1. Introduction

The first variational formula of the volume for a hypersurface M^n in the Euclidean space $\mathbb{R}^{n+1}$ is well-known. The critical points are called minimal hypersurfaces, and the associated flow is called the mean curvature flow [1]. This classical formula has been generalized by Pinl and Trapp[4], Reilly and the references cited therein [5]. Reilly derived the first variational formula of the functional of the elementary symmetric polynomials of the principal curvatures for hypersurfaces in the space forms.

In this Master's dissertation we begin by deriving the first variational formula of the functional of the traces of the second fundamental form for hypersurfaces in the Euclidean space. Let $X : M^n \to \mathbb{R}^{n+1}$ be a compact immersed hypersurface in the (n+1)-dimensional Euclidean space. Denote by $A = (h_{ij})$ the second fundamental form of M^n and by trA^m the trace of A^m, $1 \leq m \leq n$. For a given C^3 function f defined on $\mathbb{R}^n$, we consider the functional J of X,

$$J(X) = \int_M f(trA, trA^2, ..., trA^n)\ dv,$$

where the integration is with respect to the volume measure dv of M^n. Based on the Newton's formula, this functional is close related to the functional that given by Reilly [5]. We find the first variational formula of J as follows

Theorem 1.1.

$$J_t = \int_M \tau\ X_t \cdot N\ dv,$$

where

$$\begin{aligned}
\tau \ &= \ \Sigma l f_{lkm}[(trA^m)_i(trA^k)_j A_{ij}^{l-1}] + \Sigma f_{lk} l[(trA^k)_{ij} A_{ij}^{l-1} + 2(trA^k)_j \sum_{p=1}^{l-1} \frac{1}{p}(trA^p)_i A_{ij}^{l-1-p}] \\
&\quad + \Sigma f_l l[\sum_{p=1}^{l-1} \frac{1}{p}(trA^p)_{ij} A_{ij}^{l-1-p} + \sum_{p=1}^{l-1} \sum_{q=1}^{l-1-p} \frac{1}{pq}(trA^p)_i (trA^q)_j A_{ij}^{l-1-p-q} + trA^{l+1}] \\
&\quad - Hf
\end{aligned}$$

, H=trA and $(A^m)_{ij} = h_{ii_2} h_{i_2 i_3} ...... h_{i_{m-1} i_m} h_{i_m j}$.

For a given immersion $X_0 : M^n \to \mathbb{R}^{n+1}$, we consider a smooth variation $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ of X_0, then the L^2 gradient flow for this functional is a geometric evolution equation

$$X_t = -\tau N$$

with the initial condition $X|_{t=0} = X(0)$.

Let ϕ_{ij} be the trace free tensor of the second fundamental form, $\phi_{ij} = h_{ij} - \frac{H}{n}\delta_{ij}$, and $\Phi = \Sigma\phi_{ij}^2$. Then the following functional is a natural generalization of the classical Willmore functional

$$W(X) = \int_M \Phi^{\frac{n}{2}}\ dv$$

[3]. This functional is invariant under conformal mappings of $\mathbb{R}^{n+1}$. In section 3, we try to find functionals in the type of $\int_M f(trA, ...trA^n)\, dv$ which are invariant under conformal transformations of $\mathbb{R}^{n+1}$. We show that if f satisfies two first order partial equations, the translation equation

$$\sum_{m=0}^{n} m x_{m-1} f_m = 0,$$

where $x_0 = n$, and the homogeneity equation

$$\sum_{m=1}^{n} m x_m f_m = nf,$$

then the functional is invariant under all conformal transformations of $\mathbb{R}^{n+1}$. By solving these equations, we construct several explicit functionals which are invariant under all conformal transformations of $\mathbb{R}^{n+1}$. For n being even, or n being large odd, there are interesting examples.

In final section, section 4, using the general formula of Theorem 1.1, we find the first variational formula of a generalized Willmore functional as follows

$$\begin{aligned}[\int_M \Phi^{\frac{n}{2}}\ dv]_t \ &= \int_M \Phi^{\frac{n}{2}-3}\{(n-1)\Phi^2\Delta H + \Phi^3 H + n\Phi^2 \sum \phi_{ij}\phi_{jk}\phi_{ki} \\ &+(n-1)(n-2)\Phi \sum H_i\Phi_i + \frac{n(n-2)(n-4)}{4}\sum \phi_{ij}\Phi_i\Phi_j \\ &+\frac{n(n-2)}{2}\Phi \sum \phi_{ij}\Phi_{ij}\}(X_t \cdot N)\ dv,\end{aligned}$$

for all $n \neq 3, 5$. A similar formula for hypersurfaces in the sphere was given by Li [2].

2. The first variational formula

Let $X : M^n \to \mathbb{R}^{n+1}$ be an immersiom of a compact surface M^n. We shall use the following ranges of indices

$$1 \le i, j, k, l; i_1, i_2, ..., j_1, j_2, ... \le n.$$

Then in terms of an orthonormal basis $\{e_1, ..., e_n\}$ on the tangent bundle TM and its dual coframe $\{\omega_1, ...\omega_n\}$, the structure equations can be written as

$$\begin{aligned} dX &= \omega_i e_i, \\ de_i &= \omega_{ij} e_j + \omega_{i,n+1} N, \\ dN &= \omega_{n+1,i} e_i, \end{aligned}$$

where N is unit normal of X, ω_{ij} is the connection form of M^n, $\omega_{ij} + \omega_{ji} = 0$ and $\omega_{n+1,i} = -h_{ij}\omega_j$; $h_{ij} = h_{ji}$, where (h_{ij}) is the second fundamental form of M^n.

Let $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth variation of an initial immersion X_0. We denote by $\bar{d} = d + \frac{\partial}{\partial t} dt$ the differential operator on $M^n \times [0, T)$, then we have

$$\bar{d}X = \Omega_i e_i + aN, \tag{2.1}$$

$$\bar{d}e_i = \Omega_{ij} e_j + \Omega_{i,n+1} N, \tag{2.2}$$

$$\bar{d}N = \Omega_{n+1,i} e_i + \Omega N, \tag{2.3}$$

where Ω_{ij} is the connection form of $M^n \times [0, T), \Omega_{ij} + \Omega_{ji} = 0$.

