國 立 交 通 大 學

應用數學系

碩 士 論 文

擁有同構鄰子圖的正則圖

Regular Graphs with Isomorphic Neighbor-Subgraphs

研 究 生：李 昭 芳

指導老師：傅 恆 霖　教 授

中 華 民 國 九 十 三 年 六 月

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A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
In partial Fulfillment of Requirement
For the Degree of Master
In
Applied Mathematics
June 2004
Hsinchu, Taiwan, Republic of China

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摘　要

假如一個圖 G 中所有點都有相同的秩，那麼圖 G 是一個正則圖。假如一個正則圖 G 中每一個點的鄰點所生成的子圖都跟圖 H 同構，則圖 G 稱做 H-正則。

在這篇論文中，首先我們將研究哪一種圖 H 使得沒有 H-正則圖的存在(不被允許的圖 H)，接著對每一個”可能”的圖 H，我們試著去建構出 H-正則圖。最後，我們提到關於擁有最少點數的 H-正則圖的概念。那就是，對於一個給定的圖 H，討論點數最少的 H-正則圖。

中 華 民 國 九 十 三 年 六 月

Regular Graphs with Isomorphic Neighbor-Subgraphs

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Abstract

If all vertices of a graph G have the same degree, then G is a regular graph. A regular graph G is said to be H-regular if for each vertex $v \in V(G)$, the graph induced by $N_G(v)$ is isomorphic to H.

In this thesis, we shall first study for which H, an H-regular graph does not exist (forbidden $H's$) and then, for each "possible" H, we try to construct an H-regular graph. Finally, we mention the construction of H-regular graphs with smallest order, i.e., the extremal H-regular graph with a given H.

誌謝

首先感謝我的指導教授傅恆霖老師，在這些日子來不論是課業或是論文的修正對我悉心的教導和關心，而在排球場上傅老師也是我們的教練，在老師細心的教導及帶領下我們拿下不少的獎盃。最重要的是在待人處事方面，傅老師常說:「當學生不只是學習書上的知識，重點是要學習待人處事的態度和方法。」這一點也一直深深的留在我腦海中。

同時也非常感謝系上所有的老師在課業上和生活上的幫助讓我在溫馨的環境下完成學業，以及系上的助理小姐們對我的關心和照顧，讓不常回家的我有一種像是在家一樣的感覺。

感謝嚴志弘學長、嘉芬學姐、許弘松學長、郭志銘學長、張飛黃學長以及君逸、棨丰、抮君、啟賢還有所有組合和分析的同學，謝謝大家在這段期間給我的照顧。另外，也感謝交大應數系排以及系女排的所有人，讓我有很多美好的回憶。

最後感謝我的父母，是他們給我最大的支持，讓我在無憂慮的情況下完成學業，是他們陪我走過這一個重要的階段。

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1 Introduction and Preliminaries

1.1 Motivation

In the study of graph theory, regular graphs play the most important role. Almost all graphs with good structures, say connectivity, are regular. Not only this reason, when we plan to study some properties, it is better off to start with considering regular graphs. For example, bridgeless cubic graphs have been mentioned here and there. Anyway, it is nice to be regular.

In order to construct a graph with much better structure, we may also assign certain constraints to the graphs we construct. Say, if we assume that the smallest cycle length (girth) to be g, then we have an (r, g)-graph(r-regular graph with girth g). An (r, g)-graph with smallest order is the well-known (r, g)-cage. See [4] for a survey. On the other hand, if we let a k-regular graph whose adjacent pairs have λ common neighbors, and whose nonadjacent pair have μ common neighbors, then we have a strongly regular graph. The study of strongly regular graphs is also an important topic in Algebraic graph theory, see [1].

In this thesis, we shall study a new type of regular graphs. This notion was first mentioned to us by D. Hoffman a couple of years ago. The structure of such graphs is also very symmetrical. A regular graph G is said to be H-regular if for each vertex v belong to G, the graph induced by the neighbors of v is isomorphic to H. Since all graphs induced by the neighbors are isomorphic, in what follows, we call an H-regular graph a "neighbor-regular" graph.

1.2 Graph terms

In this section we present those definitions and basic properties what will be assumed throughout the rest of this thesis. For those terms not included the readers can refer to [3] for reference.

A **graph** G consists of a finite non-empty set $V(G)$ of **vertices** and a finite set $E(G)$ of distinct unordered pairs of distinct vertices called **edges**. The number of vertices of G is called the **order** of G and denoted by $v(G)$. The number of edges of G is called the **size** of G and denoted by $e(G)$. If $e = uv$ is an edge of G, then u and v are called its **endpoints**. Two or more edges joining the same pair of vertices are called **multiple edges**. A **loop** is an edge whose endpoints are equal. A graph is simple if it has no loops and multiple edges. Throughout of this thesis we consider only simple graphs.

If $e = uv$ is an edge of G, then e is said to join the vertices u and v, and these vertices are then said to be **adjacent**. If u is adjacent to v, then it is denoted by $u \sim v$. We also say that e is **incident** to u and v, and that v is a **neighbor** of u; the **neighborhood** $N_G(u)$ of u is the set of all vertices of G adjacent to u, the **closed neighborhood** $N_G[u]$ of u is the union of $N_G(u)$ and u. Two edges incident to the same vertex are **adjacent edges**. A **matching** in G is a set of edges no two of which are adjacent. Two graphs are **isomorphic** if there is a one-to-one correspondence between their vertex-sets which preserves the adjacency of vertices. An isomorphism from a graph G to itself is called an **automorphism** of G. An automorphism is therefore a permutation of the vertices of G that maps edges to edges and nonedges to nonedges.

A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, denoted $H \subseteq G$. If $V(H) = V(G)$, then H is called a spanning subgraph of G. If W is any set of vertices in G, then the subgraph induced by W is the subgraph of G obtained by joining those pairs of vertices in W what are joined in G. Any induced subgraph $G[W]$ of G is a subgraph induced by the subset W of $V(G)$.

If e is an edge of G, then the **edge-deleted subgraph** $G - e$ is the graph obtained from G by removing the edge e. Similarly, if v is a vertex of G, then the **vertex-deleted subgraph** $G - v$ is the graph obtained from G by removing the vertex v together with all its incident edges.

For each vertex v in a graph G, the number of edges incident to v is the **degree** of v, denoted by $deg(v)$ or $d(v)$. The maximum and minimum degrees in G are denoted by $\triangle(G)$ and $\delta(G)$ respectively. A vertex of degree 0 is called an **isolated vertex**, and a vertex of degree 1 is called an **end-vertex**. If all vertices of G have the same degree, then G is a **regular graph**; if each degree is k, then G is a **k-regular** graph. A 0-regular graph (that is, one with no edges) is a **null graph**, and a 3-regular graph is a **cubic graph**.

