C_n$ and a *wheel* is a graph obtained from $C_n$ by adding a new vertex and edges joining it to all the vertices of the $C_n$ (written $W_n$). A graph $G$ is called connected if there is a $(u, v)$-*path* for all $u, v \in V(G)$.

All graphs we consider are finite, simple and undirected. The *order*, *size*, *maximum* and *minimum degree* of a graph $G$ are denoted by $p(G)$, $q(G)$, $\Delta(G)$ and $\delta(G)$, respectively. The *neighborhood* of a vertex $v$, denoted by $N(v)$, is the set of vertices adjacent to $v$. The graph $S_n$ is the *complete bipartite graph* $K_{1,n}$. The graph $M_n$ is a *matching* of size $n$. The graph $G \cup H$ is the vertex disjoint union of $G$ and $H$. The graph $tH$, $t \geq 1$, is the edge disjoint union of $t$ copies of $H$. The *product* of simple graphs $G$ and $H$ is the simple graph $G \times H$ with vertex set $V(G) \times V(H)$, in which $(u, v)$ is adjacent to $(u', v')$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

A *k-edge coloring* $f$ of a loopless graph $G$ is an assignment of $k$ colors, $1, 2, \cdots, k$, to the edges of $G$. The coloring $f$ is proper if no two incident edges have the same color. A graph is $k$-edge colorable if it has a proper $k$-edge coloring. The *chromatic index* $\chi'(G)$, of a loopless graph $G$, is the minimum $k$ for which $G$ is $k$-edge colorable.

A graph $G$ is said to be *H-decomposable*, denoted by $H \mid G$, if the edge set $E(G)$ of $G$ can be partitioned into subsets such that each subsets induces a subgraph isomorphic to $H$. For convenience, we call $H$ a *divisor* of $G$ in such case. It is clear that $K_2 \mid G$ and $G \mid G$ for any graph $G$ with at least one edge. It is easy to see that if $H \mid G$, then $q(H) \mid q(G)$. In [2], Chartrand, Saba and Mynhardt made the following conjectures.

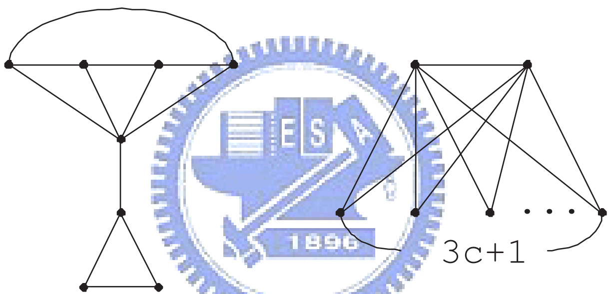
**Conjecture 1** [2] *Suppose $G$ is a graph of size $q(G) \equiv 0 (\text{mod } 3)$ and $\delta(G) \geq 2$. Then $G$ is $H$-decomposable for some graph $H$ of size* 3.

**Conjecture 2** [2] *Suppose $G$ is a 2-connected graph of order $p(G) \geq 2$ and of size $q(G) \equiv 0 (\text{mod } 3)$. Then $G$ is $P_4$-decomposable.*

These conjectures motivate our study of decomposing a graph of size $3k$ into $k$ copies of isomorphic graphs of size 3. It is worth of mentioning that Conjecture 2 has been disproved by Kumar[8]. Thus, we shall focus on the study of Conjecture 1 in Chapter 2 of this thesis.

Note here, if $q(H) = 3$, then $H = K_3$, $P_4$, $K_{1,3}$, $(P_3 \cup P_2)$ or $M_3$. Therefore, in order to prove Conjecture 1, for each given graph $G$ such that $q(G) \equiv 0 (\text{mod } 3)$ we have to find a graph $H$ of size 3 and prove that $H|G$. Here are a couple of examples.

In Figure 1(a), it is not difficult to see that the graph is $(P_3 \cup P_2)$-decomposable and the graph in Figure 2(b) is $P_4$-decomposable but not $(P_3 \cup P_2)$-decomposable. So, the plan of our proof is to characterize the graph $G$ which are $(P_3 \cup P_2)$-decomposable and for those graphs which are not $(P_3 \cup P_2)$-decomposable, we show they are either $P_4$-decomposable or $K_3$-decomposable. For this purpose, we shall first claim that if $G$ is of size $q(G) \equiv 0 (\text{mod } 3)$ and $\delta(G) \geq 2$, then $G$ is $(P_3 \cup P_2)$-decomposable if and only if $G$ is different from $K_4$ and $K_{1,1,3c+1}, c \geq 0$. And then, the proof will be obtained by the fact $K_4$ and $K_{1,1,3c+1}, c \geq 1$ are $P_4$-decomposable, and $K_3$ is $K_3$-decomposable.




(a) (b)

Figure 1.

In [8], Kumar gave a counterexample to Conjecture 2 and proved the following.

**Theorem 1.1** [8] *Suppose $G = K_{n_1,n_2,\cdots,n_r}$ a complete $r$-partite graph of size $q(G) \equiv 0 (\text{mod } 3)$, where $r \geq 2$. Then $G$ is $P_4$-decomposable except $G = K_{1,3n}$ or $K_3$.*

In Chapter 3, we study the $H$-decompositions of a graph $G$ with given $H$ of size 3, where $G$ is a complete multipartite graph, a cubic graph or a hypercube.

The followings should be mentioned.

If $q(H) = 1$, then $H = K_2$ and $K_2 \mid G$ for any graph $G$ with at least one edge. If $q(H) = 2$, then $H = P_3$ or $M_2$. For $H = P_3$, Chartrand et al. [1] showed the following theorem.

**Theorem 1.2** [1] *Every nontrivial connected graph of even size is $P_3$-decomposable.*

In Chapter 4, we study the $M_k$-decompositions of graphs. Also, we will give a necessary and sufficient condition for a graph being $M_2$-decomposable.

# Chapter 2

# $(P_3 \cup P_2)$-packings of graphs

We start this chapter with the study of $(P_3 \cup P_2)$-packings of graphs. An *H-packing* of a graph $G$ is a set of edge-disjoint subgraphs of $G$ in which each subgraph is isomorphic to $H$. An $H$-packing $\mathcal{F}$ is *maximum* if $|\mathcal{F}| \geq |\mathcal{F}'|$ for all other $H$-packings $\mathcal{F}'$ of $G$. The *leave* $L$ of an $H$-packing $\mathcal{F}$ is the subgraph induced by the set of edges of $G$ that does not occur in any subgraph of the $H$-packing $\mathcal{F}$. Therefore, a maximum packing has a minimum leave. In what follows, all the leaves we consider are minimum. It is easy to see that $H|G$ if and only if $G$ has an $H$-packing with empty leave $L$, i.e., $L$ contains no edge, or simply $L = \phi$.

The following lemmas are essential for proving the main theorem. Since they are easy to be proved, we omit the proofs.

**Lemma 2.1** *If $G = G_i$, $1 \leq i \leq 18$, given in Figure 2, then $(P_3 \cup P_2)|G$.*

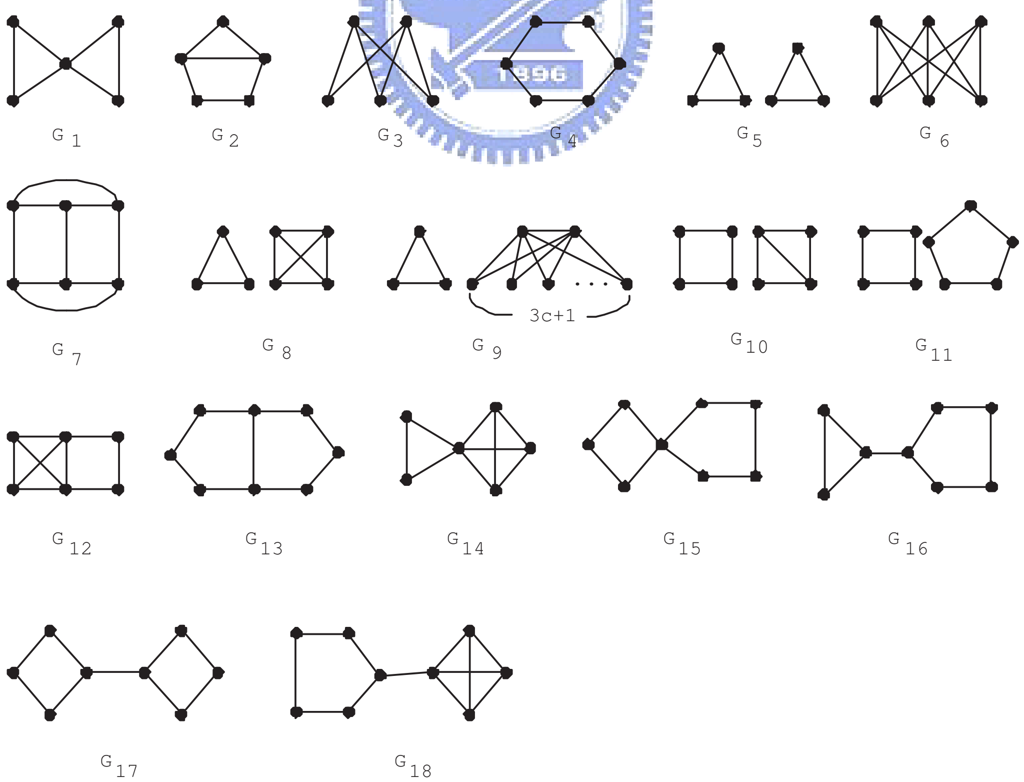



Figure 2.