On the other hand, the structure equations are given by

$$\bar{d}X = \omega_i e_i + X_t dt, \tag{2.4}$$

$$\bar{d}e_i = \omega_{ij} e_j + \omega_{i,n+1} N + (e_i)_t dt, \tag{2.5}$$

$$\bar{d}N = \omega_{n+1,i} e_i + N_t dt. \tag{2.6}$$

Comparing (2.1) with (2.4), we get

$$\begin{aligned} a &= X_t \cdot N dt, \\ \Omega_i &= \omega_i + X_t \cdot e_i dt. \end{aligned}$$

Similarly, comparing (2.2) with (2.5), and (2.3) with (2.6), we have

$$\begin{aligned} \Omega &= 0, \\ \Omega_{ij} &= \omega_{ij} + (e_i)_t \cdot e_j dt, \\ \Omega_{i,n+1} &= \omega_{i,n+1} + (e_i)_t \cdot N dt. \end{aligned}$$

For simplicity we write these relations as

$$\bar{d}\begin{pmatrix} X \\ e_1 \\ \vdots \\ e_n \\ N \end{pmatrix} = \begin{pmatrix} 0 & \Omega_1 & \cdots & \Omega_n & a \\ 0 & \Omega_{11} & \cdots & \Omega_{1n} & \Omega_{1,n+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \Omega_{n1} & \cdots & \Omega_{nn} & \Omega_{n,n+1} \\ 0 & \Omega_{n+1,1} & \cdots & \Omega_{n+1,n} & 0 \end{pmatrix} \begin{pmatrix} X \\ e_1 \\ \vdots \\ e_n \\ N \end{pmatrix}.$$

If we let

$$\omega = \begin{pmatrix} 0 & \Omega_1 & \cdots & \Omega_n & a \\ 0 & \Omega_{11} & \cdots & \Omega_{1n} & \Omega_{1,n+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \Omega_{n1} & \cdots & \Omega_{nn} & \Omega_{n,n+1} \\ 0 & \Omega_{n+1,1} & \cdots & \Omega_{n+1,n} & 0 \end{pmatrix},$$

and taking differential $\bar{d}$ to both sides,then we get

$$0 = \bar{d}\bar{d}\begin{pmatrix} X \\ e_1 \\ \vdots \\ e_n \\ N \end{pmatrix} = \bar{d}\omega \begin{pmatrix} X \\ e_1 \\ \vdots \\ e_n \\ N \end{pmatrix} - \omega \wedge \bar{d}\begin{pmatrix} X \\ e1 \\ \vdots \\ e_n \\ N \end{pmatrix}.$$

It follows that

$$\bar{d}\omega = \omega \wedge \omega.$$

Hence, we get

$$\begin{aligned} \bar{d}a &= \Omega_i \wedge \Omega_{i,n+1}, \\ \bar{d}\Omega_j &= \Omega_i \wedge \Omega_{ij} + a \wedge \Omega_{n+1,j}, \end{aligned}$$

where $1 \leq j \leq n$, and

$$\bar{d}\Omega_{ij} = \sum_{k=1}^{n+1} \Omega_{ik} \wedge \Omega_{kj},$$

where $1 \leq i, j \leq n+1$.

Since $dX_t = (dX)_t$, we compute dX_t and $(dX)_t$,

$$\begin{aligned} dX_t &= d((X_t \cdot e_i)e_i + (X_t \cdot N)N) \\ &= d(X_t \cdot e_i)e_i + (X_t \cdot e_i)(\omega_{ij}e_j + \omega_{i,n+1}N) + d(X_t \cdot N)N + (X_t \cdot N)\omega_{n+1,i}e_i \\ &= (d(X_t \cdot e_i) + (X_t \cdot e_j)\omega_{ji} + (X_t \cdot N)\omega_{n+1,i})e_i + (d(X_t \cdot N) + (X_t \cdot e_i)\omega_{n+1,i}))N. \end{aligned}$$

We define $(X_t \cdot e_i)_j$ and $(X_t \cdot N)_j$ by

$$d(X_t \cdot e_i) + (X_t \cdot e_j)\omega_{ji} = (X_t \cdot e_i)_j \omega_j,$$

and

$$d(X_t \cdot N) = (X_t \cdot N)_j \omega_j.$$

Then we have

(2.7)

$$dX_t = ((X_t \cdot e_i)_j \omega_j + (X_t \cdot N)\omega_{n+1,i})e_i + ((X_t \cdot N)_j \omega_j + (X_t \cdot e_i)\omega_{i,n+1})N.$$

On the other hand,

(2.8)
$$(dX)_t = (\omega_i)_t e_i + \omega_i (e_i)_t.$$

From (2.7), (2.8) and

$$\begin{aligned} d(X_t) \cdot e_i &= (dX)_t \cdot e_i, \\ d(X_t) \cdot N &= (dX)_t \cdot N. \end{aligned}$$

We get

$$\begin{aligned} (e_i)_t \cdot N &= (X_t \cdot N)_i + \sum h_{ij} X_t \cdot e_j, \\ (\omega_i)_t &= [(X_t \cdot e_i)_j - X_t \cdot N h_{ij}]\omega_j. \end{aligned}$$

And

(2.9)

$$\begin{aligned} (\omega_1 \wedge ...... \wedge \omega_n)_t &= \sum \omega_1 \wedge ...(\omega_i)_t ... \wedge \omega_n \\ &= ((X_t \cdot e_i)_i - H X_t \cdot N)\omega_1 \wedge ...... \wedge \omega_n. \end{aligned}$$