A sequence of edges $v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$ (sometime abbreviated to $v_0v_1 \ldots v_r$) is a **walk of length r** from v_0 to v_r. If these edges are all distinct, then the walk is a trail, and if the vertices $v_1, v_2, \ldots, v_r$ are also distinct, then it is a **path** and denoted by $[v_0, v_1, \ldots, v_r]$. A walk in which $v_0, v_1, \ldots, v_r$, are all distinct except for v_0 and v_r is a **cycle**. The **girth** of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.

The **union** $G \cup H$ of two disjoint graphs G and H is the graph having vertex set

$V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The graph obtained by taking the union of graphs G and H with disjoint vertex sets is the **disjoint union**, written $G + H$. The **join** of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G+H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$. The **Cartesian product** $G \Box H$ of two disjoint graphs G and H is the graph having the vertex set $V(G) \times V(H)$ and edge set $\{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \text{ and } v_1v_2 \in E(H) \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E(G)\}$. The Cartesian product $G \Box G \Box \cdots \Box G$(n-tuple) is denoted by G^n. A graph G is **connected** if there is a path joining each pair of vertices of G, or equivalently if G cannot be expressed as the union of two vertex disjoint graphs; a graph which is not connected is **disconnected**.

A graph in which every two vertices are adjacent is **complete graph**; the complete graph with n vertices and n(n-1)/2 edges is denoted by K_n. The **cycle graph** C_n of order n consists of the vertices and edges of a n-gon, and the **path graph** P_n is obtained by removing an edge from C_n. The **null graph** O_n is the graph with n vertices and no edges. A matching with n edges is denoted by M_n. A **bipartite graph** is one whose vertex-set can be partitioned into two sets so that each edge joins a vertex of the first set to a vertex of the second set. A **complete bipartite graph** is a bipartite graph in which every vertex in first set is adjacent to every vertex in the second set. If the two sets contain r and s vertices respectively, then the complete bipartite graph is denoted by $K_{r,s}$; a complete bipartite graph of the form $K_{1,t}$ is called a **star graph** and denoted by S_t. A connected graph which contains no cycle is a **tree**. A **complete balanced m-partite graph** with each partite set of size n is denoted by $K_{m(n)} = O_n \vee O_n \vee \cdots \vee O_n$(m-tuple).

1.3 Some special regular graphs

In this section we present some regular graphs with special conditions. A simple n-vertex graph G is **strongly regular** if there are parameters k, λ, μ such that G is k-regular, every adjacent pair of vertices have λ common neighbors, and every nonadjacent pair of vertices have μ common neighbors, written (n, k, λ, μ)-graph. For example, Petersen graph is a strongly regular graph with $n = 10, k = 3, \lambda = 0, \mu = 1$. A k-regular graph with girth g is a (k, g)-graph. A (k, g)-graph with the smallest order is called a (k, g)-**cage**. For example, Petersen graph is a $(3, 5)$-cage. For more information about cages, see [4] for reference. A regular graph G is said to be H-**regular** if for each vertex $v \in G$, the graph induced by the neighbors of v is isomorphic to H. For example, Petersen graph is a O_3-regular graph. A **Petersen graph** is the simple graph whose vertices are the 2-element subsets of 5-element set and whose edges are the pairs of disjoint 2-element subsets. The followings are examples of some (k, g)-cages, strongly regular graphs and Petersen graph.

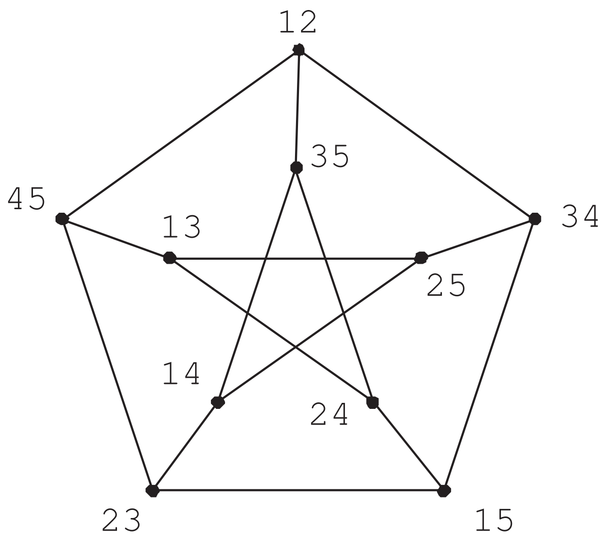


Figure 1: The Petersen graph

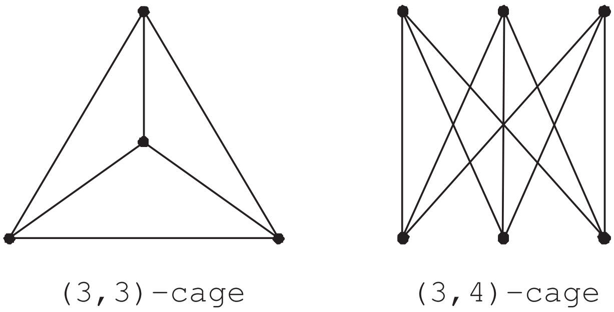


Figure 2: Examples of some (k, g)-cage

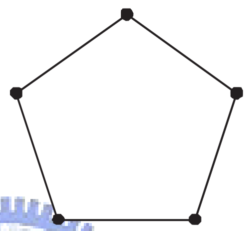
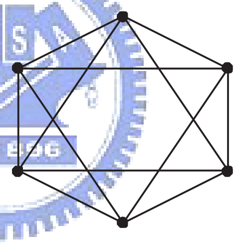
Parameter	**Graph**
(5,2,0,1)	
(6,4,2,4)	

Figure 3: Examples of strongly regular graphs

Since K_{k+1} is a $(k, 3)$-cage and a (k, g)-cage is a triangle-free graph for each $g \geq 4$. Hence, we have

Proposition 1.3.1. A (k, g)-cage is a K_k-regular graph for $g = 3$ and it is an O_k-regular graph for $g \geq 4$.

As to strongly regular graphs, it seems that all graphs obtained in literatures are H-regular graphs for some H. We list some graphs in the following table.