**Lemma 2.2** *If $G = G_i$, $19 \leq i \leq 26$, given in Figure 3, then $G$ has a $(P_3 \cup P_2)$-packing with leave an edge.*

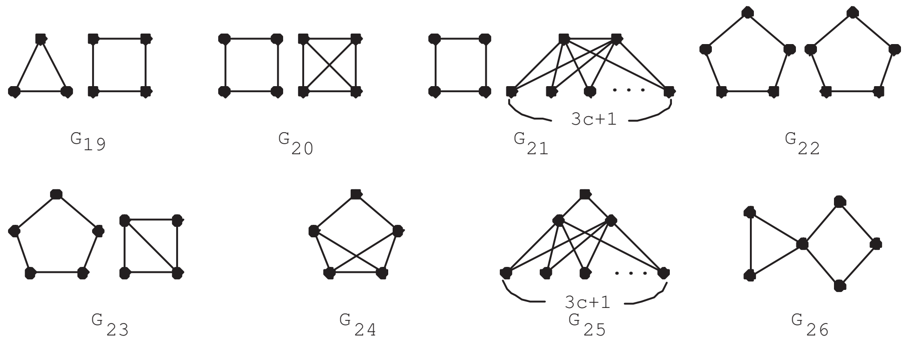



Figure 3.

**Lemma 2.3** *If $G = G_i$, $27 \leq i \leq 40$, given in Figure 4, then $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.*

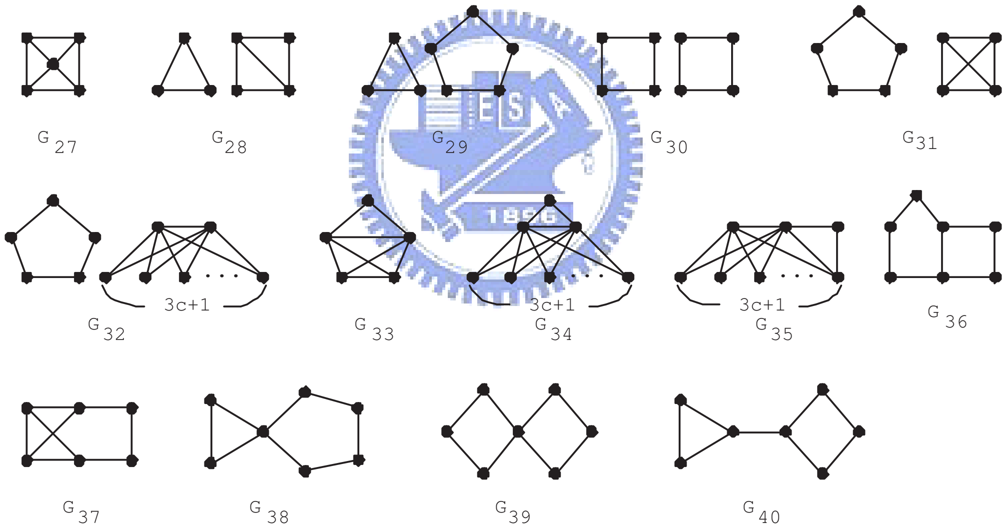



Figure 4.

The next result is our main theorem in this chapter.

**Theorem 2.4** *Suppose $G$ is a graph different from $K_{1,1,3c+1}$ with $p(G) \geq 5$, $q(G) \geq 6$ and $\delta(G) \geq 2$. Then $G$ has a $(P_3 \cup P_2)$-packing with leave $L$, where*

$$L = \begin{cases} \phi & if\ q(G) \equiv 0(\text{mod } 3); \\ P_2 & if\ q(G) \equiv 1(\text{mod } 3); \\ P_3 & if\ q(G) \equiv 2(\text{mod } 3). \end{cases}$$

**Proof.** By induction on $q(G)$.

If $q(G) = 6$, then $G = G_i$, $1 \leq i \leq 5$, given in Figure 2. By Lemma 2.1, we have $(P_3 \cup P_2)|G$.

Suppose the assertion holds for any graph $G'$ different from $K_{1,1,3c+1}$ with $p(G') \geq 5$, $\delta(G') \geq 2$ and $q(G') < q$, where $q \geq 7$. Let $G$ be a graph different from $K_{1,1,3c+1}$ with $p(G) \geq 5$, $q(G) = q$ and $\delta(G) \geq 2$. There are three cases to be considered.

**Case 1.** $\Delta(G) \geq 4$ and $\delta(G) \geq 3$.

By degree-sum formula, $q(G) = \frac{1}{2} \sum\limits_{x \in V(G)} d(x) \geq \frac{1}{2}(4 + 3\times 4) = 8$. If $q(G) = 8$, then $G = G_{27}$. We use equal sign for isomorphism. By Lemma 2.3, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

Now, suppose $q(G) > 8$. Let $v$ be a vertex with $d(v) = \Delta(G)$ and $N(v) = \{v_1, v_2, \cdots, v_{\Delta(G)}\}$. If $v_1$ is adjacent to some $v_i$ for $i \geq 2$, say $v_1v_2 \in E(G)$, let $G' = G - \{v_3vv_4, v_1v_2\}$; otherwise, let $u$ be a neighbor of $v_1$ which is different from $v$ and $G' = G - \{v_2vv_3, v_1u\}$. Then $G'$ satisfies the induction hypothesis. Since $G = G' \cup (P_3 \cup P_2)$, the assertion holds for the graph $G$.

**Case 2.** $G$ is 3-regular.

First, suppose $G$ is connected. If $p(G) = 6$, then $G = G_6$ or $G_7$. By Lemma 2.1, $(P_3 \cup P_2)|G$.

Suppose $(P_3 \cup P_2)|G'$ for any connected 3-regular graph $G'$ of order less than $p$, where $p \geq 8$. Let $G$ be a connected 3-regular graph of order $p$. It is not difficult to see that $G$ has an edge $xy$ with $N(x) = \{x_1, x_2, y\}$, $N(y) = \{y_1, y_2, x\}$ and $N(x) \bigcap N(y) = \phi$ such that $x_1y_1 \notin E(G)$ and $x_2y_2 \notin E(G)$. Let $G' = (G - \{x, y\}) \cup \{x_1y_1, x_2y_2\}$. Then $G'$ is a connected 3-regular graph of order $p-2$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Without loss of generality, we may consider the following cases.

(1) If there is an $F = \{x_1y_1v_3, x_2y_2\}$ in $\mathcal{F}$, then $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - F)$ $\cup\{x_1xx_2, yy_1\} \cup \{xyy_2, y_1v_3\}$ with empty leave.

(2) If there are $F_1 = \{v_1v_2v_3, x_1y_1\}$ and $F_2 = \{u_1u_2u_3, x_2y_2\}$ in $\mathcal{F}$, then $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xx_2, yy_1\} \cup \{v_1v_2v_3, xy\} \cup \{u_1u_2u_3, yy_2\}$ with empty leave.

(3) If there are $F_1 = \{v_1v_2v_3, x_1y_1\}$ and $F_2 = \{x_2y_2u_3, u_4u_5\}$ in $\mathcal{F}$, then $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xx_2, yy_1\} \cup \{v_1v_2v_3, xy\} \cup \{yy_2u_3, u_4u_5\}$ with empty leave.