Similarly, since

$$\bar{d}\Omega_{i,n+1} = \Omega_{ij} \wedge \Omega_{j,n+1},$$

$$\begin{aligned}
\bar{d}\Omega_{i,n+1} &= \bar{d}(\omega_{i,n+1} + (e_i)_t \cdot N dt) \\
&= \bar{d}(h_{ij}\omega_j + (e_i)_t \cdot N dt) \\
&= (dh_{ij} \wedge \omega_j + h_{ij}\omega_{jk} \wedge \omega_k) + d((e_i)_t \cdot N) \wedge dt \\
&\quad +(h_{ij})_t dt \wedge \omega_j - h_{ij}(\omega_j)_t \wedge dt \\
&= (dh_{ij} \wedge \omega_j + h_{ik}\omega_{kj} \wedge \omega_j) + d((e_i)_t \cdot N) \wedge dt + (h_{ij})_t dt \wedge \omega_j \\
&\quad -h_{ij}((X_t \cdot e_j)_k \omega_k + (e_j)_t \cdot e_k \omega_k + X_t \cdot N \omega_{n+1,j}) \wedge dt,
\end{aligned}$$

and

$$\begin{aligned}
\sum_j \Omega_{ij} \wedge \Omega_{j,n+1} &= [\omega_{ij} + (e_i)_t \cdot e_j dt] \wedge [\omega_{j,n+1} + (e_j)_t \cdot N dt] \\
&= \omega_{ij} \wedge \omega_{j,n+1} + (e_j)_t \cdot N \omega_{ij} \wedge dt - (e_i)_t \cdot e_j \omega_{j,n+1} \wedge dt \\
&= h_{jk}\omega_{ij} \wedge \omega_k + (e_j)_t \cdot N \omega_{ij} \wedge dt - (e_i)_t \cdot e_j h_{jk} \omega_k \wedge dt,
\end{aligned}$$

we have

$$\begin{aligned}
&(dh_{ij} + h_{kj}\omega_{ki} + h_{ik}\omega_{kj}) \wedge \omega_j + [d((e_i)_t \cdot N) + (e_j)_t \cdot N \omega_{ji}] \wedge dt - (h_{ij})_t \omega_j \wedge dt \\
&+[-h_{ik}(X_t \cdot e_k)_j - h_{ik}(e_k)_t \cdot e_j + X_t \cdot N h_{ik} h_{kj} + (e_i)_t \cdot e_k h_{jk}]\omega_j \wedge dt = 0.
\end{aligned}$$

We define the covariant derivative of h_{ij} as follows:

$$(dh_{ij} + h_{kj}\omega_{kj} + h_{ik}\omega_{kj}) \wedge \omega_j = \sum h_{ijk}\omega_k,$$

where $h_{ijk} = h_{ikj}$, and define $((e_i)_t \cdot N)_j$ by

$$d((e_i)_t \cdot N) + (e_j)_t \cdot N \omega_{ji} =: ((e_i)_t \cdot N)_j \omega_j.$$

So we get

$$
\begin{aligned}
- & \quad (h_{ij})_t + ((e_i)_t \cdot N)_j - h_{ik}(X_t \cdot e_k)_j - h_{ik}(e_k)_t \cdot e_j \\
- & \quad h_{kj}(e_k)_t \cdot e_i + X_t \cdot N h_{ik} h_{kj} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
- & \quad (h_{ij})_t + (X_t \cdot N)_{ij} + h_{ikj} X_t \cdot e_k + h_{ik}(X_t \cdot e_k)_j \\
- & \quad h_{ik}(X_t \cdot e_k)_j - h_{ik}(e_k)_t \cdot e_j - h_{kj}(e_k)_t \cdot e_i + X_t \cdot N h_{ik} h_{kj} = 0.
\end{aligned}
$$

Hence we have

(2.10)

$$
\begin{aligned}
(h_{ij})_t &= (X_t \cdot N)_{ij} + h_{ijk} X_t \cdot e_k - h_{ik}(e_k)_t \cdot e_j \\
&\quad - h_{kj}(e_k)_t \cdot e_i + X_t \cdot N h_{ik} h_{jk}.
\end{aligned}
$$

Let f be a C^3 function, by using (2.9) and (2.10), now we show the derivative of the functional of X, $J(X)$, with respect to the time variable t.

$$
\begin{aligned}
J_t &= [\int_M f \; dv]_t \\
&= \int_M f_l (trA^l)_t \; dv + \int_M f \; (dv)_t \\
&= \int_M f_l [l (h_{ij})_t A_{ji}^{l-1}] \; dv + \int_M f[(X_t \cdot e_i)_i - H X_t \cdot N)] \; dv \\
&= \int_M \{ l f_l [-h_{ik}(e_k)_t \cdot e_j - h_{kj}(e_k)_t \cdot e_i] A_{ji}^{l-1} + [l f_l h_{ijk} A_{ji}^{l-1} X_t \cdot e_k + f(X_t \cdot e_i)_i] \\
&\quad + l f_l A_{ji}^{l-1} (X_t \cdot N)_{ij} + (l f_l tr A^{l+1} - f H) \} X_t \cdot N \; dv.
\end{aligned}
$$

Our computation is complete through the following three steps.

(1) We claim that

$$\int_M lf_l[-h_{ik}(e_k)_t \cdot e_j - h_{kj}(e_k)_t \cdot e_i]A_{ji}^{l-1}\ dv = 0.$$

By changing the indices,

$$\begin{aligned}(h_{ik}(e_k)_t \cdot e_j)A_{ji}^{l-1} &= (h_{ik}(e_k)_t \cdot e_j)h_{ji_2}h_{i_2i_3}h_{i_3i_4}......h_{i_{l-2}i_{l-1}}h_{i_{l-1}i} \\ &= (h_{j_2j}(e_j)_t \cdot e_k)h_{ki}h_{ij_{l-1}}h_{j_{l-1}j_{l-2}}......h_{j_4j_3}h_{j_3j_2} \\ &= (-h_{ik}(e_k)_t \cdot e_j)h_{jj_2}h_{j_2j_3}......h_{j_{l-1}i} \\ &= -(h_{ik}(e_k)_t \cdot e_j)A_{ji}^{l-1}.\end{aligned}$$