Parameter	Graph	H-regular graph
(2,1,0,0)	Path graph P_2	O_1-regular graph
(3,2,1,0)	Complete graph K_3	K_2-regular graph
(4,2,0,2)	Cycle graph C_4	O_2-regular graph
(4,3,2,0)	Complete graph K_4	K_3-regular graph
(5,2,0,1)	Cycle graph C_5	O_2-regular graph
(5,4,3,0)	Complete graph K_5	K_4-regular graph
(6,3,0,3)	Circulant graph $Ci_6(1,3)$	O_3-regular graph
(6,4,2,4)	Octahedral graph	C_4-regular graph
(6,5,4,0)	Complete graph K_6	K_5-regular graph
(7,6,5,0)	Complete graph K_7	K_6-regular graph
(8,4,0,4)	Circulant graph $Ci_8(1,3)$	O_4-regular graph
(8,6,4,6)	Circulant graph $Ci_8(1,2,3)$	H_1-regular graph
(8,7,6,0)	Complete graph K_8	K_7-regular graph
(9,6,3,6)	Circulant graph $Ci_9(1,2,3)$	H_2-regular graph
(9,8,7,0)	Complete graph K_9	K_8-regular graph
(10,3,0,1)	Petersen graph	O_3-regular graph

H_1 : K_6-M_3 $\quad$ H_2 :

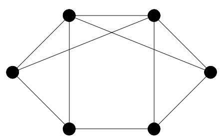


Figure 4: Some of strongly regular graphs are H-regular graph

Let D be a subset of $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. A **circulant graph** $Ci_n(D)$ is a graph with vertex set $V(G) = Z_n$, and edge set $E(G) = \{i \sim j \mid d(i, j) \in D, \forall i, j \in Z_n\}$, where $d(i, j) =_{def} min\{\mid i - j \mid, n- \mid i - j \mid\}$. The elements of D will be referred as the differences. Clearly, the circulant graph $Ci_n(1, 2, \ldots, \lfloor n/2 \rfloor)$ gives the complete graph K_n and the graph $Ci_n(1)$ gives the cyclic graph C_n. By the definition of circulant graph, we have

Proposition 1.3.2. $Ci_n(D)$ is an H-regular for some H.

Proof.

Let $D = \{a_1, a_2, \cdots, a_t\}$ be a set of differences and p, q be two distinct vertices in Z_n. Then

$$\begin{array}{l} N_G(p) = \{p \pm a_1,\ p \pm a_2,\ \cdots,\ p \pm a_t\} \\ N_G(q) = \{q \pm a_1,\ q \pm a_2,\ \cdots,\ q \pm a_t\} \end{array}$$

and

$$p \pm a_i \sim p \pm a_j \Longleftrightarrow q \pm a_i \sim q \pm a_j, \forall\ i, j$$

So, $G[N_G(p)] \cong G[N_G(q)] =_{def} H$. That is, $Ci_n(D)$ is an H-regular for some H. $\square$

A graph G is **vertex transitive** if its automorphism group acts transitively on $V(G)$. Thus for any two distinct vertices of G there is an automorphism mapping one to the other. A vertex transitive graph is necessarily regular. By definition, it is not difficult to see that a vertex transitive graph is an H-regular graph for some H. But, the converse statement may not be true. Figure 5 is a example which is an O_3-regular graph but not a vertex transitive graph. This tells us that an H-regular graph is in fact not that symmetrical sometimes.

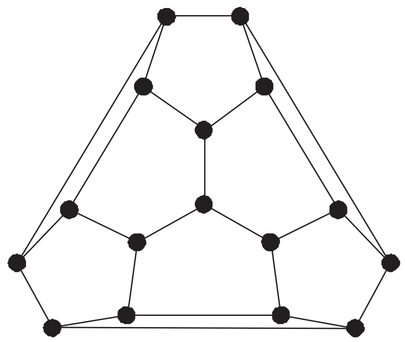


Figure 5: An O_3-regular graph which is not a vertex transitive graph

2 H-regular graphs

We start this chapter with the study of non-existence of H-regular graphs, i.e., to determine for which H, there dose not exist an H-regular graph.

2.1 Forbidden $H's$

Throughout of this section, an H which we can not find an H-regular graph is called a forbidden graph. The following lemma shows that there are quite a few connected graphs which are forbidden.

Lemma 2.1.1. Let H be a graph with two vertices x and y such that $|V(H)| \geq 3$, $d_H(x) = |V(H)| - 1$ and $d_H(y) = 1$. Then H is a forbidden graph.

Proof.

Suppose not. Let G be an H-regular graph. Then, consider an arbitrary vertex v in G. By the definition of an H-regular graph $N_G(v)$ induces a graph G' which is isomorphic to H. Let $u \in N_G(v)$ such that $d_{G'}(u) = |V(H)| - 1$ and $w \in N_G(v)$ such that $d_{G'}(w) = 1$. Now, since $w \in V(G)$, $N_G(w)$ also induces a graph G'' which is isomorphic to H. But, by the fact that $w \in N_G(v)$ and $d_{G'}(w) = 1$, $V(G'')$ contains exactly $|V(H)| - 2$ vertices which are not in $V(G') \cup \{v\}$, moreover $\{u, v\} \subseteq V(G'')$. Now, since $d_G(u) = d_G(v) = |V(H)|$, uv is an independent edge in G''. By assumption that H is connected, G'' is not isomorphic to H. Therefore, G can not be an H-regular graph, this leads to a contradiction. Hence, the proof is concluded. □

Corollary 2.1.2. Let H be a graph with two vertices x and y such that $V(H) \geq 3$, $d_H(x) = |V(H)| - 1$ and $d_H(y) = 1$. Then $H \cup O_t$ is a forbidden graph.

Proof.

The proof follows by a similar argument. □

Corollary 2.1.3. No $(S_n \cup O_t)$-regular graphs exist for $n \geq 2$ and $t \geq 0$.

Proof.

Fix $n \geq 2$. Because there exist both vertices $u, v \in S_n$ such that $d(u) = |V(S_n)| - 1$ and $d(v) = 1$. By Corollary 2.1.2, there does not exist an $(S_n \cup O_t)$-regular graph for some $t \geq 0$. □

If the connected graph in Lemma 2.1.1 we considered is a tree, then we can lower down the maximum degree.

Lemma 2.1.4. Let H be a tree of order n and $x \in V(H)$ such that $d_H(x) > (2n-2)/3$. Then H is a forbidden graph.

Proof.

Suppose not. Let G is a H-regular graph and v is an arbitrary vertex of G. By assumption $G[N_G(v)] = H$. Therefore, we let $u \in N_G(v)$ be the vertex of degree larger than $(2n-2)/3$ in $G[N_G(v)]$. Let $d_H(u) = k$, so $k > (2n-2)/3 \Rightarrow k > 2(n-k-1)$. Then there exists a vertex w such that only x and v adjacent to w in $N_G[u]$. Now, since $d_G(u) = d_G(v) = |V(H)|$, uv is an independent edge in $G[N_G(w)]$. By assumption that H is connected, $G[N_G(w)]$ is not isomorphic to H. Therefore, G can not be an H-regular graph, this leads to a contradiction. Hence, the proof is concluded. □

Lemma 2.1.5. If $H = K_n - P_s$, then H is a forbidden graph for $n \geq 3$ and $2 \leq s \leq n-1$.

Proof.

Fix $n \geq 3$, suppose G is a $(K_n - P_s)$-regular graph for some $2 \leq s \leq n-1$, and v is an arbitrary vertex of G. By assumption, $G[N_G(v)] = K_n - P_s$. Let $H = K_n - P_s$. Then there exist $x, y, z \in V(H)$ such that $d_H(x) = n-1, d_H(y) = n-2$, and $d_H(z) = n-2$. Let $H_1 = H \cup \{v\}$. Now, we consider two cases.

