Introduction

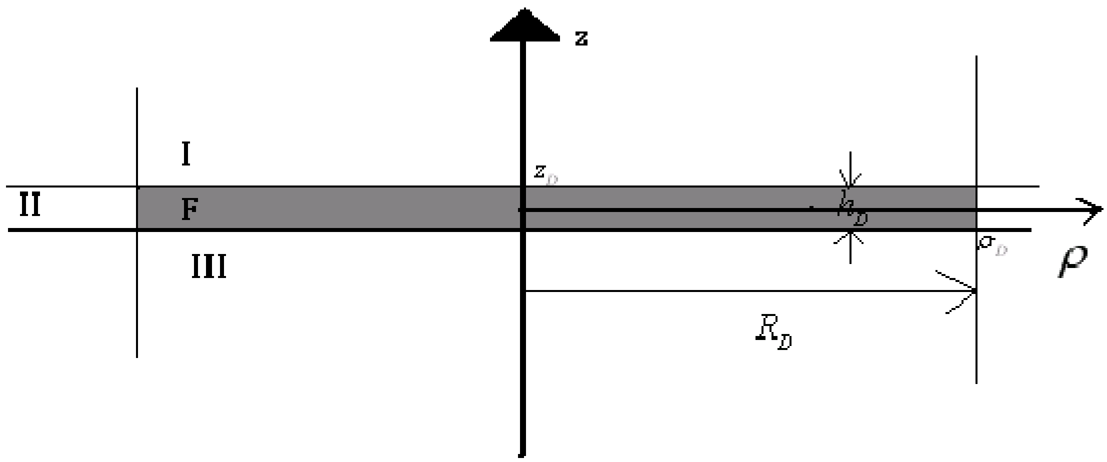

Fig.1 Disk-shaped ferrite resonator with a small thickness to diameter ratio.

Ferromagnetic resonators with magnetostrictive (MS) oscillations can be considered in microwaves as point (with respect to the external electromagnetic fields) particles. MS oscillations in a small ferrite disk resonator can be characterized by a discrete spectrum of energy levels [4]. This fact allows analyzing the MS oscillations similarly to quantum mechanical problems. A special interest of energy spectra of such a small structural element - artificial magnetic atoms - may be found in the fields of microwave artificial composite materials, microwave spectroscopy, and, probably, quantum computation.

Let sizes of ferromagnetic resonator be much less than the electromagnetic wavelength, but much more than the spin-wave wavelength taking into account the exchange interaction. Based on our model, we will consider a structure shown in Fig.1 as a section of an open cylindrical MS waveguide with the longitudinal z axis, restricted by two planes $z = 0$ and $z = h$. In a case of an axially magnetized cylinder, we have MS waveguide modes, that is, every mode propagating in the positive direction of the z axis has a counterpart-the same mode propagating in the negative direction of the z axis [1,12]. So, one can consider eigen MS oscillations in a normally magnetized ferrite disk as standing MS waves in a cylindrical waveguide. This fact will allow us to formulate the energy spectral problem for MS oscillations in a disk-form ferrite resonator. We have four main regions: region F-a ferrite and regions I-III-dielectrics. The role of the corner regions is supposed to be neglected. It is relevant to point out that in experiments [6,13] we have a multiresonance regime of MS modes just in ferrite disks with a small thickness to diameter ratio (approximately 1/15-1/20). In this case, the magnetostatic approximation can be successfully used [8]. For the irrotational rf magnetic field of the magnetostatic modes,

$$\overrightarrow{H} = -\overrightarrow{\nabla}\psi \tag{1}$$

where ψ is the magnetostatic potential and the rf magnetization $\overrightarrow{m}$ is defined as

$$\overrightarrow{m} = -\overleftrightarrow{\kappa}(\omega)\cdot\overrightarrow{\nabla}\psi \tag{2}$$

where $\overleftrightarrow{\kappa}$ is a tensor of susceptibility.