We get

$$(h_{ik}(e_k)_t \cdot e_j)A_{ji}^{l-1} = 0,$$

and hence

$$\int_M lf_l[-h_{ik}(e_k)_t \cdot e_j - h_{kj}(e_k)_t \cdot e_i]A_{ji}^{l-1}\ dv = 0.$$

(2) We claim that

$$\int_M [lf_lh_{ijk}A_{ji}^{l-1}X_t \cdot e_k + f(X_t \cdot e_i)_i]\ dv = 0.$$

Apply the Stokes' formula,

$$\begin{aligned}\int_M f(X_t \cdot e_k)_k\ dv &= -\int_M f_l(trA^l)_k(X_t \cdot e_k)\ dv \\ &= -\int_M f_l(lh_{ijk}A_{ji}^{l-1})X_t \cdot e_k\ dv,\end{aligned}$$

and hence

$$\int_M [lf_lh_{ijk}A_{ji}^{l-1}X_t \cdot e_k + f(X_t \cdot e_i)_i]\ dv = 0.$$

(3) We want to evaluate the term

$$\int_M lf_l A_{ij}^{l-1}(X_t \cdot N)_{ij}\ dv.$$

Apply the Stokes' formula again,

$$\begin{aligned} & \int_M lf_l A_{ij}^{l-1}(X_t \cdot N)_{ij}\ dv \\ = & \int_M l[f_{lk}(trA^k)_j A_{ij}^{l-1} + f_l A_{ij,j}^{l-1}]_i (X_t \cdot N)\ dv \\ = & \int_M \{l[(trA^m)_i (trA^k)_j A_{ij}^{l-1}]f_{lkm} + l[(trA^k)_{ij} A_{ij}^{l-1} \\ & +2(trA^k)_j A_{iji}^{l-1}]f_{lk} + l[A_{ijji}^{l-1}]f_l\}(X_t \cdot N)\ dv. \end{aligned}$$

Note that

$$\begin{aligned} & 2(trA^k)_j A_{iji}^{l-1} \\ = & 2(trA^k)_j [h_{ii_2i}h_{i_2i_3}...h_{i_{l-1}j} + h_{ii_2}h_{i_2i_3i}h_{i_3i_4}...h_{i_{l-1}j} + ... + h_{ii_2}h_{i_2i_3}...h_{i_{l-1}ji}] \\ = & 2(trA^k)_j [(h_{iii_2})h_{i_2i_3}...h_{i_{l-1}j} + (h_{ii_2}h_{i_2ii_3})h_{i_3i_4}...h_{i_{l-1}j} + ... + (h_{ii_2}h_{i_2i_3}...h_{i_{l-1}ij})] \\ = & 2(trA^k)_j \sum_{p=1}^{l-1}\frac{1}{p}(trA^p)_i A_{ij}^{l-1-p}, \end{aligned}$$

and

$$\begin{aligned} A_{ijji}^{l-1} &= A_{ijij}^{l-1} = (\sum_{p=1}^{l-1}\frac{1}{p}(trA^p)_i A_{ij}^{l-1-p})_j \\ &= \sum_{p=1}^{l-1}\frac{1}{p}(trA^p)_{ij}A_{ij}^{l-1-p} + \sum_{p=1}^{l-1}\sum_{q=1}^{l-1-p}\frac{1}{pq}(trA^p)_i(trA^q)_j A_{ij}^{l-1-p-q}. \end{aligned}$$

Hence

$$\begin{aligned}
&\int_M \ l f_l A_{ij}^{l-1}(X_t \cdot N)_{ij} \ dv \\
= &\int_M \{l[(trA^m)_i(trA^k)_j A_{ij}^{l-1}] f_{lkm} + l[(trA^k)_{ij}A_{ij}^{l-1} + 2(trA^k)_j \sum_{p=1}^{l-1}\frac{1}{p}(trA^p)_i A_{ij}^{l-1-p}] f_{lk} \\
&+ l[\sum_{p=1}^{l-1}\frac{1}{p}(trA^p)_{ij}A_{ij}^{l-1-p} + \sum_{p=1}^{l-1}\sum_{q=1}^{l-1-p}\frac{1}{pq}(trA^p)_i(trA^q)_j A_{ij}^{l-1-p-q}] f_l\}(X_t \cdot N) \ dv.
\end{aligned}$$

From above three steps, finally we get the first variational formula

$$\begin{aligned}
[\int_M \ f \ dv]_t \ = \ &\int_M \{\Sigma l f_{lkm}[(trA^m)_i(trA^k)_j A_{ij}^{l-1}] + \Sigma f_{lk} l[(trA^k)_{ij}A_{ij}^{l-1} + 2(trA^k)_j \sum_{p=1}^{l-1}\frac{1}{p}(trA^p)_i A_{ij}^{l-1-p}] \\
&+ \Sigma f_l l[\sum_{p=1}^{l-1}\frac{1}{p}(trA^p)_{ij}A_{ij}^{l-1-p} + \sum_{p=1}^{l-1}\sum_{q=1}^{l-1-p}\frac{1}{pq}(trA^p)_i(trA^q)_j A_{ij}^{l-1-p-q} + trA^{l+1}] \\
&- Hf\}(X_t \cdot N) \ dv.
\end{aligned}$$

3. Conformally invariant functionals

Assume that $\overline{X} : M^n \to \mathbb{R}^{n+1}$ is conformal to $X : M^n \to \mathbb{R}^{n+1}$. Let $\{\overline{e_1}, ..., \overline{e_n}\}$ be an orthonormal basis, and $\{\overline{\omega_1}, ..., \overline{\omega_n}\}$, the dual coframe. Then $\overline{e_i} = \frac{1}{\rho} e_i$, and $\overline{\omega_i} = \rho\omega_i$. for some positive function ρ, $1 \le i \le n$. The volume forms of $\overline{X}$ and X are related by

$$d\overline{v} = \overline{\omega_1} \wedge ... \wedge \overline{\omega_n} = \rho^n \omega_1 \wedge ... \wedge \omega_n = \rho^n dv.$$