Case 1. $s = 2$

Consider the vertex y. Because $y \sim v$, so $d_{H_1}(y) = n-2+1 = n-1$. Since $n-1$ neighbors of y which are of full degrees, $G[N_G(y)] \neq K_n - P_2$.

Case 2. $3 \leq s \leq n-1$

Let $G_1 = G[N_{H_1}(y)]$ and consider the vertex y. Because $y \sim v$, so $d_{H_1}(y) = n-2+1 = n-1$, $d_{G_1}(v) = 2+n-4 = n-2$, $d_{G_1}(x) = 2+n-4 = n-2$, and the vertices of $G_1 - \{v, x\}$ are of degree at most $n-2$ in G_1. Since $y \sim z$ and $d_{H_1}(y) = n-2+1 = n-1$, there exists a vertex w which is not in H_1, and $w \sim z$. As to the vertex $u \in G[N_G(y)]$, $d_{G[N_G(y)]}(u) \leq n-2$. Now, consider the vertex w. Since $d_G(y) = d_G(z) = n$, $G[N_G(w)] \neq K_n - P_s$. Both cases lead to a contradiction. Hence, the proof is concluded. □

Corollary 2.1.6. No $((K_n - P_s) \cup O_t)$-regular graphs exist, where $n \geq 3$, $2 \leq s \leq n-1$ and $t \geq 0$.

Proof.

The proof follows by a similar argument. □

Lemma 2.1.7. If $H = K_{m,n}$ and $m \neq n$, then H is a forbidden graph.

Proof.

Suppose not. Let G be an H-regular graph and v be an arbitrary vertex of G. By assumption, $G[N_G(v)] = H$. Suppose that H consists of X and Y, where $|X| = m$, $|Y| = n$ and $m > n$. Let $G_1 = H \cup \{v\}$. Then $d_{G_1}(x) = n + 1$ for all $x \in X$ and $G[N_{G_1}(x)] = K_{1,n}$. Since $G[N_G(x)]$ is isomorphic to H, each vertex of A joins each vertex of Y, where $A = N_G(x) \setminus (Y \cup \{v\})$. But $d_G(y) = (m+1) + (m-1) = 2m > m+n$ for all $y \in Y$, this leads to a contradiction. Hence, the proof is concluded. $\square$

Corollary 2.1.8. If $H = K_{n_1,n_2,\cdots,n_r}$ and $n_i \neq n_j$, for some $i \neq j$, then H is a forbidden graph.

Proof.

The proof follows by a similar argument. $\square$

Lemma 2.1.9. If $H = K_n - K_s$, then H is a forbidden graph for $n \geq 3$ and $2 \leq s \leq n-1$.

Proof.

Fix $n \geq 3$. Suppose G is a $(K_n - K_s)$-regular graph for some $2 \leq s \leq n-1$ and v is an arbitrary vertex of G. By assumption, $G[N_G(v)] = K_n - K_s$. Let $K_n - K_s = H_1 \vee H_2$, where $H_1 = O_s$ and $H_2 = K_{n-s}$. Consider an arbitrary vertex x of H_1. By assumption, $G[N_G(x)]$ is isomorphic to H. But the neighbors of x in $H \cup \{v\}$ are of degree $n-1$(major vertices), so $G[N_G(x)]$ is disconnected. That is, $G[N_G(x)] \neq K_n - K_s$. This leads to a contradiction. Hence, the proof is concluded. $\square$

2.2 Constructions of H-regular graphs

In this section, we will use "join" and "Cartesian product" of graphs to discuss the structure of H-regular graphs.

Lemma 2.2.1. If G is an H-regular graph, then $G \vee G$ is a $(G \vee H)$-regular graph.

Proof.

Let v be an arbitrary vertex of $G \vee G$. Then $G[N_{G \vee G}(v)] = G \vee G[N_G(v)] = G \vee H$. $\square$

Corollary 2.2.2. $C_n \vee C_n$ is a K_5-regular graph for $n = 3$ and it is a $(C_n \vee O_2)$-regular graph for all $n \geq 4$.

Proof.

By Lemma 2.2.1, since C_3 is an P_2-regular graph, $C_3 \vee C_3$ is a $(C_3 \vee P_2)$-regular graph, i.e., K_5-regular graph. On the other hand, C_n is an O_2-regular graph, for all $n \geq 4$, $C_n \vee C_n$ is a $(C_n \vee O_2)$-regular graph, for all $n \geq 4$. $\square$

Lemma 2.2.3. $K_{m(n)}$ is a $K_{m-1(n)}$-regular graph for all $m \geq 2$ and $n \geq 1$.

Proof.

Fix $n \geq 1$ and $m \geq 2$, choose $x \in K_{m(n)}$. Then $G[N_{K_{m(n)}}(x)] = O_n \vee O_n \vee \cdots \vee O_n$ (m-1 tuple)$= K_{m-1(n)}$. $\square$

Corollary 2.2.4. $K_{t,t}$ is an O_t-regular graph for $t \geq 1$.

Proof.

By Lemma 2.2.3, let $m = 2$ and $n = t$. $\square$

Lemma 2.2.5. If G_1 is an H_1-regular graph and G_2 is an H_2-regular graph, then $G_1 \square G_2$ is an $(H_1 \cup H_2)$-regular graph.

Proof.

Choose a vertex $x \in V(G_1 \square G_2)$. By definition of Cartesian product, $N_{G_1 \square G_2}(x) = N_{G_1}(x) \cup N_{G_2}(x)$. Hence $G[N_{G_1 \square G_2}(x)] = G[N_{G_1}(x) \cup N_{G_2}(x)] = H_1 \cup H_2$. $\square$

Corollary 2.2.6. If H-regular graphs exist, then $(H \cup O_t)$-regular graphs exist for $t \geq 1$.

Proof.

Let G be an H-regular graph. Because $K_{t,t}$ is an O_t-regular graph for each $t \geq 1$, by Lemma 2.2.5, $G \square K_{t,t}$ is an $(H \cup O_t)$-regular graph. $\square$

Lemma 2.2.7. If G is an H-regular graph, then G^t is a $(\bigcup^t H)$-regular graph for each $t \geq 1$, where $\bigcup^t H$ is $H \cup H \cup \cdots \cup H$ (t tuple).

Proof.

By Lemma 2.2.5, G^t is an $(\bigcup^t H)$-regular graph for each $t \geq 1$. $\square$

Corollary 2.2.8. $(K_3)^t$ is an M_t-regular graph for each $t \geq 1$.

Proof.