Taking into account Eq. (1) for the rf magnetic field, the equation for rf magnetic flux density

$$\overrightarrow{B} = -\overleftrightarrow{\mu}(\omega)\cdot\overrightarrow{\nabla}\psi \tag{3}$$

where $\overleftrightarrow{\mu}(\omega) = \overleftrightarrow{I} + 4\pi\overleftrightarrow{\kappa}(\omega)$ is the tensor of permeability, $\overleftrightarrow{I}$ is the unit matrix and the equation

$$\overrightarrow{\nabla}\cdot\overrightarrow{B} = 0 \tag{4}$$

one can write the following operator equation [3]:

$$\widehat{L}(\omega)\,V = 0 \tag{5}$$

where

$$\widehat{L}(\omega) \equiv \begin{pmatrix} (\overleftrightarrow{\mu}(\omega))^{-1} & \overrightarrow{\nabla} \\ -\overrightarrow{\nabla} & 0 \end{pmatrix} \tag{6}$$

is a differential-matrix operator,

$$V\equiv\begin{pmatrix}\overrightarrow{B}\\ \psi\end{pmatrix} \tag{7}$$

is a vector function included in the domain of definition of the operator $\widehat{L}$. Equation (6) describes the field inside a ferrite. Outside of a ferrite medium we have the same equation, but with $\overleftrightarrow{\mu}=\overleftrightarrow{I}$. Based on the Eq. (6) inside a ferrite, analogous equation with $\overleftrightarrow{\mu}=\overleftrightarrow{I}$ outside a ferrite and taking into account homogeneous boundary condition, one can formulate a spectral problem for MS waveguide modes.

A special featuer of an MS waveguide structure based on axially magnetized ferrite cylinder (that does not take place in such types of electromagnetic-wave waveguide structures as closed hollow or open dielectric waveguides) is the fact that in the frequency region $\omega_1 \leq \omega \leq \omega_2$ between two cutoff frequencies $\omega_1=\gamma' H_i$ and $\omega_2=\gamma'\left[H_i\left(H_i+4\pi M_s\right)\right]^{1/2}$, where γ' is the gyromagnetic ratio, H_i is the internal dc magnetic field, and M_s is the saturation magnetization, we have a complete discrete spectrum of propagating ($\gamma=i\beta$) MS modes [1,12].

Together with a system of two first-order homogeneous Eq. (6), a second-order homogeneous differential equation for a MS waveguide can be considered as well. This is the so-called Walker equation in a ferrite [8]:

$$\widehat{G}\psi=0 \tag{8}$$

where

$$\widehat{G}=-\overrightarrow{\nabla}\cdot\left(\overleftrightarrow{\mu}\,\overrightarrow{\nabla}\right) \tag{9}$$

is the Walker operator. For a ferrite magnetized along the z axis the tensor of permeability has a form [1]

$$\overleftrightarrow{\mu}=\mu_0\begin{bmatrix}\mu & i\mu_a & 0\\ -i\mu_a & \mu & 0\\ 0 & 0 & 1\end{bmatrix} \tag{10}$$

where $\mu=1-\omega_1\omega_m/\left(\omega^2-\omega_1^2\right)$, $\mu_a=\omega\omega_m/\left(\omega_1^2-\omega^2\right)$, $\omega_m=\gamma' 4\pi M_s$, and μ_0 is the permeability of vacuum.

In some cases of MS waveguides, the knowledge of the MS potential wave function ψ give a possibility to define every state of the physical quantities.

1 Derivation of Numerical Model

In [4], the walker equation (8) is simplified to 1D equation by means of separation of variables. That is an approximate model. We consider here instead the full model of (8) in cylindrical coordinates due to the disk shape of the resonator. We begin by writing (8) in the rectangular coordinates in a more detailed form :

$$\begin{aligned}
\overrightarrow{B} &= -\overleftrightarrow{\mu} \cdot \overrightarrow{\nabla}\psi \\
&= -\mu_0 \begin{bmatrix} \mu & i\mu_a & 0 \\ -i\mu_a & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial\psi}{\partial x} \\ \frac{\partial\psi}{\partial y} \\ \frac{\partial\psi}{\partial z} \end{bmatrix} \\
&= -\mu_0 \begin{bmatrix} \mu\frac{\partial\psi}{\partial x} + i\mu_a\frac{\partial\psi}{\partial y} \\ -i\mu_a\frac{\partial\psi}{\partial x} + \mu\frac{\partial\psi}{\partial y} \\ \frac{\partial\psi}{\partial z} \end{bmatrix}
\end{aligned}$$