Since,

(3.1)

$$\begin{aligned}
d\overline{\omega_i} &= d(\rho\omega_i) \\
&= d\rho \wedge \omega_i + \rho d\omega_i \\
&= \rho_j\omega_j \wedge \omega_i + \rho\omega_{ij} \wedge \omega_j \\
&= (\rho\omega_{ij} - \rho_j\omega_i + \rho_i\omega_j) \wedge \omega_j,
\end{aligned}$$

and

$$d\overline{\omega_i} = \overline{\omega_{ij}} \wedge \overline{\omega_j} = \rho\overline{\omega_{ij}} \wedge \omega_j. \tag{3.2}$$

Comparing (3.1) with (3.2), the connection forms $\overline{\omega_{ij}}$ and ω_{ij} are related by

$$\overline{\omega_{ij}} = \omega_{ij} + (\log\rho)_i\omega_j - (\log\rho)_j\omega_i.$$

On the other hand,

$$\begin{aligned}
\overline{\omega_{i,n+1}} &= \omega_{i,n+1} - (\log\rho)_{n+1}\omega_i \\
&= h_{ij}\omega_j - (\log\rho)_{n+1}\delta_{ij}\omega_j.
\end{aligned}$$

We then have the relation between h_{ij} and $\overline{h_{ij}}$,

$$\overline{h_{ij}} = \frac{1}{\rho}(h_{ij} - \lambda\delta_{ij}),$$

and hence

$$\overline{A} = [\overline{h_{ij}}] = \frac{1}{\rho}(A - \lambda I),$$

where $\lambda = (\log\rho)_{n+1}$ and I is the identity matrix.

We consider the transformation, from A maps to $\overline{A}$, as the action which is a compostion of a translation and a multiplication. If a C^3 function f, satisfies

$$f(trA, trA^2, ...trA^n) = \rho^n f(tr\overline{A}, tr\overline{A}^2, ...tr\overline{A}^n),$$

then the corresponding functional is invariant under conformal transformations of $\mathbb{R}^{n+1}$,

$$\int_M f(trA, trA^2, ...trA^n)\ dv = \int_M f(tr\overline{A}, tr\overline{A}^2, ...tr\overline{A}^n)\ d\overline{v}.$$

We need the following two conditions:

(1) The homogeneity condition:

Let $\overline{A} = \frac{1}{\rho}A$, where ρ is positive. If $\rho^n f(tr\overline{A}, ...tr\overline{A}^n) = f(trA, trA^2, ...trA^n)$ for all positive ρ, then we have

$$\begin{aligned} 0 &= (\rho^n f(tr\overline{A}, ...tr\overline{A}^n))_\rho \\ &= nf(tr\overline{A}, ..., tr\overline{A}^n)\rho^{n-1} \\ &\quad +\rho^n \sum_{m=1}^{n} f_m(tr\overline{A}, ..., tr\overline{A}^n)(-\frac{k}{\rho})tr\overline{A}^m. \end{aligned}$$

We get

$$nf(x_1...x_n) = \sum_{m=1}^{n} mx_m f_m(x_1...x_n). \tag{3.3}$$

(2) The translation condition:

Let $\overline{A} = A - \lambda I$. If $f(tr\overline{A}, ...tr\overline{A}^n) = f(trA, trA^2, ...trA^n)$, for all λ, then we have

$$\begin{aligned} 0 &= (f(tr\overline{A}, ...tr\overline{A}^n))_\lambda \\ &= \sum_m f_m(tr\overline{A}, ...tr\overline{A}^n)(tr\overline{A}^m)_\lambda \\ &= -\sum_m mf_m(tr\overline{A}, ...tr\overline{A}^n)(tr\overline{A}^{m-1}). \end{aligned}$$

Thus we obtain

$$\sum_{m=1}^{n} mx_{m-1} f_m(x_1...x_n) = 0, \tag{3.4}$$

where $x_0 = n$.

We now solve the homogeneity equation (3.3) and the translation equation (3.4).

For the special case $n = 2$, the homogeneity equation and the translation equation are given by

$$\begin{aligned} x_1 f_1 + 2x_2 f_2 &= 2f, \\ 2f_1 + 2x_1 2f_1 &= 0. \end{aligned}$$

These partial differential equations can be written as

$$\begin{pmatrix} x_1 & 2x_2 \\ 2 & 2x_1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 2f \\ 0 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \frac{1}{2x_1^2 - 4x_2} \begin{pmatrix} 2x_1 & -2x_2 \\ -2 & x_1 \end{pmatrix} \begin{pmatrix} 2f \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} (\log f)_1 &= \frac{2x_1}{x_1^2 - 2x_2} = (\log |x_1^2 - 2x_2|)_1, \\ (\log f)_2 &= \frac{-2}{x_1^2 - 2x_2} = (\log |x_1^2 - 2x_2|)_2. \end{aligned}$$

This implies

$$f(x_1, x_2) = c(x_2 - \frac{1}{2}x_1^2),$$

for constant c.

We conclude that in the case n=2,

$$f(trA, trA^2) = c(trA^2 - \frac{1}{2}(trA)^2),$$

for constant c.

For the case, $n = 3$, the homogeneity equation and the translation equation are given by

$$x_1 f_1 + 2x_2 f_2 + 3x_3 f_3 = 3f, \tag{3.5}$$

$$3f_1 + 2x_1 f_2 + 3x_2 f_3 = 0. \tag{3.6}$$

Consider he characteristic curve of (3.5) and (3.6)

$$\begin{cases} x_1(t) = 3t, \\ x_2(t) = 3t^2 + c_2, \\ x_3(t) = 3t^3 + 3c_2 t + c_3. \end{cases} \tag{3.7}$$

Then we have

$$\begin{aligned} \frac{d}{dt} f(x_1(t), x_2(t), x_3(t)) &= f_1 x_1'(t) + f_2 x_2'(t) + f_3 x_3'(t) \\ &= 3f_1 + 2x_1 f_2 + 3x_2 f_3 \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} f(x_1(t), x_2(t), x_3(t)) &= f(0, c_2, c_3) \\ &= u(c_1, c_2), \end{aligned}$$

for all t.