Because K_3 is an M_1-regular graph, by Lemma 2.2.7, we conclude that $(K_3)^t$ is an M_t-regular graph. $\square$

Corollary 2.2.9. If G is an H-regular graph, then $G \square (K_3)^t$ is an $(H \cup M_t)$-regular graph.

Proof.

Because $(K_3)^t$ is an M_t-regular graph, by Lemma 2.2.5, we get $G\Box(K_3)^t$ is an $(H\cup M_t)$-regular graph. $\Box$

2.3 H-regular graphs of small orders

In section 2.3, we shall consider the graphs H with order ≤ 5 and $H \cong C_n$ or P_n for $n \leq 7$.

Proposition 2.3.1. A C_n-regular graph exists for $n = 3, 4, 5, 6, 7$.

Proof.

The followings are easy to check.

- $n = 3$ Tetrahedron is a C_3-regular graph.

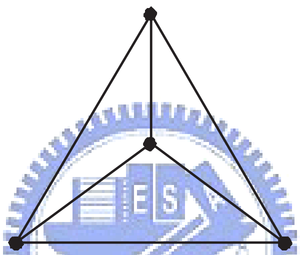


Figure 6: C_3-regular graph

- $n = 4$ Octahedron is a C_4-regular graph.

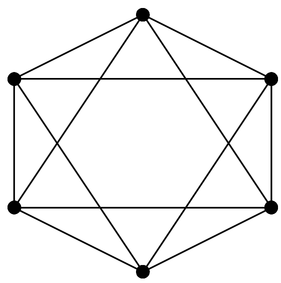


Figure 7: C_4-regular graph

• $n = 5$ Icosahedron is a C_5-regular graph.

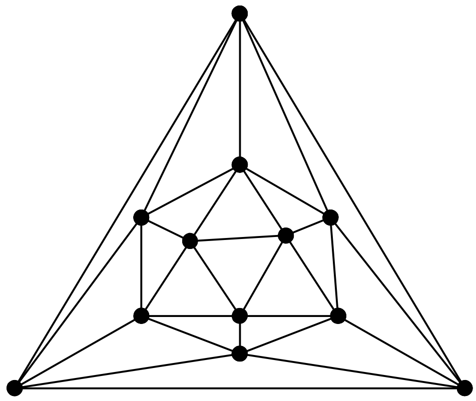


Figure 8: C_5-regular graph

• $n = 6$ $Ci_{12}(1, 2, 5)$ is an C_6-regular graph.

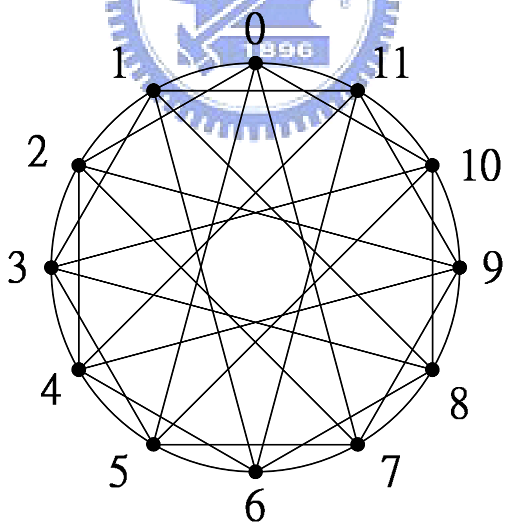
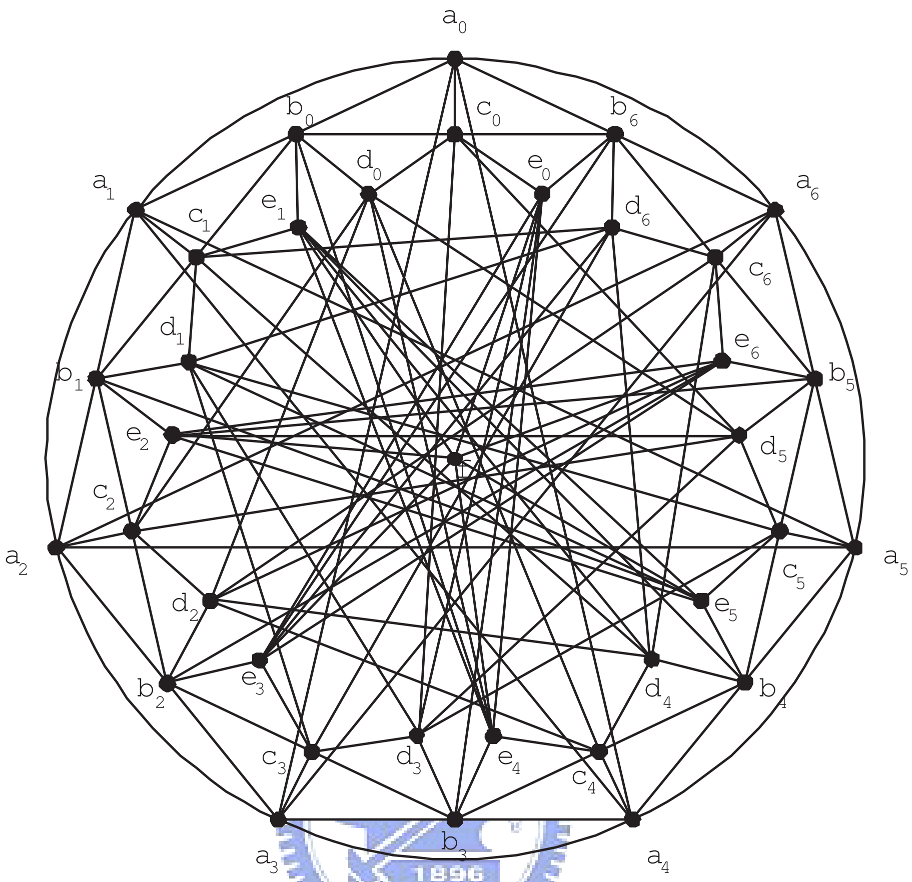


Figure 9: C_6-regular graph

- $n = 7$ $\qquad$ G is a C_7-regular graph.

G:



$V(G) = \{a_i, b_i, c_i, d_i, e_i, f \mid i \in Z_7\}$, Edges of G are :

$a_i \sim [a_{i+1}, a_{i+3}, a_{i+4}, a_{i+6}, b_i, c_i, b_{i+6}]$; $b_i \sim [a_i, c_i, c_{i+1}, a_{i+1}, d_i, e_{i+1}, e_{i+4}]$;

$c_i \sim [a_i, b_i, b_{i+6}, d_i, d_{i+3}, d_{i+5}, e_i]$; $d_i \sim [b_i, c_i, c_{i+2}, c_{i+4}, d_{i+2}, d_{i+5}, e_{i+4}]$;

$e_i \sim [b_{i+3}, b_{i+6}, c_i, d_{i+3}, e_{i+3}, e_{i+4}, f]$; $f \sim [e_i, e_{i+3}, e_{i+6}, e_{i+2}, e_{i+5}, e_{i+1}, e_{i+4}]$.