$$\widehat{G} = -\overrightarrow{\nabla} \cdot \left(\overleftrightarrow{\mu}\overrightarrow{\nabla}\right)$$

$$\widehat{G}\psi = 0 \Rightarrow -\overrightarrow{\nabla} \cdot \left(\overleftrightarrow{\mu}\overrightarrow{\nabla}\right)\psi = 0$$

$$\Rightarrow -\overrightarrow{\nabla} \cdot \left(\mu_0 \begin{bmatrix} \mu & i\mu_a & 0 \\ -i\mu_a & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \right) \psi = 0$$

$$\begin{aligned}
&\Rightarrow \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{pmatrix} \mu\frac{\partial}{\partial x} + i\mu_a\frac{\partial}{\partial y} \\ -i\mu_a\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \psi = 0 \\
&\Rightarrow \left(\mu\frac{\partial^2}{\partial x^2} + i\mu_a\frac{\partial^2}{\partial x\partial y} - i\mu_a\frac{\partial^2}{\partial x\partial y} + \mu\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)\psi = 0
\end{aligned}$$

We have

$$\left[\mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{\partial^2}{\partial z^2} \right]\psi = 0 \tag{11}$$

The boundary conditions at surfaces of disk are continuity of MS potential ψ and the normal components of magnetic flux density $\overrightarrow{B}$. Further, $\psi \to 0$ at infinity. Note that a physically acceptable solution for (11) is possible only for $\mu < 0$ in the ferrite region. This makes Eq. (11) mathematically interesting and challenging since it is a mixed type PDE, i.e., it changes its type form hyperbolic in the ferrite to elliptic in the dielectric. Moreover, we together with the interface continuity conditions for both ψ and $\overrightarrow{B}$. We now transform (11) to cylindrical coordinates.