From (3.7), we get

$$\begin{aligned} t &= \frac{x_1}{3}, \\ c_2 &= x_2 - \frac{x_1^2}{3}, \\ c_3 &= x_3 + \frac{2}{9}x_1^3 - x_1x_2. \end{aligned}$$

Hence we have

$$f(x_1, x_2, x_3) = u(x_2 - \frac{x_1^2}{3}, x_3 + \frac{2}{9}x_1^3 - x_1x_2). \tag{3.8}$$

By substituting (3.8) into (3.5), we get

$$\begin{aligned} & 3u(x_2 - \frac{x_1^2}{3}, x_3 + \frac{2}{9}x_1^3 - x_1x_2) \\ = \; & 2(x_2 - \frac{x_1^3}{3})u_1 + 3(x_3 + \frac{2}{9}x_1^3 - x_1x_2)u_2. \end{aligned}$$

Let

$$\begin{aligned} x &= c_2 = x_2 - \frac{x_1^2}{3}, \\ y &= c_3 = x_3 + \frac{2}{9}x_1^3 - x_1x_2, \end{aligned}$$

and

$$g(x, y) = \log u(x, y).$$

(3.5) becomes

$$2xg_x(x, y) + 3yg_y(x, y) = 3. \tag{3.9}$$

Consider the characteristic curve of (3.9)

$$\begin{cases} x(t) = e^{2t}\cos^{\frac{2}{3}}\theta, \\ y(t) = e^{3t}\sin\theta. \end{cases} \tag{3.10}$$

This gives

$$\begin{aligned} & \frac{d}{dt} g(x(t), y(t)) \\ = \; & g_x x'(t) + g_y y'(t) \\ = \; & 2x(t) g_x + 3y(t) g_y \\ = \; & 3, \end{aligned}$$

and

$$g(x_1(t), x_2(t)) = 3t + g(\cos^{\frac{2}{3}}\theta, \sin\theta),$$

for all t.

From (3.10), we get

$$\begin{aligned} e^t &= (x^3 + y^2)^{\frac{1}{6}}, \\ \cos\theta &= (x^3 + y^2)^{-\frac{1}{2}} x^{\frac{3}{2}}, \\ \sin\theta &= (x^3 + y^2)^{-\frac{1}{2}} y. \end{aligned}$$

Hence we have

$$g(x, y) = \frac{1}{2}\log|x^3 + y^2| + u_0(\tan^{-1}\frac{y}{x^{3/2}}).$$

The solutions take the form

$$u = c\sqrt{(x_2 - \frac{x_1^2}{3})^3 + (x_3 + \frac{2}{9}x_1^3 - x_1 x_2)^2},$$

where c is constant. Here let u_0 be a constant function. Howerever, this solution u is not a C^3 function, and is not the desired functional.

Now we solve the homogeneity equation (3.3) and the translation equation(3.4) for general n. Consider the curve

(3.11)

$$\begin{aligned} x_0(t) &= c_0, \\ x_1(t) &= nt + c_1, \\ x_2(t) &= nt^2 + c_2, \\ &\cdots \\ x_m(t) &= \sum_{k=0}^{m} \binom{m}{k} c_k t^{m-k}, \\ &\cdots \\ x_n(t) &= \sum_{k=0}^{n} \binom{n}{k} c_k t^{n-k}, \end{aligned}$$

for $2 < m \leq n$, where $c_0 = n$ and $c_1 = 0$, then

$$\begin{aligned} \frac{d}{dt} f(x_1(t), ...x_n(t)) &= \sum_{m=1}^{n} x'_m(t) f_m \\ &= \sum_{m=1}^{n} m x_{m-1}(t) f_m = 0, \end{aligned}$$

where $x_0 = n$. This shows that

$$\begin{aligned} f(x_1(t), ..., x_n(t)) &= f(x_1(0)...x_n(0)) \\ &= f(0, c_2, ..., c_n) = u(c_2, ..., c_n), \end{aligned}$$

for all t.

From (3.11), we get

$$\begin{aligned} t &= \frac{x_1}{n}, \\ c_0 &= n, \\ c_1 &= 0, \end{aligned}$$

and

$$\begin{aligned} c_m &= x_m - \sum_{k=0}^{m-1} \binom{m}{k} c_k t^{m-k} \\ &= \sum_{k=0}^{m-2} \binom{m}{m-k} x_{m-k}(-t)^k \\ &\quad -[\binom{m}{0} - \binom{m}{1} + ... + (-1)^{m-2}\binom{m}{m-2}]nt^m \\ &= \sum_{k=0}^{m} \binom{m}{m-k} x_{m-k}(-t)^k, \end{aligned}$$

for $2 \leq m \leq n$, where $x_0 = n$.

Let

$$y_m = c_m,$$

for $2 < m \leq n$, and

$$g(y_2, ..., y_n) = \log u(y_2, ..., y_n).$$

Since

$$f(x_1, ..., x_n) = u(y_2, ..., y_n),$$

by the homogeneity condition and using the fact that $\frac{\partial y_m}{\partial x_k} = 0$, for all $k > m$, then we have

$$\begin{aligned} nu(y_2, ..., y_n) &= nf(x_1, ..., x_n) \\ &= \sum_{k=1}^{n} kx_k f_k(x_1, ..., x_n) \\ &= \sum_{k=1}^{n} \sum_{m=2}^{n} kx_k (\frac{\partial u}{\partial y_m})(\frac{\partial y_m}{\partial x_k}) \\ &= \sum_{m=2}^{n} \sum_{k=1}^{m} kx_k (\frac{\partial y_m}{\partial x_k})(\frac{\partial u}{\partial y_m}) \\ &= \sum_{m=2}^{n} my_m (\frac{\partial u}{\partial y_m}). \end{aligned}$$

The equation (3.3) can be written as

$$\sum_{m=2}^{n} m y_m (\frac{\partial g}{\partial y_m}) = n. \tag{3.12}$$

Consider the characteristic curve of (3.12)

$$\begin{cases} y_2(t) = c_2' e^{2t}, \\ \cdots \\ y_n(t) = c_n' e^{nt}. \end{cases} \tag{3.13}$$

This gives

$$\begin{aligned} & \frac{d}{dt} g(y_2(t), ..., y_n(t)) \\ = & \sum_{m=2}^{n} (\frac{\partial g}{\partial y_m}) y_m'(t) \\ = & \sum_{m=2}^{n} m y_m (\frac{\partial g}{\partial y_m}) \\ = & n, \end{aligned}$$

and hence

$$g(y_1(t), ..., y_n(t)) = nt + g(c_2', ..., c_n'),$$

for all t.