(4) Suppose there are $F_1 = \{x_1y_1v_3, v_4v_5\}$ and $F_2 = \{x_2y_2u_3, u_4u_5\}$(or $F_2 = \{y_2x_2u_3, u_4u_5\}$) in $\mathcal{F}$. If $x_1 \notin \{u_4, u_5\}$, then $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xy, u_4u_5\} \cup \{yy_1v_3, v_4v_5\} \cup \{yy_2u_3, xx_2\}$(or $\{xx_2u_3, yy_2\}$) with empty leave.

If $x_1 = u_4$(or $u_5$) and $u_5 \neq v_3$, then $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{xx_1u_5, y_1v_3\} \cup \{xyy_1, v_4v_5\} \cup \{yy_2u_3, xx_2\}$(or $\{xx_2u_3, yy_2\}$) with empty leave.

If $x_1 = u_4$(or $u_5$) and $u_5 = v_3$, then $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xy, y_2u_3(\text{or } x_2u_3)\} \cup \{x_1v_3y_1, v_4v_5\} \cup \{y_1yy_2, xx_2\}$ with empty leave.

Hence, by induction, $(P_3 \cup P_2)|G$ for any connected 3-regular graph $G$ except $G = K_4$.

Secondly, let $G = (mK_4) \cup G_1 \cup \cdots \cup G_n$ be a disconnected 3-regular graph, where $m \geq 0$ and $G_i \neq K_4$ for $1 \leq i \leq n$. Since $P_3 \cup P_2 | G_i$, $G - mK_4$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave.

If $m = 1$, choose an $F$ in $\mathcal{F}$. It is easy to see that $K_4 \cup F = 3(P_3 \cup P_2)$. Hence, $(P_3 \cup P_2)|G$.

If $m \neq 1$, then $G = \frac{m}{2}(2K_4) \cup G_1 \cup \cdots \cup G_n$ when $m$ is even and $G = \frac{m-3}{2}(2K_4) \cup (3K_4) \cup G_1 \cup \cdots \cup G_n$ when $m$ is odd. It is easy to see that $(P_3 \cup P_2)|(tK_4)$ for $t = 2$ or $3$. Hence $(P_3 \cup P_2)|(mK_4)$ for $m \geq 2$ and then $(P_3 \cup P_2)|G$.

**Case 3.** $\delta(G) = 2$.

Suppose $G$ has a cycle-component. Let $C_n = x_1x_2 \cdots x_nx_1$ be the minimum cycle-component. If $3 \leq n \leq 5$, let $G' = G - C_n$.

Suppose $n = 3$ and $C_n = x_1x_2x_3x_1$. If $G = G_8, G_9, G_{19}, G_{28}$ or $G_{29}$, by Lemmas 2.1, 2.2 and 2.3, the assertion holds for these graphs $G$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $L$. Choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. Hence, $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - F) \cup \{x_1x_2x_3, v_4v_5\} \cup \{v_1v_2v_3, x_1x_3\}$ with leave $L$.

Suppose $n = 4$ and $C_n = x_1x_2x_3x_4x_1$. If $G = G_{10}, G_{11}, G_{20}, G_{21}$ or $G_{30}$, by Lemmas 2.1, 2.2 and 2.3, the assertion holds for these graphs $G$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $L$. For $L = \phi$, choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. Then $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - F) \cup \{x_1x_2x_3, v_4v_5\} \cup \{v_1v_2v_3, x_3x_4\}$ with leave $x_1x_4$. For $L = v_1v_2$, $G$ has a $(P_3 \cup P_2)$ -packing $\mathcal{F} \cup \{x_1x_2x_3, v_1v_2\}$ with leave $x_3x_4x_1$. For $L = v_1v_2v_3$, $G$ has a $(P_3 \cup P_2)$-packing $\mathcal{F} \cup \{x_1x_2x_3, v_1v_2\} \cup \{x_3x_4x_1, v_2v_3\}$ with empty leave.

Suppose $n = 5$ and $C_n = x_1x_2x_3x_4x_5x_1$. If $G = G_{22}, G_{23}, G_{31}$ or $G_{32}$, by Lemmas 2.2 and 2.3, the assertion holds for these graphs $G$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $L$. Choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. For $L = \phi$, $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - F) \cup \{x_1x_2x_3, v_4v_5\} \cup \{v_1v_2v_3, x_3x_4\}$ with leave $x_4x_5x_1$. For $L = u_1u_2$, $G$ has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - F) \cup \{x_1x_2x_3, v_4v_5\} \cup \{x_3x_4x_5, u_1u_2\} \cup \{v_1v_2v_3, x_1x_5\}$ with empty leave. For $L = u_1u_2u_3$, $G$ has a $(P_3 \cup P_2)$-packing $\mathcal{F} \cup \{x_1x_2x_3, u_1u_2\} \cup \{x_3x_4x_5, u_2u_3\}$ with leave $x_1x_5$.

For $n \geq 6$, let $C_n = x_1x_2 \cdots x_nx_1$. If $q(G) \equiv 0 (\text{mod } 3)$, let $G' = (G - \{x_2, x_3, x_4\}) \cup x_1x_5$. Then $q(G') = q(G) - 3 \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_1x_5 \in F$. Since $F = \{x_1x_5x_6, v_4v_5\}$, $\{x_nx_1x_5, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_5\}$, it is not difficult to see that $(F - x_1x_5) \cup x_1x_2x_3x_4x_5 = 2(P_3 \cup P_2)$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with empty leave.

If $q(G) \equiv 1 (\text{mod } 3)$, let $G' = (G - x_2) \cup x_1x_3$. Then $q(G') = q(G) - 1 \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ such that $x_1x_3 \in F$. Since $F = \{x_1x_3x_4, v_4v_5\}$, $\{x_nx_1x_3, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_3\}$, it is not difficult to see that $(F - x_1x_3) \cup x_1x_2x_3 = (P_3 \cup P_2) \cup L$, where $L = x_1x_2$ or $x_2x_3$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave $L$.

If $q(G) \equiv 2 (\text{mod } 3)$, let $G' = (G - \{x_2, x_3\}) \cup x_1x_4$. Then $q(G') = q(G) - 2 \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ such that $x_1x_4 \in F$. Since $F = \{x_1x_4x_5, v_4v_5\}$, $\{x_nx_1x_4, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_4\}$, it is not difficult to see that $(F - x_1x_4) \cup x_1x_2x_3x_4 = (P_3 \cup P_2) \cup L$, where $L = x_1x_2x_3$ or $x_2x_3x_4$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

Suppose $G$ has no cycle-component. Since $\delta(G) = 2$, there is a shortest path $x_0x_1x_2\cdots x_t$(not necessary open) in $G$ with $d(x_0) \geq 3, d(x_t) \geq 3$ and $d(x_i) = 2$ for $1 \leq i < t$, where $t \geq 2$. Consider the following cases.

(1) $x_0x_t \in E(G)$.

Suppose $q(G) \equiv 2 (\mathrm{mod}\ 3)$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 0(\mathrm{mod}\ 3)$. If $G = G_{33}, G_{34}$ or $G_{35}$, by Lemma 2.3, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$. Otherwise, by induction hypothesis, $(P_3 \cup P_2)|G'$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave $x_0x_1x_2$.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 2(\mathrm{mod}\ 3)$. If $G = G_{36}$, by Lemma 2.3, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$- packing $\mathcal{F}$ with leave a $L' = P_3$. It is easy to check that $L' \cup x_0x_1x_2x_3 = (P_3 \cup P_2) \cup P_3$ except $L' = x_0vx_3$. For $L' = x_0vx_3$, choose an $F$ in $\mathcal{F}$ with $x_0x_3 \in F$. It is easy to check that $F \cup x_0x_1x_2x_3vx_0 = 2(P_3 \cup P_2) \cup L$, where $L = x_0x_3x_2$ or $x_1x_0x_3$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

If $t \geq 4$, let $G' = (G - \{x_1, x_2\}) \cup x_0x_3$. Then $q(G') \equiv 0(\mathrm{mod}\ 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_0x_3 \in F$. It is a routine matter to check that $(F - x_0x_3) \cup x_0x_1x_2x_3 = (P_3 \cup P_2) \cup L$, where $L = x_0x_1x_2$ or $x_1x_2x_3$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

Suppose $q(G) \equiv 1(\mathrm{mod}\ 3)$. Let $G' = G - x_0x_t$. Then $q(G') \equiv 0(\mathrm{mod}\ 3)$. Since $x_1$ is of degree two in $G'$ and $x_0x_t \notin E(G')$, $G'$ is neither $K_4$ nor $K_{1,1,3c+1}$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing with empty leave. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave $x_0x_t$.