Let

$$x = \rho\cos\theta, y = \rho\sin\theta, z = z$$

We have

$$\begin{aligned}
\frac{\partial x}{\partial \rho} &= \cos\theta, \frac{\partial y}{\partial \rho} = \sin\theta \\
\frac{\partial x}{\partial \theta} &= -\rho\sin\theta, \frac{\partial y}{\partial \theta} = \rho\cos\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \psi}{\partial \rho} &= \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \rho} \\
&= \cos\theta\frac{\partial \psi}{\partial x} + \sin\theta\frac{\partial \psi}{\partial y} \\
\frac{\partial \psi}{\partial \theta} &= \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -\rho\sin\theta\frac{\partial \psi}{\partial x} + \rho\cos\theta\frac{\partial \psi}{\partial y}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left\langle \begin{array}{c} \cos\theta\frac{\partial\psi}{\partial x} + \sin\theta\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\rho} \cdots (1) \times \rho\cos\theta \\ -\rho\sin\theta\frac{\partial\psi}{\partial x} + \rho\cos\theta\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\theta} \cdots (2) \times \sin\theta \end{array} \right. \\
&\Rightarrow \left\langle \begin{array}{c} \rho\cos^2\theta\frac{\partial\psi}{\partial x} + \rho\sin\theta\cos\theta\frac{\partial\psi}{\partial y} = \rho\cos\theta\frac{\partial\psi}{\partial\rho} \\ -\rho\sin^2\theta\frac{\partial\psi}{\partial x} + \rho\sin\theta\cos\theta\frac{\partial\psi}{\partial y} = \sin\theta\frac{\partial\psi}{\partial\theta} \end{array} \right. \\
&\Rightarrow \rho\frac{\partial \psi}{\partial x} = \rho\cos\theta\frac{\partial \psi}{\partial \rho} - \sin\theta\frac{\partial \psi}{\partial \theta} \\
&\Rightarrow \frac{\partial \psi}{\partial x} = \cos\theta\frac{\partial \psi}{\partial \rho} - \frac{\sin\theta}{\rho}\frac{\partial \psi}{\partial \theta} \qquad (12)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&\Rightarrow \left\langle \begin{array}{c} \cos\theta\frac{\partial\psi}{\partial x} + \sin\theta\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\rho} \cdots (1) \times \rho\sin\theta \\ -\rho\sin\theta\frac{\partial\psi}{\partial x} + \rho\cos\theta\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\theta} \cdots (2) \times \cos\theta \end{array} \right. \\
&\Rightarrow \left\langle \begin{array}{c} \rho\sin\theta\cos\theta\frac{\partial\psi}{\partial x} + \rho\sin^2\theta\frac{\partial\psi}{\partial y} = \rho\sin\theta\frac{\partial\psi}{\partial\rho} \\ -\rho\sin\theta\cos\theta\frac{\partial\psi}{\partial x} + \rho\cos^2\theta\frac{\partial\psi}{\partial y} = \cos\theta\frac{\partial\psi}{\partial\theta} \end{array} \right. \\
&\Rightarrow \rho\frac{\partial\psi}{\partial y} = \rho\sin\theta\frac{\partial\psi}{\partial\rho} + \cos\theta\frac{\partial\psi}{\partial\theta} \\
&\Rightarrow \frac{\partial\psi}{\partial y} = \sin\theta\frac{\partial\psi}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial\psi}{\partial\theta} \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial\rho}\right)\cos\theta - \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial\theta}\right)\frac{\sin\theta}{\rho} \qquad (14) \\
&= \frac{\partial}{\partial\rho}\left(\frac{\partial\psi}{\partial x}\right)\cos\theta - \frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial x}\right)\frac{\sin\theta}{\rho} \\
&= \frac{\partial}{\partial\rho}\left(\cos\theta\frac{\partial\psi}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right)\cos\theta - \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\psi}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right)\frac{\sin\theta}{\rho} \\
&= \cos^2\theta\frac{\partial^2\psi}{\partial\rho^2} - \frac{2\sin\theta\cos\theta}{\rho}\frac{\partial^2\psi}{\partial\theta\partial\rho} + \frac{\sin^2\theta}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{2\sin\theta\cos\theta}{\rho^2}\frac{\partial\psi}{\partial\theta} + \frac{\sin^2\theta}{\rho}\frac{\partial\psi}{\partial\rho}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial\rho}\right)\sin\theta + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial\theta}\right)\frac{\cos\theta}{\rho} \qquad (15) \\
&= \frac{\partial}{\partial\rho}\left(\frac{\partial\psi}{\partial y}\right)\sin\theta + \frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial y}\right)\frac{\cos\theta}{\rho} \\
&= \frac{\partial}{\partial\rho}\left(\sin\theta\frac{\partial\psi}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right)\sin\theta + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right)\frac{\cos\theta}{\rho} \\
&= \sin^2\theta\frac{\partial^2\psi}{\partial\rho^2} + \frac{2\sin\theta\cos\theta}{\rho}\frac{\partial^2\psi}{\partial\theta\partial\rho} + \frac{\cos^2\theta}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{\rho^2}\frac{\partial\psi}{\partial\theta} + \frac{\cos^2\theta}{\rho}\frac{\partial\psi}{\partial\rho}
\end{aligned}
$$

Putting (14) and (15) together, we thus obtain

$$
\left[\mu\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right) + \frac{\partial^2}{\partial z^2}\right]\psi = 0 \qquad (16)
$$

Using the fact that the ferrite is cylindrical and letting

$$\psi = \phi(\rho, z)e^{il\theta}, l = 0, \pm 1, \pm 2, ...$$

Eq. (16) can thus be written as

$$\left[\mu\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{l^2}{\rho^2}\right) + \frac{\partial^2}{\partial z^2}\right]\phi = 0 \tag{17}$$