Let $p_1, ..., p_n$ are nonnegative numbers such that $p_1 + ... + p_n = n$, and $c_2'^{\frac{p_2}{2}} \cdots c_n'^{\frac{p_n}{n}} = 1$. (3.13) gives

$$\begin{aligned} e^t &= \exp(\frac{p_2 + ... + p_n}{n} t) \\ &= (y_2^{\frac{p_2}{2n}} c_2'^{\frac{-p_2}{2n}}) \cdots (y_n^{\frac{p_n}{n^2}} c_n'^{\frac{-p_n}{n^2}}) \\ &= y_2^{\frac{p_2}{2n}} \cdots y_n^{\frac{p_n}{n^2}} \end{aligned}$$

and

$$c'_m = y_m(y_2^{\frac{p_2}{2n}} \cdots y_n^{\frac{p_n}{n^2}})^{-m},$$

for all $2 \leq m \leq n$.

In this case, let g is conctant function, then we have

$$\begin{aligned} u(y_2, ..., y_n) &= e^{g(y_2,...,y_n)} \\ &= e^{nt} e^{g(c'_2,...,c'_n)} \\ &= c y_2^{\frac{p_2}{2}} \cdots y_n^{\frac{p_n}{n}}. \end{aligned}$$

where c is constant, $p_1, ..., p_n$ are nonnegative, $p_1 + ... + p_n = n$, and

$$y_m = \sum_{k=0}^{m} \binom{m}{k} x_k (-\frac{x_1}{n})^{m-k},$$

for all $m \geq 2$.

We list some interesting functionals for the case $2 \leq n \leq 7$, as follows:

(1) $n = 2$, $tr(A - \frac{H}{2}I)^2$.

(2) $n = 3$, we have no C^3 functionals.

(3) $n = 4$, $tr(A - \frac{H}{4}I)^4$, and $[tr(A - \frac{H}{4}I)^2]^2$.

(4) $n = 5$, we have no C^3 functionals.

(5) $n = 6$, $tr(A - \frac{H}{6}I)^6$, $[tr(A - \frac{H}{6}I)^3]^2$, $[tr(A - \frac{H}{6}I)^2]^3$, $(tr(A - \frac{H}{6}I)^2)(tr(A - \frac{H}{6}I)^4)$.

(6) $n = 7$, $[tr(A - \frac{H}{7}I)^2]^{\frac{7}{2}}$.

For the special cases of $n \geq 6$, $p_2 = n$, and $p_i = 0$ for all $i \neq 2$, the functional takes the form,

$$\Phi^{\frac{n}{2}} = [tr(A - \frac{H}{n}I)^2]^{\frac{n}{2}}.$$

4. The variation of the generalized Willmore functional

In this finial section, we want to derive the first variation of the generalized Willmore functional,

$$\begin{aligned}
[\int_M \Phi^{\frac{n}{2}}\ dv]_t &= \int_M \Phi^{\frac{n}{2}-3}\{(n-1)\Phi^2\Delta H + \Phi^3 H + n\Phi^2 \sum \phi_{ij}\phi_{jk}\phi_{ki} \\
&\quad +(n-1)(n-2)\Phi \sum H_i\Phi_i + \frac{n(n-2)(n-4)}{4}\sum \phi_{ij}\Phi_i\Phi_j \\
&\quad +\frac{n(n-2)}{2}\Phi \sum \phi_{ij}\Phi_{ij}\}(X_t \cdot N)\ dv.
\end{aligned}$$

First we note that

$$\Phi = \sum \phi_{ij}^2 = tr(A - \frac{H}{n}I)^2 = S - \frac{H^2}{n}, \tag{4.1}$$

where

$$S = trA^2 = \Phi + \frac{H^2}{n}. \tag{4.2}$$

For simplifying the computation, we let $\frac{\partial \Phi^{\frac{n}{2}}}{\partial H} = (\Phi^{\frac{n}{2}})_1$, $\frac{\partial \Phi^{\frac{n}{2}}}{\partial S} = (\Phi^{\frac{n}{2}})_2$, $\frac{\partial^2 \Phi^{\frac{n}{2}}}{\partial H^2} = (\Phi^{\frac{n}{2}})_{11}$, and so on. Then we have

(4.3)

$$
\begin{aligned}
(\Phi^{\frac{n}{2}})_1 &= -H\Phi^{\frac{n}{2}-1}, \\
(\Phi^{\frac{n}{2}})_2 &= \frac{n}{2}\Phi^{\frac{n}{2}-1}, \\
(\Phi^{\frac{n}{2}})_{11} &= [(1-\frac{2}{n})H^2 - \Phi]\Phi^{\frac{n}{2}-2}, \\
(\Phi^{\frac{n}{2}})_{12} &= (\Phi^{\frac{n}{2}})_{21} = -(\frac{n}{2}-1)H\Phi^{\frac{n}{2}-2}, \\
(\Phi^{\frac{n}{2}})_{22} &= \frac{n}{2}(\frac{n}{2}-1)\Phi^{\frac{n}{2}-2}, \\
(\Phi^{\frac{n}{2}})_{111} &= [-(1-\frac{2}{n})(1-\frac{4}{n})H^3 + 3(1-\frac{2}{n})H\Phi]\Phi^{\frac{n}{2}-3}, \\
(\Phi^{\frac{n}{2}})_{112} &= (\Phi^{\frac{n}{2}})_{121} = (\Phi^{\frac{n}{2}})_{211} = [(\frac{n}{2}-2)(1-\frac{2}{n})H^2 + (1-\frac{n}{2})\Phi]\Phi^{\frac{n}{2}-3}, \\
(\Phi^{\frac{n}{2}})_{122} &= (\Phi^{\frac{n}{2}})_{212} = (\Phi^{\frac{n}{2}})_{221} = -(\frac{n}{2}-1)(\frac{n}{2}-2)H\Phi^{\frac{n}{2}-3}, \\
(\Phi^{\frac{n}{2}})_{222} &= \frac{n}{2}(\frac{n}{2}-1)(\frac{n}{2}-2)\Phi^{\frac{n}{2}-3}.
\end{aligned}
$$