Suppose $q(G) \equiv 0(\mathrm{mod}\ 3)$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 1(\mathrm{mod}\ 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave an edge $e$.

If $\{x_0x_1x_2, e\}$ forms a $(P_3 \cup P_2)$, then $(P_3 \cup P_2)|G$.

If $e = x_0z, z \neq x_2$(similarly if $e = x_2z, z \neq x_0$), choose an $F$ in $\mathcal{F}$ with $x_0x_2 \in F$. It is a routine matter to check that $F \cup zx_0x_1x_2 = 2(P_3 \cup P_2)$ except $F = \{x_0x_2z, v_4v_5\}$. For $F = \{x_0x_2z, v_4v_5\}$, choose an $F_1$ in $\mathcal{F} - F$. It is a routine matter to check that $F_1 \cup zx_0x_1x_2 = 2(P_3 \cup P_2)$ except $F_1 = \{x_0u_2u_3, zu_5\}$ or $\{x_2u_2z, u_4u_5\}$, where $x_0$ is neither $u_4$ nor $u_5$. If $F_1 = \{x_0u_2u_3, zu_5\}$, then $F \cup F_1 \cup zx_0x_1x_2 = \{x_1x_0x_2, zu_5\} \cup \{x_0zx_2, v_4v_5\} \cup \{x_0u_2u_3, x_1x_2\}$.

If $F_1 = \{x_2u_2z, u_4u_5\}$, then $F \cup F_1 \cup zx_0x_1x_2 = \{x_0x_1x_2, zu_2\} \cup \{x_0zx_2, v_4v_5\} \cup \{x_0x_2u_2, u_4u_5\}$. Hence, $(P_3 \cup P_2)|G$.

Suppose $e = x_0x_2$. Since $G$ is different from $K_{1,1,3c+1}$, there is an edge $v_4v_5$ such that $e$ and $v_4v_5$ are vertex disjoint edges. Choose an $F$ in $\mathcal{F}$ with $v_4v_5 \in F$. It is a routine matter to check that $F \cup x_0x_1x_2x_0 = 2(P_3 \cup P_2)$ except $F = \{x_0v_2x_2, v_4v_5\}$. For $F = \{x_0v_2x_2, v_4v_5\}$, choose an $F_1$ in $\mathcal{F} - F$. It is a routine matter to check that $F_1 \cup x_0x_1x_2x_0 = 2(P_3 \cup P_2)$ except $F_1 = \{x_0u_2x_2, u_4u_5\}$, $\{w_1x_0w_3, x_2w_5\}$ or $\{z_1x_2z_3, x_0z_5\}$. If $F_1 = \{x_0u_2x_2, u_4u_5\}$, then $F \cup F_1 \cup x_0x_1x_2x_0 = \{x_1x_0u_2, x_2v_2\} \cup \{x_1x_2u_2, u_4u_5\} \cup \{x_2x_0v_2, v_4v_5\}$. If $F_1 = \{w_1x_0w_3, x_2w_5\}$ (similarly if $F_1 = \{z_1x_2z_3, x_0z_5\}$), then $F \cup F_1 \cup zx_0x_1x_2 = \{x_2x_0v_2, v_4v_5\} \cup \{w_1x_0w_3, x_1x_2\} \cup \{v_2x_2w_5, x_0x_1\}$. Hence, $(P_3 \cup P_2)|G$.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 0 (\text{mod } 3)$. If $G = G_{12}$, by Lemma 2.1, $(P_3 \cup P_2)|G$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_0x_3 \in F$. Thus, $F \cup x_0x_1x_2x_3 = 2(P_3 \cup P_2)$ and we have $(P_3 \cup P_2)|G$.

If $t = 4$, let $G' = G - \{x_1, x_2, x_3\}$. Then $q(G') \equiv 2 (\text{mod } 3)$. If $G = G_{13}$, by Lemma 2.1, $(P_3 \cup P_2)|G$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $v_1v_2v_3$. Since $v_1v_2v_3 \cup x_0x_1x_2x_3x_4 = \{v_1v_2v_3, x_2x_3\} \cup \{x_0x_1x_2, x_3x_4\}$, $(P_3 \cup P_2)|G$.

If $t \geq 5$, let $G' = (G - \{x_1, x_2, x_3\}) \cup x_0x_4$. Then $q(G') \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_0x_4 \in F$. It is a routine matter to check that $(F - x_0x_4) \cup x_0x_1x_2x_3x_4 = 2(P_3 \cup P_2)$. Hence, $(P_3 \cup P_2)|G$.

(2) $x_0x_t \notin E(G)$ and $x_0 \neq x_t$.

Suppose $q(G) \equiv 2 (\text{mod } 3)$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 0 (\text{mod } 3))$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Hence, $G$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $x_0x_1x_2$.

If $t \geq 3$, let $G' = (G - \{x_1, x_2\}) \cup x_0x_3$. Then $q(G') \equiv 0 (\text{mod } 3)$. If $G = G_{37}$, by Lemma 2.3, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_0x_3 \in F$. It is a routine matter to check that $(F - x_0x_3) \cup x_0x_1x_2x_3 = (P_3 \cup P_2) \cup L$, where $L = x_0x_1x_2$

or $x_1x_2x_3$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

Suppose $q(G) \equiv 1 (\text{mod } 3)$. Let $G'$=$(G - x_1)\cup x_0x_2$. Then $q(G') \equiv 0(\text{mod } 3)$. If $G = G_{24}$ or $G_{25}$, by Lemma 2.2, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_2$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_0x_2 \in F$. It is a routine matter to check that $(F - x_0x_2) \cup x_0x_1x_2 = (P_3 \cup P_2) \cup L$, where $L = x_0x_1$ or $x_1x_2$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_2$.

Suppose $q(G) \equiv 0(\text{mod } 3)$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 1(\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave an edge $e$. If $\{x_0x_1x_2, e\}$ forms a $P_3 \cup P_2$, then $(P_3 \cup P_2)|G$. Let $e = x_0z$(similarly $e = x_2z$). Choose an $F$ in $\mathcal{F}$ with $x_2 \in F$. It is a routine matter to check that $F \cup zx_0x_1x_2 = 2(P_3 \cup P_2)$ except $F = \{zv_2x_2, v_4v_5\}$. Since $d(x_2) \geq 3$, there is some $F_1$ in $\mathcal{F} - F$ with $x_2 \in F_1$. Similarly, $F_1 \cup zx_0x_1x_2 = 2(P_3 \cup P_2)$ where $F_1 = \{zu_2x_2, u_4u_5\}$. In such case, if $v_2$ is incident with $u_4u_5$, say $v_2 = u_4$, then $F \cup F_1 \cup zx_0x_1x_2 = \{x_0x_1x_2, v_4v_5\} \cup \{x_2v_2u_5, u_2z\} \cup \{x_0zv_2, x_2u_2\}$; otherwise, $F \cup F_1 \cup zx_0x_1x_2 = \{x_0x_1x_2, v_4v_5\} \cup \{u_2x_2v_2, x_0z\} \cup \{u_2zv_2, u_4u_5\}$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with empty leave.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 0(\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_0 \in F$. It is a routine matter to check that $F \cup x_0x_1x_2x_3 = 2(P_3 \cup P_2)$ except $F = \{x_0v_2x_3, v_4v_5\}$. For $F = \{x_0v_2x_3, v_4v_5\}$, by the same argument as above, $G$ has a $(P_3 \cup P_2)$-packing with empty leave.

If $t \geq 4$, let $G' = G - \{x_1, x_2, x_3\} \cup x_0x_4$. Then $q(G') \equiv 0(\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$- packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_0x_4 \in F$. It is a routine matter to check that $(F - x_0x_4) \cup x_0x_1x_2x_3x_4 = 2(P_3 \cup P_2)$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with empty leave.

(3) $x_0 = x_t$ and $t \geq 3$.