Note : $x \sim [\alpha_1, \alpha_2, \ldots, \alpha_k] =_{def} \{x \sim \alpha_i \mid i = 1, 2, \ldots, k\}$.

Figure 10: C_7-regular graph

Proposition 2.3.2. A P_n-regular graph exists for $n = 2, 4, 5, 6, 7$.

Proof.

The following is easy to check,

- $n = 2$ C_3 is a P_2-regular graph.
- $n = 3$ No P_3-regular graph, by Corollary 2.1.3.
- $n = 4$

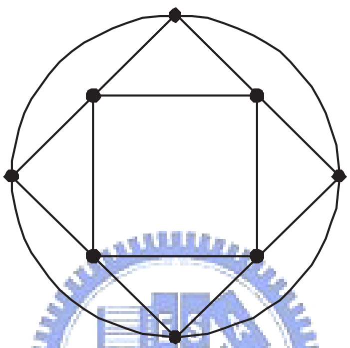


Figure 11: P_4-regular graph

- $n = 5$

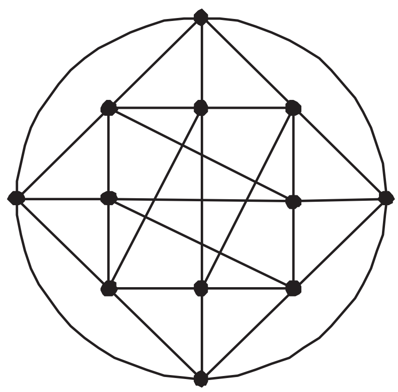
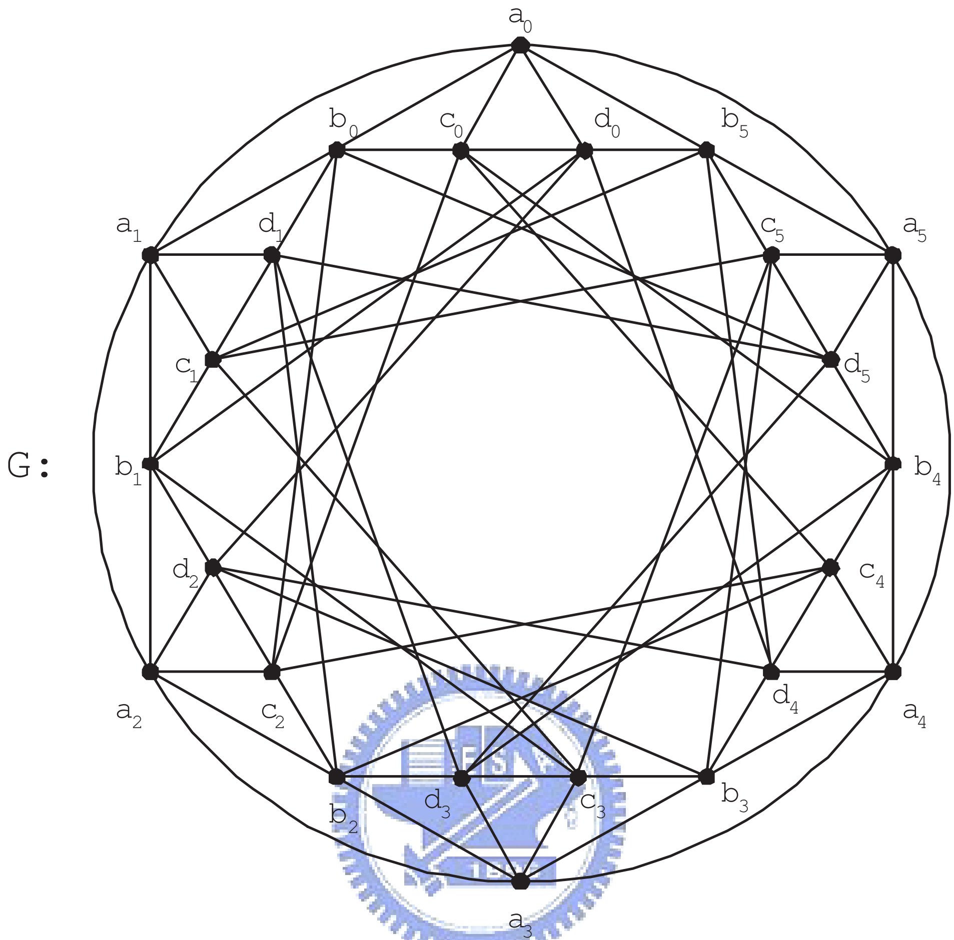


Figure 12: P_5-regular graph

• $n = 6$ $\quad$ G is a P_6-regular graph.



$V(G) = \{a_i, b_i, c_i, d_i |\ i \in Z_6\}$, Edges of G are :

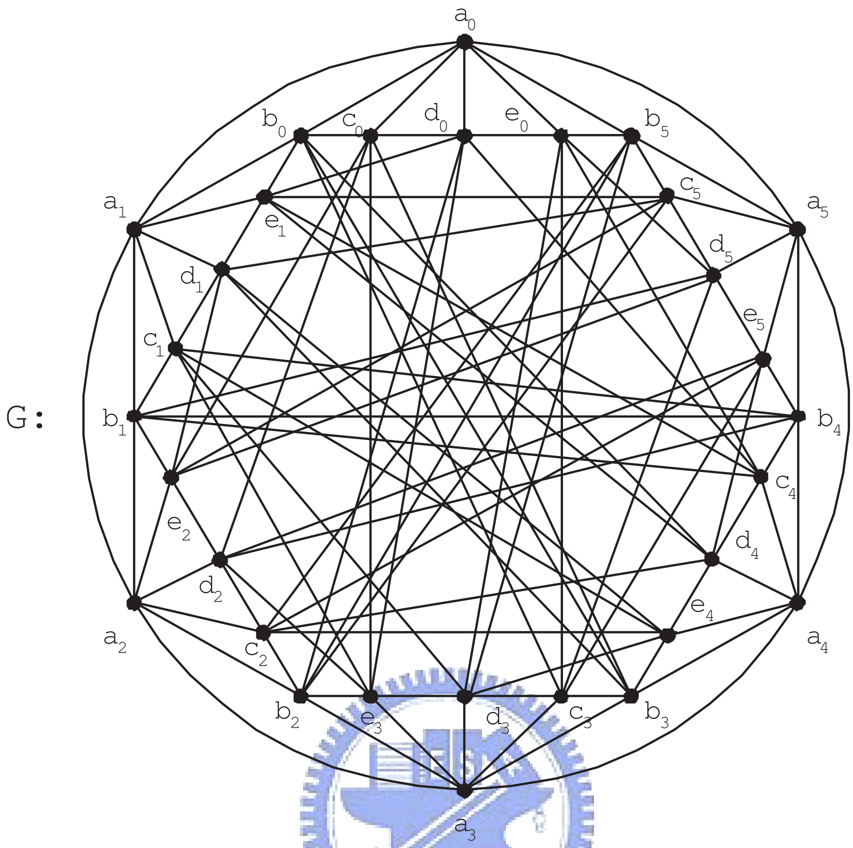
$a_i \sim [a_{i+1}, b_i, c_i, d_i, b_{i+5}, a_{i+5}]; b_i \sim [c_{i+2}, c_i, a_i, a_{i+1}, d_{i+1}, d_{i+5}];$