By Theorem 1.1 in section 1, the first variation fomula of the generalized Willmore functional is

$$
\begin{aligned}
[\int_M \Phi^{\frac{n}{2}}\ dv]_t &= \int_M \{\sum_{l,k,m=1}^{2} l(trA^k)_i(trA^m)_j A_{ij}^{l-1}(\Phi^{\frac{n}{2}})_{lkm} + \sum_{l,k=1}^{2} l(trA^k)_{ij}A_{ij}^{l-1}(\Phi^{\frac{n}{2}})_{lk} \\
&\quad +4\sum_{k=1}^{2}(trA^k)_i(trA)_i(\Phi^{\frac{n}{2}})_{2k} + 2\Delta H(\Phi^{\frac{n}{2}})_2 + \sum_{l=1}^{2} l(trA^{l+1})(\Phi^{\frac{n}{2}})_l - H\Phi^{\frac{n}{2}}\}\ (X_t\cdot N)\ dv.
\end{aligned}
$$

Now we compute term by term directly by using (4.1), (4.2) and (4.3) as follows:

(4.4)

$$
\begin{aligned}
&\sum_{l,k,m=1}^{2} l(trA^k)_i(trA^m)_j A_{ij}^{l-1}(\Phi^{\frac{n}{2}})_{lkm} \\
=\ & H_i^2(\Phi^{\frac{n}{2}})_{111} + 2H_iS_i(\Phi^{\frac{n}{2}})_{112} + S_i^2(\Phi^{\frac{n}{2}})_{122} \\
&+2(H_iH_jh_{ij})(\Phi^{\frac{n}{2}})_{211} + 4(H_iS_jh_{ij})(\Phi^{\frac{n}{2}})_{212} + 2(S_iS_jh_{ij})(\Phi^{\frac{n}{2}})_{222} \\
=\ & \Phi^{\frac{n}{2}-3}\{[2\phi_{ij}(1-\frac{n}{2})]\Phi H_iH_j + [2(1-\frac{n}{2})]H_i\Phi_i + [2\phi_{ij}\frac{n}{2}(\frac{n}{2}-1)(\frac{n}{2}-2)]\Phi_i\Phi_j\},
\end{aligned}
$$

(4.5)

$$\sum_{l,k=1}^{2} l(trA^k)_{ij}A_{ij}^{l-1}(\Phi^{\frac{n}{2}})_{lk}$$
$$= \Delta H(\Phi^{\frac{n}{2}})_{11} + S_{ii}(\Phi^{\frac{n}{2}})_{12} + 2H_{ij}h_{ij}(\Phi^{\frac{n}{2})}_{21} + 2S_{ij}^2 h_{ij}(\Phi^{\frac{n}{2}})_{22}$$
$$= \Phi^{\frac{n}{2}-2}\{-\Phi\Delta H + n(\frac{n}{2}-1)\Phi_{ij}\phi_{ij} + 2(\frac{n}{2}-1)\phi_{ij}H_iH_j\},$$

(4.6)

$$4\sum_{k=1}^{2}(trA^k)_i(trA)_i(\Phi^{\frac{n}{2}})_{2k}$$
$$= 4H_i^2\Phi_{21} + 4S_iH_i\Phi_{22}$$
$$= \Phi^{\frac{n}{2}-3}\{[2n(\frac{n}{2}-1)\Phi]H_i\Phi_i\},$$

(4.7)

$$\sum_{l=1}^{2} l(trA^{l+1})(\Phi^{\frac{n}{2}})_l$$
$$= S(\Phi^{\frac{n}{2}})_1 + 2h_{ij}h_{jk}h_{ki}(\Phi^{\frac{n}{2}})_2$$
$$= (\Phi + \frac{H^2}{n})(-H\Phi^{\frac{n}{2}-1}) + 2(\phi_{ij} + \frac{H}{n}\delta_{ij})(\phi_{jk} + \frac{H}{n}\delta_{jk})(\phi_{ki} + \frac{H}{n}\delta_{ki})(\frac{n}{2}\Phi^{\frac{n}{2}-1})$$
$$= \Phi^{\frac{n}{2}-3}\{-H\Phi^3 - \frac{H^3}{n}\Phi^2 + n\Phi^2[\phi_{ij}\phi_{jk}\phi_{kl} + \frac{3}{n}H\Phi + \frac{H^3}{n^2}]\}$$
$$= \Phi^{\frac{n}{2}-3}\{2H\Phi^3 + n\Phi^2\phi_{ij}\phi_{jk}\phi_{ki}\},$$

and

(4.8) $$2\Delta H(\Phi^{\frac{n}{2}})_2 = \Phi^{\frac{n}{2}-3}\{n\Phi^2\Delta H\}.$$

From (4.4) to (4.8), we conclude that

$$[\int_M \Phi^{\frac{n}{2}}\ dv]_t = \int_M \Phi^{\frac{n}{2}-3}\{(n-1)\Phi^2\Delta H + \Phi^3 H + n\Phi^2\sum\phi_{ij}\phi_{jk}\phi_{ki}$$
$$+(n-1)(n-2)\Phi\sum H_i\Phi_i + \frac{n(n-2)(n-4)}{4}\sum\phi_{ij}\Phi_i\Phi_j$$
$$+\frac{n(n-2)}{2}\Phi\sum\phi_{ij}\Phi_{ij}\}(X_t\cdot N)\ dv.$$

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