Suppose $q(G) \equiv 2(\text{mod } 3)$. For $t = 3$ or $4$, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2, \cdots, x_{t-1}\}$. If $G = G_{38}$ or $G_{39}$, by Lemma 2.3, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $L$. If $t = 3$, then $L = P_3$ and $L \cup x_0x_1x_2x_0 = \{L, x_1x_2\} \cup x_1x_0x_2$. If $t = 4$, then $L = P_2$ and

$L \cup x_0x_1x_2x_3x_0 = \{x_1x_2x_3, L\} \cup x_1x_0x_3$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_{t-1}, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - \{x_0, x_1, \cdots, x_{t-1}\}$. If $G = G_{40}$, by Lemma 2.3, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $L$. If $t = 3$, then $L = P_2$ and $L \cup x_0x_1x_2x_0 \cup x_0z = \{x_0x_1x_2, L\} \cup x_2x_0z$. If $t = 4$, then $L = \phi$ and $x_0x_1x_2x_3x_0 \cup x_0z = \{x_1x_2x_3, x_0z\} \cup x_1x_0x_3$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

For $t \geq 5$, let $G' = (G - \{x_2, x_3\}) \cup x_1x_4$. Then $q(G') \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_1x_4 \in F$. It is a routine matter to check that $(F - x_1x_4) \cup x_1x_2x_3x_4 = (P_3 \cup P_2) \cup L$, where $L = x_1x_2x_3$ or $x_2x_3x_4$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_3$.

Suppose $q(G) \equiv 1 (\text{mod } 3)$. For $t = 3$, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2\}$. If $G = G_{26}$, by Lemma 2.2, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_2$. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave a $P_2$. Choose an $F$ in $\mathcal{F}$. It is a routine matter to check that $F \cup x_0x_1x_2x_0 = 2(P_3 \cup P_2)$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_2$.

Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_2, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - x_0z$. Then $q(G') \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave $x_0z$.

For $t \geq 4$, let $G' = (G - x_2) \cup x_1x_3$. Then $q(G') \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_1x_3 \in F$. It is a routine matter to check that $(F - x_1x_3) \cup x_1x_2x_3 = (P_3 \cup P_2) \cup L$, where $L = x_1x_2$ or $x_2x_3$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with leave a $P_2$.

Suppose $q(G) \equiv 0 (\text{mod } 3)$. For $3 \leq t \leq 5$, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2, \cdots, x_{t-1}\}$. If $G = G_{14}$ or $G_{15}$, by Lemma 2.1, $G$ has a $(P_3 \cup P_2)$-packing with empty leave. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $L$. If $t = 3$, then $L = \phi$. Choose an $F$ in $\mathcal{F}$. It is a routine matter to check that $F \cup x_0x_1x_2x_0 = 2(P_3 \cup P_2)$. If $t = 4$, then $L = P_3$. It is a routine matter to check that $L \cup x_0x_1x_2x_3x_0 = 2(P_3 \cup P_2)$.

If $t = 5$, then $L = uv$. If $x_0$ is incident with $uv$, say $x_0 = u$, then $x_0x_1x_2x_3x_4x_0 \cup uv = \{x_0x_1x_2, x_3x_4\} \cup \{vx_0x_4, x_2x_3\}$. Otherwise, choose an $F = \{z_1z_2z_3, z_4z_5\}$ in $\mathcal{F}$ with $x_0 \in F$. If $x_0 = z_4$ or $z_5$, then $F \cup x_0x_1x_2x_3x_4x_0 \cup uv = \{x_0x_1x_2, uv\} \cup \{x_2x_3x_4, z_4z_5\} \cup \{z_1z_2z_3, x_4x_0\}$. If $x_0 = z_1, z_2$ or $z_3$, then $F \cup x_0x_1x_2x_3x_4x_0 \cup uv = \{x_0x_1x_2, uv\} \cup \{x_3x_4x_0, z_4z_5\} \cup \{z_1z_2z_3, x_2x_3\}$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with empty leave.

Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_{t-1}, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - \{x_0, x_1, \cdots, x_{t-1}\}$. If $G = G_{16}, G_{17}$ or $G_{18}$, by Lemma 2.1, $G$ has a $(P_3 \cup P_2)$-packing with empty leave. Otherwise, by induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $L$. It is a routine matter to check that $L \cup x_0x_1 \cdots x_tx_0 \cup x_0z = 2(P_3 \cup P_2)$ for $3 \leq t \leq 5$. Hence, $G$ has a $(P_3 \cup P_2)$-packing with empty leave.

Finally, for $t \geq 6$, let $G' = (G - \{x_2, x_3, x_4\}) \cup x_1x_5$. Then $q(G') \equiv 0 (\text{mod } 3)$. By induction hypothesis, $G'$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an $F$ in $\mathcal{F}$ with $x_1x_5 \in F$. It is a routine matter to check that $(F - x_1x_5) \cup x_1x_2x_3x_4x_5 = 2(P_3 \cup P_2)$. Hence $G$ has a $(P_3 \cup P_2)$-packing with empty leave.

Therefore, the proof concludes by induction. ■

Now, we are ready to prove the Conjecture 1.

**Theorem 2.5** *If $G$ is a graph with $q(G) \equiv 0 (\text{mod } 3)$ and $\delta(G) \geq 2$, then $H|G$ for some graph $H$ of size 3.*

**Proof.** If $q(G) = 3$, then it is trivial that $G|G$. In [8], Kumar proved that $P_4|G$ if $G = K_4$ or $K_{1,1,3c+1}$. For the rest, by Theorem 2.4, we have $(P_3 \cup P_2)|G$. Therefore, we complete the proof. ■

The result of Theorem 2.4 generalizes the following result proved in [5].

**Theorem 2.6** [5] *The necessary and sufficient conditions for a simple graph $G$ having a $(P_3 \cup P_2)$-decomposition are the following :*

(1) $q(G) \equiv 0(\mathrm{mod}\ 3)$*;*

(2) $\Delta(G) \leq \frac{2}{3}q(G)$*;*

(3) $c(G) \leq \frac{1}{3}q(G)$*, where* $c(G)$ *denotes the number of odd components of* $G$*;*

(4) *the edges of* $G$ *cannot be covered by two adjacent vertices.*

# Chapter 3

# $H$-decompositions of Special Classes of Graphs

## 3.1 $H$-decompositions of complete multipartite Graphs

First, we consider the $S_3$-decomposition of a complete $r$-partite graph $G = K_{n_1,\cdots,n_r}$ of size $q(G) \equiv 0(\text{mod } 3)$ and $r \geq 2$.

**Lemma 3.1** *The graph $K_{m,n}$ is $S_3$-decomposable if and only if $mn \equiv 0(\text{mod } 3)$.*

**Proof.** The condition $mn \equiv 0(\text{mod } 3)$ is clearly necessary. Conversely, assume $3 \mid m$. Since $E(K_{m,n})$=$E(nK_{m,1}) = E(\frac{mn}{3}K_{3,1})$, $K_{m,n}$ is $S_3$-decomposable. ■

**Lemma 3.2** *The complete graph $K_n$ is $S_3$-decomposable if and only if $n > 5$ and $q(K_n) \equiv 0(\text{mod } 3)$.*

**Proof.** It is clear that $K_n$ is not $S_3$-decomposable if $n \leq 5$ or $q(K_n) \not\equiv 0(\text{mod } 3)$. Suppose $n \geq 6$ and $q(K_n) \equiv 0(\text{mod } 3)$, i.e., $n \equiv 0$ or $1(\text{mod } 3)$. It is a routine matter to check that $K_6$ and $K_9$ are $S_3$-decomposable. If $n \equiv 0(\text{mod } 3)$ and $n \geq 12$, then $E(K_n) = E(K_6) \cup E(K_{n-6,6}) \cup E(K_{n-6})$. If $n \equiv 1(\text{mod } 3)$ and $n \geq 7$, then $E(K_n)$=$E(K_{n-1})$∪$E(K_{n-1,1})$. By Lemma 3.1 and mathematical induction, $K_n$ is $S_3$-decomposable. ■

**Theorem 3.3** *The graph $G = K_{n_1,\cdots,n_r}$ with $r \geq 2$ is $S_3$-decomposable if and only if $q(G) \equiv 0(\text{mod } 3)$ and $G$ is different from $K_4$ and $K_{1,1,3c+1}$, $c \geq 0$.*

**Proof.** The condition $q(G) \equiv 0(\text{mod } 3)$ is clearly necessary and $K_4 = K_{1,1,1,1}$ is not $S_3$-decomposable. For $G = K_{1,1,3c+1}$, there is a unique edge $e = xy$ in $E(G)$ which is incident with other edges in $E(G)$. If $G$ is $S_3$-decomposable, since $deg(v) = 2$ if $v$ is neither $x$ nor $y$, the center of each $S_3$ of an $S_3$-decomposition of $G$ must be $x$ or $y$. Hence, $\frac{q(G)}{3} = 2c+1 \leq \lfloor\frac{deg(x)}{3}\rfloor + \lfloor\frac{deg(y)}{3}\rfloor = c+c = 2c$. It is impossible. Therefore, $G = K_{1,1,3c+1}$ is not $S_3$-decomposable.