We proceed to derive the interface conditions. By definition,

$$\begin{aligned}
\overrightarrow{B} &= -\overleftrightarrow{\mu} \cdot \overrightarrow{\nabla}\psi \\
&= -\mu_0 \begin{bmatrix} \mu & i\mu_a & 0 \\ -i\mu_a & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta\frac{\partial\psi}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial\psi}{\partial\theta} \\ \sin\theta\frac{\partial\psi}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial\psi}{\partial\theta} \\ \frac{\partial\psi}{\partial z} \end{bmatrix} \\
&= -\mu_0 \begin{bmatrix} \mu\left(\cos\theta\frac{\partial\psi}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right) + i\mu_a\left(\sin\theta\frac{\partial\psi}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right) \\ -i\mu_a\left(\cos\theta\frac{\partial\psi}{\partial\rho} - \frac{\sin\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right) + \mu\left(\sin\theta\frac{\partial\psi}{\partial\rho} + \frac{\cos\theta}{\rho}\frac{\partial\psi}{\partial\theta}\right) \\ \frac{\partial\psi}{\partial z} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{n} &= \langle x, y, z\rangle = \langle 0, 1, 0\rangle \\
\overrightarrow{n} &= \langle \rho, \theta, z\rangle = \left\langle 1, \frac{\pi}{2}, 0\right\rangle
\end{aligned}$$

$$\begin{aligned}
B_{\rho_D^-} &= -\mu_0\left[i\mu_a\frac{1}{\rho_D}\frac{\partial\psi}{\partial\theta} + \mu\frac{\partial\psi}{\partial\rho}\right] \\
&= -\mu_0\left[\frac{-\mu_a l}{\rho_D}\phi e^{il\theta} + \mu e^{il\theta}\frac{\partial\phi}{\partial\rho}\right] \\
B_{\rho_D^+} &= -\mu_0\left[\frac{\partial\psi}{\partial\rho}\right] \\
&= -\mu_0 e^{il\theta}\frac{\partial\phi}{\partial\rho} \\
B_{\rho_D^-} &= B_{\rho_D^+}
\end{aligned}$$

$$\Rightarrow \mu\frac{\partial\phi}{\partial\rho} - \frac{\mu_a l}{\rho_D}\phi \mid_{\rho_D^-} = \frac{\partial\phi}{\partial\rho} \mid_{\rho_D^+}$$

$z - direction:$

$$\frac{\partial \phi}{\partial z}(\rho, z_{D^-}) = \frac{\partial \phi}{\partial z}(\rho, z_{D^+}) \tag{18}$$

$\rho - direction:$

$$\mu \frac{\partial \phi}{\partial \rho}(\rho_{D^-}, z) - \frac{\mu_a l}{\rho_D}\phi(\rho, z) = \frac{\partial \phi}{\partial \rho}\left(\rho_{D^+}, z\right) \tag{19}$$

Boundary conditions:

$$\phi(\rho, z_{\max}) = \phi(\rho_{\max}, z) = 0 \tag{20}$$

$$\frac{\partial \phi}{\partial \rho}(0, z) = 0,\ \frac{\partial \phi}{\partial z}(\rho, 0) = 0 \tag{21}$$

2 Numerical methods

2.1 Finite Difference discretization

We now use the central difference method to approximate the model (17) $\sim$ (19). Let the domain of the model

$$\Omega = (0, \rho_{\max}) \times (0, z_{\max}) = \{(\rho, z) : 0 < \rho < \rho_{\max},\ 0 < z < z_{\max}\}$$

be partioned as

$$\rho_1 = 0 < \rho_2 < ... < \rho_i < ... < \rho_{m-1} < \rho_m = \rho_{\max},$$
$$z_1 = 0 < z_2 < ... < z_j < ... < z_{n-1} < z_n = z_{\max}$$

Where h, k are mesh sizes of ρ-axis, and z-axis with $m > 0$ and $n > 0$, such that $h = \frac{\rho_{\max}}{m}$, $k = \frac{z_{\max}}{n}$, $\rho_i = ih$ and $z_j = jk$ for each $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. At any interior mesh point (ρ_i, z_j), the Eq. (17) becomes

$$\mu \left(\frac{\partial^2 \phi}{\partial \rho^2} (\rho_i, z_j) + \frac{1}{\rho_i} \frac{\partial \phi}{\partial \rho} (\rho_i, z_j) - \frac{l^2}{\rho_i^2} \phi (\rho_i, z_j) \right) + \frac{\partial^2 \phi}{\partial z^2} (\rho_i, z_j) = 0 \qquad (22)$$