$c_i \sim [d_i, a_i, b_i, c_{i+2}, c_{i+4}, b_{i+4}]; d_i \sim [c_i, a_i, b_{i+5}, d_{i+4}, d_{i+2}, b_{i+1}].$

Note : $x \sim [\alpha_1, \alpha_2, \ldots, \alpha_k] =_{def} \{x \sim \alpha_i |\ i = 1, 2, \ldots, k\}.$

Figure 13: P_6-regular graph

- $n = 7$ $\qquad$ G is a P_7-regular graph.



$V(G) = \{a_i, b_i, c_i, d_i, e_i | \ i \in Z_6\}$, Edges of G are :

$a_i \sim [a_{i+1}, a_{i+3}, a_{i+4}, a_{i+6}, b_i, c_i, b_{i+6}]$; $b_i \sim [a_i, c_i, c_{i+1}, a_{i+1}, d_i, e_{i+1}, e_{i+4}]$;

$c_i \sim [a_i, b_i, b_{i+6}, d_i, d_{i+3}, d_{i+5}, e_i]$; $d_i \sim [b_i, c_i, c_{i+2}, c_{i+4}, d_{i+2}, d_{i+5}, e_{i+4}]$;

$e_i \sim [b_{i+3}, b_{i+6}, c_i, d_{i+3}, e_{i+3}, e_{i+4}, f]$; $f \sim [e_i, e_{i+3}, e_{i+6}, e_{i+2}, e_{i+5}, e_{i+1}, e_{i+4}]$.

Note : $x \sim [\alpha_1, \alpha_2, \ldots, \alpha_k] =_{def} \{x \sim \alpha_i | \ i = 1, 2, \ldots, k\}$.

Figure 14: P_7-regular graph

Proposition 2.3.3. For each graph of order 2, H, there exists an H-regular graph.

Proof.

Since H is of order 2, $H = P_2$ or O_2. The proof follows by letting the H-regular graphs be K_3 and C_4 respectively. $\square$

Proposition 2.3.4. There exists an H-regular graph for each graph H of order 3 except $H = P_3$.

Proof.

We consider the following cases.

- $H = O_3$

  $K_{3,3}$ is an O_3-regular graph.

- $H = P_2 \cup O_1$

  Since K_3 is an P_2-regular graph, by Lemma 2.2.5, $K_3 \square K_2$ is a $P_2 \cup O_1$-regular graph.

- $H = P_3$

  Because $P_3 = K_3 - P_2$, by Lemma 2.1.5, no P_3-regular graphs exist.

- $H = K_3$

  K_4 is a K_3-regular graph. $\square$

Proposition 2.3.5. There exists an H-regular graph for the graphs H of order 4 except $H = K_4 - P_2, K_4 - P_3, S_3$ or $P_3 \cup O_1$.

Proof.

We consider the following cases.

- $H = O_4$

  $K_{4,4}$ is a a O_4-regular graph.

- $H = P_2 \cup O_2$

Since $K_3 \square K_2$ is a $P_2 \cup O_1$-regular graph, by Lemma 2.2.5, $(K_3 \square K_2) \square K_2$ is a $P_2 \cup O_2$-regular graph.

- $H = M_2$

  $(K_3)^2$ is an M_2-regular graph. (Corollary 2.2.8)

- $H = C_3 \cup O_1$

  Since K_4 is a C_3-regular graph, by Lemma 2.2.5, $K_4 \square K_2$ is a $C_3 \cup O_1$-regular graph.

- $H = P_4$ or C_4

  By Proposition 2.3.1 and Proposition 2.3.2.

- $H = K_4$

  K_5 is a K_4-regular graph.

- $H = K_4 - P_2$ or $K_4 - P_3$

  By Lemma 2.1.5, no $(K_4 - P_2)$-regular graphs and $(K_4 - P_3)$-regular graphs exist.

- $H = S_3$ or $P_3 \cup O_1$

  By Corollary 2.1.3, no S_3-regular graphs and $(P_3 \cup O_1)$-regular graphs exist. □

Proposition 2.3.6 Let H be a graph of order 5. Then an H-regular graph exists if and only if $H = G_1, G_2, G_4, G_5, G_7, G_8, G_{10}, G_{13}, G_{14}, G_{20}, G_{21}, G_{24}, G_{25}, G_{34}$, see Figure 15. [2]

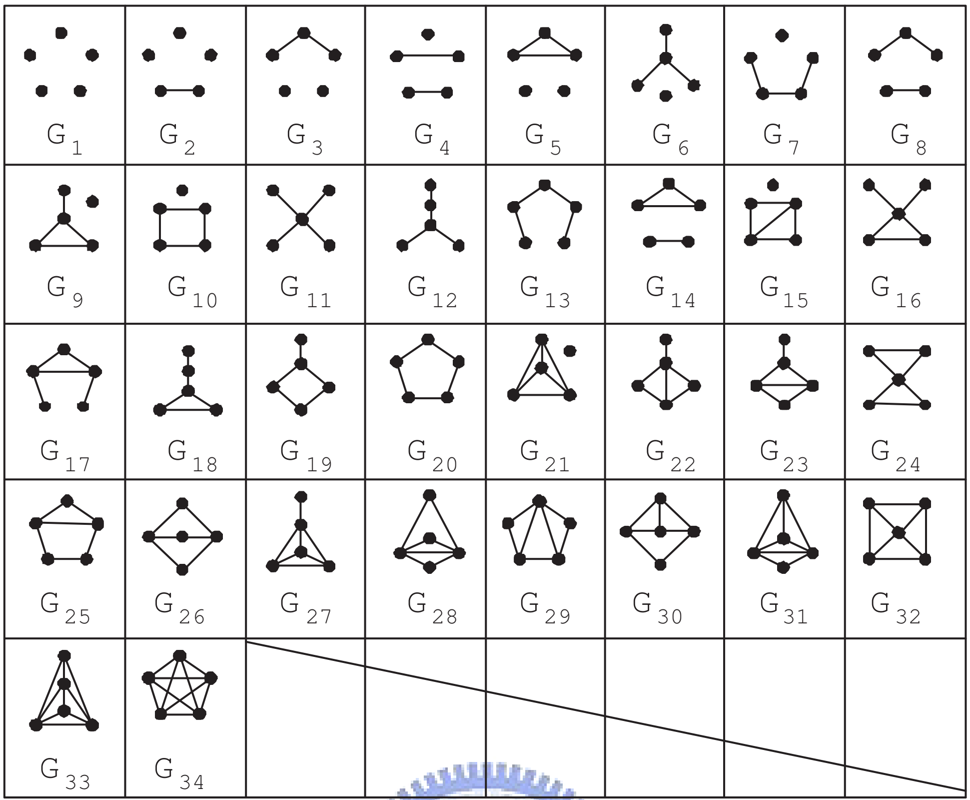


Figure 15: All graphs of order 5

Proof.