Conversely, we shall prove the assertion by induction on $r \geq 2$. By Lemma 3.1, $K_{n_1,n_2}$ is $S_3$-decomposable if $n_1 n_2 \equiv 0(\text{mod } 3)$. Suppose any graph $G' = K_{n_1,\cdots,n_{r'}}$ of size $q(G') \equiv 0(\text{mod } 3)$ and different from $K_4$ and $K_{1,1,3c+1}$ is $S_3$-decomposable for $r' < r$, where $r \geq 3$. Let $G = K_{n_1,\cdots,n_r}$ of size $q(G) \equiv 0(\text{mod } 3)$ and different from $K_4$ and $K_{1,1,3c+1}$. Consider the following disjoint cases.

Case 1. At least one $n_i'$s $\equiv 0(\text{mod } 3)$.

We may assume $n_1 = 3a, a \geq 1$. By Lemma 3.1, $S_3 | K_{n_1,n_2+\cdots+n_r}$. By induction hypothesis, $S_3 | K_{n_2,\cdots,n_r}$ except $K_{n_2,\cdots,n_r} = K_4$ or $K_{1,1,3c+1}$. Since $E(G) = E(K_{n_1,n_2+\cdots+n_r}) \cup E(K_{n_2,\cdots,n_r})$, $S_3 \mid G$ except $G = K_{3a,1,1,1,1}$ or $K_{3a,1,1,3c+1}$.

If $G = K_{3a,1,1,1,1}$, then $E(G) = E(K_{3,1,1,1,1}) \cup K_{3(a-1),4}$. It is a routine matter to check that $S_3 \mid K_{3,1,1,1,1}$. By Lemma 3.1, $S_3 \mid K_{3(a-1),4}$. Hence, $S_3 \mid G$.

If $G = K_{3a,1,1,3c+1}$, then $E(G) = E(K_{3,1,1,1}) \cup E(K_{3(a-1),1+1+3c+1}) \cup E(K_{3c,3+1+1})$. It is a routine matter to check that $S_3 \mid K_{3,1,1,1}$. By Lemma 3.1, $S_3 \mid K_{3(a-1),1+1+3c+1}$ and $S_3 \mid K_{3c,3+1+1}$. Hence, $S_3 \mid G$.

Case 2. At least three of $n_i'$s $\equiv 2(\text{mod } 3)$.

We may assume $n_1 = 3a + 2$, $n_2 = 3b + 2$ and $n_3 = 3c + 2$. Then $E(G) = E(K_{n_1,n_2,n_3}) \cup E(K_{n_1+n_2+n_3,n_4,\cdots,n_r})$. By induction hypothesis, $S_3 \mid K_{n_1+n_2+n_3,n_4,\cdots,n_r}$. Moreover, $E(K_{n_1,n_2,n_3}) = E(K_{2,2,2}) \cup E(K_{3a,n_2+n_3}) \cup E(K_{3b,2+n_3}) \cup E(K_{3c,2+2})$. It is a routine matter to check that $S_3 \mid K_{2,2,2}$. By Lemma 3.1, $K_{3a,n_2+n_3}, K_{3b,2+n_3}$ and $K_{3c,2+2}$ are $S_3$-decomposable. Hence, $S_3 \mid K_{n_1,n_2,n_3}$ and then $S_3 \mid G$.

Case 3. Exactly two of $n_i'$s $\equiv 2(\text{mod } 3)$.

Suppose $n_1 \equiv n_2 \equiv 2(\text{mod } 3)$ and $n_i \equiv 1(\text{mod } 3)$ for $i \geq 3$. Then $q(G) \equiv (\binom{r-2}{2} + 4(r-2) + 4)(\text{mod } 3) \equiv \frac{r^2+3r-2}{2}(\text{mod } 3) \not\equiv 0(\text{mod } 3)$ for $r \geq 3$. Hence, there are no graphs $G = K_{n_1,\cdots,n_r}$ of size $q(G) \equiv 0(\text{mod } 3)$ in this case.

Case 4. Exactly one of $n_i'$s $\equiv 2(\text{mod } 3)$.

We may assume $n_1 = 3a + 2$ and $n_i \equiv 1(\text{mod } 3)$ for $i \geq 2$. Then $q(G) \equiv (\binom{r-1}{2} + 2(r-1))(\text{mod } 3) \equiv \frac{(r-1)(r+2)}{2}(\text{mod } 3)$. Since $q(G) \equiv 0(\text{mod } 3)$, we obtain $r \equiv 1(\text{mod } 3)$. Hence, $n_2 + \cdots + n_r \equiv (r-1)(\text{mod } 3) \equiv 0(\text{mod } 3)$. Moreover, $E(G) = E(K_{n_1,n_2+\cdots+n_r}) \cup$

$E(K_{n_2,\cdots,n_r})$. Then $S_3 \mid K_{n_1,n_2+\cdots+n_r}$ by Lemma 3.1. By induction hypothesis, $S_3 \mid K_{n_2,\cdots,n_r}$ except $K_{n_2,\cdots,n_r} = K_{1,1,1}$. Hence, $S_3|G$ except $G = K_{3a+2,1,1,1}$.

If $G = K_{3a+2,1,1,1}$, then $E(G) = E(K_{2,1,1,1}) \cup E(K_{3a,1+1+1})$. It is a routine matter to check that $S_3 \mid K_{2,1,1,1}$. By Lemma 3.1, $S_3 \mid K_{3a,1+1+1}$. Hence, $S_3 \mid G$.

Case 5. $n_i \equiv 1 \pmod 3$ for $1 \leq i \leq r$.

Then $q(G) \equiv \frac{r(r-1)}{2} \pmod 3$. Since $q(G) \equiv 0 \pmod 3$, we obtain $r \equiv 0$ or $1 \pmod 3$. For $r = 3$, let $G = K_{n_1,n_2,n_3}$ with $n_1 \geq n_2 \geq 4$ (since $G \neq K_{1,1,3c+1}$). Then $E(G) = E(K_{4,4,1}) \cup E(K_{n_1-4,n_2+n_3}) \cup E(K_{n_2-4,4+n_3}) \cup E(K_{n_3-1,4+4})$. It is a routine matter to check that $S_3 \mid K_{4,4,1}$. By Lemma 3.1, $K_{n_1-4,n_2+n_3}, K_{n_2-4,4+n_3}$ and $K_{n_3-1,4+4}$ are all $S_3$-decomposable. Hence, $S_3 \mid G$.

For $r = 4$, let $G = K_{n_1,n_2,n_3,n_4}$ with $n_1 \geq 4$ (since $G \neq K_4$). Then $E(G) = E(K_{4,1,1,1}) \cup E(K_{n_1-4,n_2+n_3+n_4}) \cup E(K_{n_2-1,4+n_3+n_4}) \cup E(K_{n_3-1,4+1+n_4}) \cup E(K_{n_4-1,4+1+1})$. It is a routine matter to check that $S_3 \mid K_{4,1,1,1}$. By Lemma 3.1, $K_{n_1-4,n_2+n_3+n_4}$, $K_{n_2-1,4+n_3+n_4}$, $K_{n_3-1,4+1+n_4}$ and $K_{n_4-1,4+1+1}$ are all $S_3$-decomposable. Hence, $S_3 \mid G$.

For $r > 5$, $E(G) = E(K_r) \cup E(K_{n_1-1,n_2+\cdots+n_r}) \cup (\cup_{i=2}^{r} E(K_{n_i-1,r-1})) \cup E(K_{n_2-1,\cdots,n_r-1})$. By Lemma 3.2, $S_3 \mid K_r$. By Lemma 3.1, $S_3 \mid K_{n_1-1,n_2+\cdots+n_r}$ and $S_3 \mid K_{n_i-1,r-1}$ for $2 \leq i \leq r$. By induction hypothesis, $S_3 \mid K_{n_2-1,\cdots,n_r-1}$. Hence, $S_3 \mid G$.