The difference method is obtained using the centered-difference quotient for the partial derivatives given by

$$\frac{\partial \phi}{\partial \rho} (\rho_i, z_j) = \frac{\phi (\rho_{i+1}, z_j) - \phi (\rho_{i-1}, z_j)}{2h} - \frac{h^2}{6} \frac{\partial^3 \phi}{\partial \rho^3} (\eta_i, z_j)$$

where $\eta_i \in [\rho_{i-1}, \rho_{i+1}]$,

$$\frac{\partial^2 \phi}{\partial \rho^2} (\rho_i, z_j) = \frac{\phi (\rho_{i+1}, z_j) - 2\phi (\rho_i, z_j) + \phi (\rho_{i-1}, z_j)}{h^2} - \frac{h^2}{12} \frac{\partial^4 \phi}{\partial \rho^4} (\xi_i, z_j)$$

where $\xi_i \in [\rho_{i-1}, \rho_{i+1}]$, and

$$\frac{\partial^2 \phi}{\partial z^2} (\rho_i, z_j) = \frac{\phi (\rho_i, z_{j+1}) - 2\phi (\rho_i, z_j) + \phi (\rho_i, z_{j-1})}{k^2} - \frac{k^2}{12} \frac{\partial^4 \phi}{\partial z^4} (\rho_i, \zeta_j)$$

where $\zeta_j \in [z_{j-1}, z_{j+1}]$. Substituting these into Eq. (22) gives

$$\mu \left(\frac{\phi (\rho_{i+1}, z_j) - 2\phi (\rho_i, z_j) + \phi (\rho_{i-1}, z_j)}{h^2} + \frac{1}{\rho_i} \frac{\phi (\rho_{i+1}, z_j) - \phi (\rho_{i-1}, z_j)}{2h} - \frac{l^2 \phi}{\rho_i^2} (\rho_i, z_j) \right)$$

$$+\frac{\phi(\rho_i, z_{j+1}) - 2\phi(\rho_i, z_j) + \phi(\rho_i, z_{j-1})}{k^2} = \mu\left(\frac{h^2}{6}\frac{\partial^3\phi}{\partial\rho^3}(\eta_i, z_j) + \frac{h^2}{12}\frac{\partial^4\phi}{\partial\rho^4}(\xi_i, z_j)\right)$$

$$+\frac{k^2}{12}\frac{\partial^4\phi}{\partial z^4}(\rho_i, \zeta_j)$$

Neglecting the error term

$$\tau_{i,j} = \mu\left(\frac{h^2}{6}\frac{\partial^3\phi}{\partial\rho^3}(\eta_i, z_j) + \frac{h^2}{12}\frac{\partial^4\phi}{\partial\rho^4}(\xi_i, z_j)\right) + \frac{k^2}{12}\frac{\partial^4\phi}{\partial z^4}(\rho_i, \zeta_j)$$

Let $\phi_{i,j}$ be an approximation of ϕ at (ρ_i, z_j), the finite difference equation at (ρ_i, z_j) then reads

$$\mu\left(\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h^2} + \frac{1}{\rho_i}\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h} - \frac{l^2\phi_{i,j}}{\rho_i^2}\right) + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{k^2} = 0$$

$$\Rightarrow \left(\frac{-2\mu}{h^2} - \frac{l^2\mu}{\rho_i^2} - \frac{2}{k^2}\right)\phi_{i,j} + \left(\frac{\mu}{h^2} + \frac{\mu}{2h\rho_i}\right)\phi_{i+1,j} + \left(\frac{\mu}{h^2} - \frac{\mu}{2h\rho_i}\right)\phi_{i-1,j}$$

$$+\left(\frac{1}{k^2}\right)\phi_{i,j+1} + \left(\frac{1}{k^2}\right)\phi_{i,j-1} = 0 \qquad (23)$$