We consider the following cases.

- $H = G_1$ and G_{34}

  $K_{5,5}$ is a G_1-regular graph and K_6 is a G_{34}-regular graph.

- $H = G_2, G_4, G_5, G_7, G_{10}, G_{14}$ and G_{21}

  By Corollary 2.2.6, $G_2, G_4, G_5, G_7, G_{10}, G_{14}$ and G_{21}-regular graphs exist respectively.

- $H = G_8$

  See Figure 16, $(P_3 \cup P_2)$-regular graphs exist.

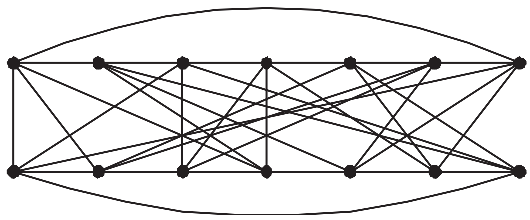


Figure 16: $(P_3 \cup P_2)$-regular graph

- $H = G_{13}$ and G_{20}

  By Proposition 2.3.1 and Proposition 2.3.2, G_{13} and G_{20}-regular graphs exist respectively.

- $H = G_{24}$ and G_{25}

  The graph $Ci_8(1,3,4)$ is a G_{24}-regular graph and the graph $Ci_8(1,2,4)$ is a G_{25}-regular graph.

- $H = G_3, G_6$ and G_{11}

  By Corollary 2.1.3, G_3, G_6 and G_{11} are forbidden graphs.

- $H = G_9, G_{15}, G_{29}, G_{31}$ and G_{33}

  By Lemma 2.1.6, $G_9, G_{15}, G_{29}, G_{31}$ and G_{33} are forbidden graphs.

- $H = G_{16}, G_{22}$ and G_{27}

  By Lemma 2.1.1, G_{16}, G_{22} and G_{27} are forbidden graphs.

- $H = G_{26}$

  Because G_{26} is $K_{3,2}$. By Lemma 2.1.7, no G_{26}-regular graphs exist.

- $H = G_{28}$

By Lemma 2.1.9, no G_{28}-regular graphs exist.

For the followings cases, we shall use similar technique to prove the nonexistence of an H-regular graph for $H = G_{17}, G_{18}, G_{19}, G_{23}, G_{30}$ and G_{32}. Since their proofs are similar, we show the proofs of the first two cases.

- $H = G_{17}$

  Let G be a G_{17}-regular graph and $v \in V(G)$ such that $G[N_G(v)] = G_{17}$. Let $N_G(v) = \{x, y, z, w, u\}$ such that $x \sim y$, $x \sim z$, $x \sim w$, $z \sim w$ and $w \sim u$. By assumption $G[N_G(z)] = G_{17}$, there exist two vertices p and q which are not in $N_G(v)$ such that $p \sim x$ and $q \sim y$. Now, consider $G[N_G(y)]$. Since $d_G(v)$ and $d_G(x)$ are of degree 5, $G[N_G(y)]$ is disconnected. Hence, $G[N_G(y)] \neq G_{17}$. This is a contradiction and thus G_{17} is forbidden.

- $H = G_{18}$

  Let G be a G_{18}-regular graph and $v \in V(G)$ such that $G[N_G(v)] = G_{18}$. Let $N_G(v) = \{x, y, z, w, u\}$ such that $x \sim y$, $x \sim w$, $x \sim u$, $y \sim z$ and $w \sim u$. By assumption $G[N_G(x)] = G_{18}$, there exists a vertex p which is not in $N_G(v)$ such that $p \sim x$ and $p \sim y$. Consider $G[N_G(y)]$. Since $d_G(v)$ and $d_G(x)$ are of degree 5, $G[N_G(y)] \neq G_{18}$. This is a contradiction. Hence, G_{18} is forbidden.

Since we have checked all cases of graphs of order 5, the proof is concluded. $\square$

2.4 Extremal problem

Let G be an H-regular graph. Then, we define $f(H)$ as the smallest order for all possible H-regular graphs. The following result is a trivial lower bound.

Lemma 2.4.1. If a graph G is an H-regular graph, then $f(H) \geq 2|V(H)| - \delta(H)$.

Proof.

$$f(H) \geq |V(H)| + 1 + (|V(H)| - (\delta(H) + 1)) = 2|V(H)| - \delta(H). \qquad \square$$

Corollary 2.4.2. $f(C_3) = 4$, $f(C_4) = 6$, $f(P_2) = 3$, $f(P_4) = 7$.

Proof.

By Lemma 2.4.1, $f(C_3) \geq 4$, $f(C_4) \geq 6$, $f(P_2) \geq 3$, and $f(P_4) \geq 7$. Since K_4 is a C_3-regular graph, $Ci_6(1,2)$ is a C_4-regular graph, K_3 is a P_2-regular graph and $Ci_7(1,2)$ is a P_4-regular graph, $f(C_3) \leq 4$, $f(C_4) \leq 6$, $f(P_2) \leq 3$ and $f(P_4) \leq 7$ respectively. This concludes the proof. $\square$

Proposition 2.4.3. $f(O_t) = 2t$, for each $t \geq 1$.

Proof.

By Lemma 2.4.1, $f(O_t) \geq 2t$, and $K_{t,t}$ is an O_t-regular graph, hence $f(O_t) = 2t$. $\square$

Proposition 2.4.4. $f(K_n) = n + 1$, for each $n \geq 1$.

Proof.

By Lemma 2.4.1, $f(K_n) \geq n + 1$, and K_{n+1} is K_n-regular graph, hence $f(K_n) = n + 1$. $\square$

3 Concluding Remark

The study of neighbor-regular graphs has just begun. So far, not much is known. In this thesis, we manage to obtain several classes of graphs which are forbidden and for quite a few graphs H we construct an H-regular graph. But, we also realize the difficulty of obtaining general results. For example, we can construct H-regular graphs for $H = C_n$ or P_n whenever $n \leq 7$. How about $n \geq 8$? On the other hand, we are able to say something about forbidden graphs, but there are quite a few forbidden graphs remained undiscovered. To conclude this thesis, we would like to pose a conjecture on finding forbidden graphs.

Conjecture. Let H be a tree which is not a path. Then H is a forbidden graph.

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