Therefore, the assertion holds by the mathematical induction. ∎

**Theorem 3.4** *The graph $G = K_{n_1,\cdots,n_r}$ with $r \geq 2$ is $(P_3 \cup P_2)$-decomposable if and only if $q(G) \equiv 0 \pmod 3$ and $G$ is different from $K_4$ and $K_{1,1,3c+1}, c \geq 0$.*

**Proof.** It follows by Theorem 2.4. ∎

We may use the same argument as in Theorem 3.3 to prove the next theorem. However, we will give an alternative proof in Chapter 4.

**Theorem 3.5** *The graph $G = K_{n_1,\cdots,n_r}$ with $r \geq 2$ is $M_3$-decomposable if and only if $q(G) \equiv 0 \pmod 3$ and $G$ is different from $K_{1,3n}, K_{2,3n}, K_{3,3,1}, K_{1,1,3c+1}$ and $K_{1,1,1,m}$, where $n \geq 1, c \geq 0$ and $m \geq 1$.*

Remark here, the problem of determining the graph $G = K_{n_1,\cdots,n_r}$ being $K_3$-decomposable is still widely open, see [3, 4].

## 3.2 $H$-decompositions of Cubic graphs

A *cubic* graph is a 3-regular graph. Let $G$ be a cubic graph. By the *degree-sum formula*, we obtain $2q(G) = 3p(G)$. Hence, $q(G) \equiv 0 (\text{mod } 3)$.

**Theorem 3.6** *Suppose $G$ is a cubic graph.*

(1) *$G$ is not $K_3$-decomposable.*

(2) *$G$ is $P_4$-decomposable if $G$ is 2-connected.*

(3) *$G$ is $S_3$-decomposable if and only if it is bipartite.*

(4) *$G$ is $(P_3 \cup P_2)$-decomposable except $G = K_4$.*

(5) *$G$ is $M_3$-decomposable except $G = K_4$.*

**Proof.**

(1) It is easy to see that a graph which is $K_3$-decomposable must be eulerian, i.e., the degree of each vertex is even. Hence, $G$ is not $K_3$-decomposable.

(2) If $G$ is 2-connected, then $G$ has a perfect matching $M$ (see [10]). Let $M = \{e_1, \cdots, e_t\}$, where $t = \frac{p(G)}{2}$. Since $G$ is cubic, $G \backslash M$ is a disjoint union of cycles. For each cycle, we assign a oritation on it. Secondly, each $e_i$ in $M$ and the two arcs that point to the end vertices of $e_i$ form a $P_4$. It is not difficult to check that $E(G)$ is partitioned into $tP_4$ in such a way. Therefore, $G$ is $P_4$-decomposable.

(3) Suppose $G = (X, Y)$ is a cubic bipartite graph. For each vertex $v$ in $X$, the three edges that incident with $v$ form an $S_3$. Hence, $G$ is $S_3$-decomposable.

Conversely, suppose $E(G)$ can be partitioned into $t = \frac{q(G)}{3}$ $S_3'$s, say $S_3^1$, $S_3^2$, $\cdots$, $S_3^t$. Since $G$ is cubic, each vertex in $V(G)$ is either the center of some $S_3^i$ or a leaf of $S_3^i$, $S_3^j$ and $S_3^k$. Hence the sets $X = \{v \in V(G) \mid v \text{ is the center of some } S_3^i\}$

and $Y = \{v \in V(G) \mid v$ is a leaf of some $S_3^i\}$ are independent sets. Moreover, each edge in $E(G)$ has one end in $X$ and one end in $Y$ since it is an edge of some $S_3^i$. Therefore, $G$ is a bipartite graph with bipartition $(X, Y)$.

(4) It follows by Theorem 1.1.

(5) If $p(G) = 6$, then $G = K_{3,3}$ or $K_3 \times K_2$. It is easy to check that $G$ is $M_3$-decomposable. If $p(G) \geq 8$, then $q(G) - 3\Delta(G) \geq \frac{1}{2} \times 3 \times 8 - 3 \times 3 = 3 > 0$. By Theorem 4.5, $G$ is $M_3$-decomposable. ■

In [8], Kumar constructed a 2-connected graph $G$ of size $q(G) \equiv 0 (\bmod 3)$ and $\delta(G) = 2$ which is not $P_4$-decomposable. See Figure 5. Combining with Theorem 3.5(2), we give a modified conjecture as follows.

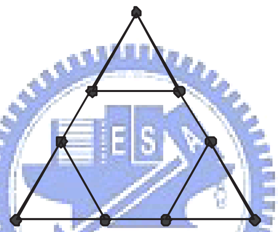


Figure 5.

**Conjecture 3** *Any 2-connected graph $G$ of size $q(G) \equiv 0 (\bmod 3)$ and $\delta(G) \geq 3$ is $P_4$-decomposable.*

## 3.3 $H$-decompositions of Hypercubes

An $n$-cube, denoted by $Q_n$, is defined recursively as follows : $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ for $n \geq 2$. It is well-known that $Q_n$ is bipartite. Hence, $Q_n$ is not $K_3$-decomposable. Moreover, $q(Q_n) = n \cdot 2^{n-1} \equiv 0 (\bmod 3)$ if and only if $n \equiv 0 (\bmod 3)$. It is a routine matter to check that $Q_3$ is $H$-decomposable with $H$ of size 3 if $H$ is different from $K_3$. If we replace each vertex of $Q_3$ by a $Q_n$ and each edge of $Q_3$ by a matching of size $p(Q_n) = 2^n$ such that the corresponding vertices of 8 $Q_n'$s form a $Q_3$, then it yields a $Q_{n+3}$. Hence, $E(Q_{n+3}) = E(8Q_n) \cup E(2^n Q_3)$. Therefore, by induction on $n \equiv 0 (\bmod 3)$, the following result is easy to see.

**Theorem 3.7** *Suppose* $n \equiv 0 (\text{mod } 3)$ *and* $H$ *is a graph different from* $K_3$ *and of size* 3. *Then* $Q_n$ *is* $H$*-decomposable.*

# Chapter 4

## $M_k$-decompositions of graphs

In this chapter, we mainly obtain necessary and sufficient conditions for graphs which are $M_2$-decomposable. But for completeness, we also present some result on the decomposition of $G$ into matchings of size $k$, $k \geq 3$.

A graph $G$ is said to be *$n$-edge colorable* if its edge set $E(G)$ can be partitioned into $n$ disjoint matchings $E_1$, $E_2$, $\cdots$, $E_n$ and it is *equitably* $n$-edge colorable if the sizes of $E_i$ and $E_j$ differ by at most one for all $1 \leq i \leq j \leq n$. The *chromatic index* of $G$, denoted by $\chi'(G)$, is the minimum number $n$ such that $G$ is $n$-edge colorable.

The followings are useful in this chapter.

**Theorem 4.1** [10] *For a simple graph $G$, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

A simple graph is *of class* 1 if $\Delta(G) = \chi'(G)$, otherwise it is *of class* 2.

**Theorem 4.2** [11] *Suppose $G$ is a simple graph $G$ and $n \geq \chi'(G)$. Then $G$ is equitably $n$-edge colorable.*

**Theorem 4.3** [6] *Suppose $G = (X, Y)$ a bipartite graph. Then $G$ has a matching that saturates every vertex in $X$ if and only if $| S | \leq | N(S) |$ for all $S \subseteq X$, where $N(S) = \{y \in Y \mid xy \in E(G)$ for some $x \in S\}$.*

**Theorem 4.4** [7] *The graph $G = K_{n_1,n_2,\cdots,n_r}$ is of class 2 if and only if $|E(G)| > \Delta(G)\lfloor \frac{p(G)}{2} \rfloor$.*

Now, we are ready to prove the main results of this chapter.

**Theorem 4.5** *Suppose $G$ is a graph of size $q(G) \equiv 0 (\text{mod } k)$, where $k \geq 1$.*

(1) *If $q(G) > k\Delta(G)$, then $G$ is $M_k$-decomposable.*

(2) *If $q(G) < k\Delta(G)$, then $G$ is not $M_k$-decomposable.*

(3) *If $q(G) = k\Delta(G)$, then $G$ is $M_k$-decomposable if and only if $\chi'(G) = \Delta(G)$.*

**Proof.** Let $q(G) = nk$. It is clear that $G$ is $M_k$-decomposable if and only if $G$ is equitably $n$-edge colorable. If $q(G) > k\Delta(G)$, by Theorem 4.1, $n \geq \Delta + 1 \geq \chi'(G)$. By Theorem 4.2, $G$ is equitably $n$-edge colorable. Hence $G$ is $M_k$-decomposable.