When $\mu = 1$

$$\left(\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h^2} + \frac{1}{\rho_i}\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h} - \frac{l^2\phi_{i,j}}{\rho_i^2}\right) + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{k^2} = 0$$

$$\Rightarrow \left(\frac{-2}{h^2} - \frac{l^2}{\rho_i^2} - \frac{2}{k^2}\right)\phi_{i,j} + \left(\frac{1}{h^2} + \frac{1}{2h\rho_i}\right)\phi_{i+1,j} + \left(\frac{1}{h^2} - \frac{1}{2h\rho_i}\right)\phi_{i-1,j}$$

$$+\left(\frac{1}{k^2}\right)\phi_{i,j+1} + \left(\frac{1}{k^2}\right)\phi_{i,j-1} = 0 \qquad (24)$$

For the interface condition:

$z - direction:$

$$\frac{\phi_{i,j+1}-\phi_{i,j}}{k}=\frac{\phi_{i,j}-\phi_{i,j-1}}{k}$$

$$\Rightarrow \frac{1}{k}\phi_{i,j+1}-\frac{2}{k}\phi_{i,j}+\frac{1}{k}\phi_{i,j-1}=0 \tag{25}$$

$\rho - direction:$

$$\mu\left(\frac{\phi_{i,j}-\phi_{i-1,j}}{h}\right)-\frac{\mu_a l}{\rho_D}\phi_{i,j}=\frac{\phi_{i+1,j}-\phi_{i,j}}{h}$$

$$\Rightarrow \left(\frac{\mu+1}{h}-\frac{\mu_a l}{\rho_D}\right)\phi_{i,j}-\frac{\mu}{h}\phi_{i-1,j}-\frac{1}{h}\phi_{i+1,j}=0 \tag{26}$$

2.2 Eigenvalue problem

Written in matrix form,

$$A\overrightarrow{x}=0 \tag{27}$$

We move the matrix components of A that involve the unknown coefficient μ in the ferrite to Eq. (27) right side to get the following generalized eigenvalue problem

$$\widetilde{A}\overrightarrow{x}=\mu B\overrightarrow{x}$$

Note that most of components in B are zeros. By rearranging the order of the components in $\overrightarrow{x}$, we can decompose the matrixs $\widetilde{A}$ and B are $\widetilde{A}=\begin{pmatrix}\widetilde{A}_{11} & \widetilde{A}_{12}\\ \widetilde{A}_{21} & \widetilde{A}_{22}\end{pmatrix}$, $B=\begin{pmatrix}B_{11} & 0\\ 0 & 0\end{pmatrix}$, $\overrightarrow{x}=\begin{pmatrix}\overrightarrow{x}_1\\ \overrightarrow{x}_2\end{pmatrix}$, where $\widetilde{A}_{11}$ and B_{11} are submatrixs correspond to the ferrite region, $\widetilde{A}_{12}$ and $\widetilde{A}_{21}$ corresponding to the interface, and $\widetilde{A}_{22}$ corresponding to the dielectric region.

The equation can thus be decomposed to one eigenvalue problem and one algebraic problem

$$\widetilde{A}_{11}\overrightarrow{x}_1+\widetilde{A}_{12}\overrightarrow{x}_2=\mu B_{11}\overrightarrow{x}_1 \tag{28}$$

$$\widetilde{A}_{21}\overrightarrow{x}_1+\widetilde{A}_{22}\overrightarrow{x}_2=0 \tag{29}$$

Move $\widetilde{A}_{21}\overrightarrow{x}_1$ to right side of equation (29). We have $\widetilde{A}_{22}\overrightarrow{x}_2 = -\widetilde{A}_{21}\overrightarrow{x}_1$. Then, we have

$$\overrightarrow{x}_2 = \widetilde{A}_{22}^{-1}\left(-\widetilde{A}_{21}\right)\overrightarrow{x}_1 \tag{30}$$

By solving the smaller-size generalized eigenvalue problem for $\overrightarrow{x}_1$ and the substituting back to (30), Eq. (28) can consequently be solved.

$$\left[\widetilde{A}_{11} + \widetilde{A}_{12}\widetilde{A}_{22}^{-1}\left(-\widetilde{A}_{21}\right)\right]\overrightarrow{x}_1 = \mu B_{11}\overrightarrow{x}_1 \tag{31}$$

3 Numerical results

The numerical data of various parameters of the model problem are shown in Table 1. We divide our numerical results.

Table 1 Numerical data for parameters

$Disk\ data\ 2R_D$	$3.98mm$
$Disk\ data\ h_D$	$0.284mm$
$4\pi M_s$	$1792Gass$
$The\ working\ ferquency\ \frac{\omega}{2\pi}$	$9.51GH_Z$
μ	$1-\frac{\omega_1\omega_m}{\omega^2-\omega_1^2}$
ω_1	rH
ω_m	$r4\pi M_s$
r	$g\mu_B$
g	2
μ_B	9.27×10^{-24}
E	$4\pi M_s H$
E	$\frac{joule}{meter^3}\left(\frac{joule}{meter^3}=\frac{\frac{1}{100}erg}{mm^3}\right)$

As noted, the eigenvalue μ is unknow and it is correct only for negative value. This means that the admissible frequency region is restricted as $\omega_1 \leq \omega \leq \omega_2$. We now give a few energy levels in the solutions. Here, the energy is defined as $E = 4\pi M_s H$ where $H = \omega_1/r$ and ω_1 is implicitly given in μ.

The energy levels are shown in Table 2. The corresponding MS potential wave function are shown in Fig.2, 3, 4, 5, 6 and 7.

♦ $l = 0$

Table 2 Energy spectrum and MS potential

permeability μ	magnetic field H	Energy E	ψ
-0.1595	$2710.5; -4256$	4.857216×10^6	$Fig.2(a), Fig.2(b)$
-0.2004	$2731.1; -4223.9$	4.8941312×10^6	$Fig.3(a), Fig.3(b)$
-0.2875	$2771.1; -4162.9$	4.9658112×10^6	$Fig.4(a), Fig.4(b)$
-0.2973	$2775.3; -4156.6$	4.9733376×10^6	$Fig.5(a), Fig.5(b)$
-0.2994	$2776.2; -4155.2$	4.9749504×10^6	$Fig.6(a), Fig.6(b)$
-0.9710	$2972.1; -3881.3$	5.3260032×10^6	$Fig.7(a), Fig.7(b)$

References

[1] A. Gurevich and G. Melkov, Magnetic Oscillations and Waves, CRC, New York (1996)

[2] A. S. Borovik-Romanov, Spin waves and magnetic excitations, Elsevier (1988)

[3] E. O. Kamenetskii, Microwave Opt. Technol. Lett. 11, 99 (1996)

[4] E. O. Kamenetskii, Energy Eigenstates of Magnetostatic Waves and Oscillations. Phys. Rev. E 63, 066612 (2001)

[5] E. O. Kamenetskill, R. Shavit, and M. Sigalov, Magnetic artificial atoms based on thin-film ferrite disk particles

[6] J. F. Dillon, Jr., J. Appl. Phys. 31, 1605 (1960)

[7] Kraus/ Fleisch, Electromagnetics With Applications, McGraw-Hill (1999)

[8] L. R. Walker, Magnetostatic Modes in Ferromagnetic Resonance. Phys. Rev. 105, 390 (1957)

[9] M. G. Cottam, Linear and nonlinear spin waves in magnetic films and superlattices, World Scientific (1994)

[10] Patel, Numerical Analysis, Saunders College Pub (1994)

[11] R. L. Burden/ J. D. Faires, Numerical analysis, Brooks/ Cole (2001)

[12] R. I. Joseph and E. Schlőmann, J. Appl. Phys 32, 1001 (1961)

[13] T. Yukawa and K. Abe, J. Appl. Phys. 45, 3146 (1974)

[14] W. F. Ames, Numerical Methods for Partial Differential Equations (1977)