If $q(G) < k\Delta(G)$, by Theorem 4.1, $n < \Delta \leq \chi'(G)$. Hence, $G$ is not $n$-edge colorable and then $G$ is not $M_k$-decomposable.

Suppose $q(G) = k\Delta(G)$, i.e., $n = \Delta(G)$. If $G$ is $M_k$-decomposable, then $G$ is $n$-edge colorable. By Theorem 4.1, $\Delta(G) \leq \chi'(G) \leq n = \Delta(G)$ and then $\chi'(G) = \Delta(G)$. Conversely, if $\chi'(G) = \Delta(G) = n$, by Theorem 4.2, $G$ is equitably $n$-edge colorable. Therefore, $G$ is $M_k$-decomposable. ■

Recall that, in [1], the authors gave a necessary and sufficient condition for a graph being $P_3$-decomposable. Here, we characterize graphs which are $M_2$-decomposable.

**Theorem 4.6** *Suppose $G$ is a graph different from $K_3 \cup K_2$ and of even size. Then $G$ is $M_2$-decomposable if and only if $q(G) \geq 2\Delta(G)$.*

**Proof.** Let $q(G) = 2\Delta(G)$. By Theorem 4.5, $G$ is $M_2$-decomposable if $q(G) > 2\Delta(G)$ and $G$ is not $M_2$-decomposable if $q(G) < 2\Delta(G)$. Now, let $q(G) = 2\Delta(G)$. For $\Delta(G) = 1$, then $G = M_2$. For $\Delta(G) = 2$, then $G = C_4$, $K_3 \cup K_2$, $P_5$, $P_4 \cup P_2$, $2P_3$ or $P_3 \cup 2P_2$. Hence, $G$ is $M_2$-decomposable except $G = K_3 \cup K_2$. Suppose $\Delta(G) \geq 3$. Choose a vertex $v$ with $deg(v) = \Delta(G)$ and $v_1$, $v_2$, $\cdots$, $v_{\Delta(G)}$ are adjacent to $v$. Let $E(G) \backslash \{vv_1, \cdots, vv_{\Delta(G)}\} = \{e_1, \cdots, e_{\Delta(G)}\}$. Consider the bipartite graph $H = (X, Y)$ with vertex set $V(H) = X \cup Y$, where $X = \{e_1, \cdots, e_{\Delta(G)}\}$ and $Y = \{v_1, \cdots, v_{\Delta(G)}\}$, and edge set $E(H) = \{e_i v_j \mid e_i$ is not incident with $v_j\}$. Let $S \subseteq X$. It is easy to see that $deg_H(e_i) \geq \Delta(G) - 2$. Hence, if $\mid S \mid \leq \Delta(G) - 2$, then $\mid N(S) \mid \geq \Delta(G) - 2 \geq \mid S \mid$. For $\mid S \mid = \Delta(G) - 1 \geq 2$, if

$\mid N(S) \mid = \Delta(G) - 2$. then all $e_i'$s in $S$ have the same end vertices. It is impossible since $G$ is simple. Hence $\mid N(S) \mid \geq \Delta(G) - 1 = \mid S \mid$. For $\mid S \mid = \Delta(G)$, we have $S = X$ and $\mid N(S) \mid = \mid Y \mid = \Delta(G)$. Therefore, $\mid N(S) \mid \geq \mid S \mid$ for all $S \subseteq X$. By Theorem 4.3, $H$ has a matching $M$ that saturates every vertex in $X$. Without loss of generality, assume $M = \{e_1v_1, e_2v_2, \cdots, e_{\Delta(G)}v_{\Delta(G)}\}$. Color the edges $e_i$ and $vv_i$ by the color $i$ for $1 \leq i \leq \Delta(G)$. Then $G$ is $\Delta(G)$-colorable. By Theorem 4.1, $\Delta(G) \leq \chi'(G) \leq \Delta(G)$ and then $\chi'(G) = \Delta(G)$. By Theorem 4.5, $G$ is $M_2$-decomposable. ■

Next, we will give an alternative proof to Theorem 3.5.

**Theorem 3.5** *Suppose $G = K_{n_1,n_2,\cdots,n_r}$ of size $q(G) \equiv 0(\text{mod } 3)$ with $r \geq 2$. Then $G$ is $M_3$-decomposable if and only if $G$ is different from $K_{1,3n}$, $K_{2,3n}$, $K_{1,3,3}$, $K_{1,1,3c+1}$ and $K_{1,1,1,m}$, where $n \geq 1$, $c \geq 0$ and $m \geq 1$.*

**Proof.** Let $q(G) = 3\Delta(G)$. By directed computing, $q(G) < 3\Delta(G)$ if $(n_1, \cdots, n_r) = (1, 3n)$, $(2, 3n)$, $(1, 3, 3)$, $(1, 1, 3c+1)$ or $(1, 1, 1, m)$, $q(G) = 3\Delta(G)$ if $(n_1, \cdots, n_r) = (3, n)$, $n \geq 3$, (2,2,2), (1,3,6), (1,4,4), (1,1,2,4), (1,2,2,2), (1,1,1,1,3) or (1,1,1,1,1,1) and $q(G) > 3\Delta(G)$ for other cases. By Theorem 4.5, $G$ is $M_3$-decomposable if $q(G) > 3\Delta(G)$ and $G$ is not $M_3$-decomposable if G is one of $K_{1,3n}$, $K_{2,3n}$, $K_{1,3,3}$, $K_{1,1,3c+1}$, and $K_{1,1,1,m}$. For $q(G) = 3\Delta(G)$, it is easy to see that $\mid E(G) \mid \leq \Delta(G) \lfloor \frac{p(G)}{2} \rfloor$ for each possible graph $G$. By Theorem 4.4, $G$ is of class 1, i.e., $\chi'(G) = \Delta(G)$. By Theorem 4.5, $G$ is $M_3$-decomposable. ■

Before we put an end of this chapter, we would like to point out the relationship of $M_k$-decomposition of a graph $G$ with $k\Delta(G)$ edges. Clearly, if $G$ is of Class 1,then an $M_k$-decomposition exists. Unfortunately, for smaller $\Delta(G)$, we may not be able to guarantee that $G$ is of Class 1.

**Example 1** *For $k \geq 1$, there exists a graph $G$ such that $q(G) = k\Delta(G)$, $1 < \Delta(G) < 2k - 1$ and $\chi'(G) = \Delta(G) + 1$.*

**Proof.** Let $\Delta = 2k - i < 2k - 1$, where $2 \leq i < 2k - 1$.

If $i$ is even, let $G = K_{2k-i+1} \cup P_{n+1}$, where $n = k(2k-i) - \binom{2k-i+1}{2} = \frac{(i-1)(2k-i)}{2}$. It is easy to see that $q(G) = k\Delta(G)$ and $\chi'(G) = \chi'(K_{2k-i+1}) = 2k - i + 1 = \Delta(G) + 1$.

If $i > 1$ and odd, let $G = (K_{2k-i+2} \backslash H) \cup P_{n+1}$, where $H = P_3 \cup M_{k-\frac{i+1}{2}}$ and $n = k(2k-i+1) - \binom{2k-i+2}{2} + |E(H)| = (i-2)k - \frac{1}{2}(i^2 - 2i - 1)$. It is easy to see that $q(G) = k\Delta(G)$ and $\chi'(G) = \chi'(K_{2k-i+2} \backslash H) \geq \left\lceil \frac{\binom{2k-i+2}{2} - |E(H)|}{\frac{2k-i+1}{2}} \right\rceil = 2k - i + 1 = \Delta(G) + 1$ or $\chi'(G) = \Delta(G) + 1$ by Theorem 4.1. ■

From above example, we have constructed a graph of Class 2 which satisfies the conditions $q(G) = k\Delta(G)$ and $1 < \Delta(G) < 2k - 1$. But if $k = 1$, then $q(G) = \Delta(G)$ and this $G = S_\Delta$. Clearly $\chi'(G) = \Delta(G)$. Forthemore for $k = 2$ and 3, if $\Delta(G) \geq 3$ and $\Delta(G) \geq 5$ respectively, then $\chi'(G) = \Delta(G)$. Hence, it is reasonable to make the following conjecture to conclude this thesis.

**Conjecture 4** *If $G$ is a simple graph with $q(G) = k\Delta(G)$ and $\Delta(G) \geq 2k - 1$, then $\chi'(G) = \Delta(G)$.*

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