GARCH Models With Jumps

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Abstract

In this paper, we mainly use the GARCH model with Jumps to describe the exchange rates market and compare the performance of models with jumps and without jumps. In addition, we will use the martingale theory and the argument of the utility maximization to derive the risk-neutral process and use the Monte Carlo simulation to find the option price.

Contents

1 Introduction

Up to now, there are many studies concerned on the dynamic of finances. Early, Black, Sholes and Merton(1973) assume that the stock price is followed by the geometric Brownian motion and derive the European pricing formula. Subsequently, Many studies argue that the distribution of log returns of stock prices does not follow the normal distribution because of the phenomenons of skewness and fat tails. In addition, the volatility, σ, may not be a constant and has the phenomenon of clustering. These findings make us understand that the finance market is very complex especially in volatilities because it is not observable.

Modelling the dynamic of volatilities is very difficult until 1982. Engle brings up the autoregressive conditional heteroscedastic(ARCH) model to describe the dynamic of the volatilities and get good performances. Later, there are many models that improve the ARCH model appearing. For example, the generalized ARCH(GARCH) model of Bollerslev(1986), the exponential GARCH(EGARCH) model of Nelson(1991), the conditional heteroscedastic autoregressive moving-average(CHARMA) model of Tsay(1987), and so on. Further, the family of ARCH models will be improved such that the model can present the non-symmetric impact, for example, the non-linear GARCH(NGARCH) model of Duan(1995), the GARCH model of Heston and Nandi(2000) and the the asymmetric ARCH of Engle(2004).

To improve Black-Scholes-Merton model, in addition to the volatility parameter, σ, is not a constant and the distribution of log-returns does not follow the normal distribution, recent researches suggest that the path of the log-price is not continuous and use a model containing jumps to describe the path should be fitter. For example, Duan(2004) uses a model whose jumps followed a compounded Poisson process to describe the dynamic of the S&P 500 index and gets satisfied results in the convergence and option pricing.

The phenomenons of volatilities we mentioned above are not only in the stock market, we can also find these properties in other financial markets. J. Duan and Jason Z. Wei(1999) use GARCH models to describe the exchange rates market and follow the assumption of LRNVR(locally risk-neutral valuation relationship) to price the foreign currency and cross-currency option. Engle(2002) uses the GARCH model with a random variable followed the exponential distribution to test the realized volatilities of the exchange rates market and also have some better results. In this paper, we will continue using Engle's(2002) suggest and Duan's(2004) idea to construct a GARCH Model with jumps to describe the dynamic of the exchange rate.

The paper proceeds as follows: In section 2, we will describe our model that will use to test the exchange rate market and calculate some basic statistics about our model such that we can make our model more better as we restrict the parameter of our model. In section 3, we will introduce the martingale theory to price the option and find the risk-neutral process of our model in some assumption. In section 4, we will use the real data to test the performance of our model. Finally, we will conclude in section 5.

2 Model

2.1 Setup

Assume that the economical environment we consider is discrete time for a period $[0, T]$ where the uncertainty is described by a complete filtered probability space $(\Omega, \{F_t\}_{t=0}^{T}, P)$ where F_0 contains all P-null sets. Moreover, we assume that the market is friction-free. Denote that E_t is the exchange rate between two countries, which are called Country A and Country B, that is represented as the value of B currencies that per unit A currency can be exchanged ($i.e\ Currency_B/Currency_A$) at time t. Furthermore, we assume that the process of $\{E_t\}_{t=1,2,\cdots,T}$ follows:

$$\begin{aligned}
\ln E_t &= \ln E_{t-1} + R_t + h_t X_t && (1)\\
h_t^2 &= \sigma_t^2 J_t \\
\sigma_t^2 &= \omega + \alpha\sigma_{t-1}^2 + \beta(h_{t-1}X_{t-1} - c)^2 && (2)\\
J_t &= \sum_{j=1}^{N_t+1} \Im_t^j
\end{aligned}$$

where

$$\begin{aligned}
\Im_t^j &\sim Exponential(\lambda)\ (i.i.d.)\\
N_t &\sim Poisson(\mu)\ (i.i.d.)\\
X_t &\sim N(0,1)\ (i.i.d.)
\end{aligned}$$

In addition, we assume that $\Im_t^j, N_t, X_t$ are independent for all t and j. and R_t, an

$F_{t-1}-$measurable random variable, is the expected logarithm return at time t-1.

Let r_t^A and r_t^B ,$F_{t-1}-$measurable random variables, are represented as the continuous risk-free interest rates in Country A and Country B, respectively. As an investor in Country B holds E_{t-1} units of B currencies at time t-1, if he invests to the risk-less bond in Country B, then at time t, the value will become $E_{t-1}e^{r_t^B}$. But, if he exchanges to A currencies and invests to the risk-less bond in Country A, at time t, he exchanges the A currencies back to B, then the value will become $E_t e^{r_t^A}$. Because in the physical world, the latter will exposure to the exchange rate risk, therefore, it is natural to expect that

$$E_t e^{r_t^A} \geq E_{t-1} e^{r_t^B}$$

which implies

$$\ln \frac{E_t}{E_{t-1}} \geq (r_t^B - r_t^A)$$

therefore, R_t must be greater than $(r_t^B - r_t^A)$. The result is very similar to the stock market if we view per unit of A currency as a share of stock and the risk-less interest rate as $(r_t^B - r_t^A)$.

But on the other hand, for the investor in Country A, if he holds one unit of A currency at time $t-1$, then he will have $e^{r_t^B}$ at time t as he invest in the risk-less bond at time $t-1$. If he exchanges the one currency to B currencies and then invests to the risk-less investment in Country B. At time t, he exchanges the B currencies he earned back to A currencies. then the value of A currencies that he has are $E_t e^{r_t^B}/E_{t+1}$. Of course, we will expect that

$$E_{t-1} e^{r_t^B}/E_t \geq e^{r_t^A}$$

which implies

$$\ln \frac{E_t}{E_{t-1}} \leq (r_t^B - r_t^A)$$

Combine the two results, we can get that

$$R_t = \mathbf{E}_{t-1}^P[ln \frac{E_t}{E_{t-1}}] = r_t^B - r_t^A \tag{3}$$

The result is not like other financial market such as the stock market. Many models describing the stock market will assume that the trend, R_t is equal to the risk-less interest rate plus a term of the risk premium. But in the exchange rate market, the drift term is only the difference between r_t^B and r_t^A. This implies that as we invest in the exchange rate market, the expected return seems still only a "risk-less" interest rate, not adding any additional risk premiums. But In fact, if we want to earn the "risk-less" return $r_t^B - r_t^A$, we must exposure in the risk of the exchange rates not like the risk-less interest rate in the stock market. Therefore, $r_t^B - r_t^A$ is not a risk-less rate because the risk premium is included in this term.

h_t^2, the square of the volatility term of logarithm return of exchange rates, follows a specially form of GARCH(1,1). In order to guarantee $h_t \geq 0$ for all t, we need restrict the parameters $\omega > 0$, $\alpha, \beta \geq 0$, and in addition adding $\alpha + \beta\lambda(\mu + 1) < 1$ for the stationarity. The 'leverage effect' parameter, c, plays a role that make the residuals have non-symmetry affections to $\{\sigma_t^2\}$. Many finance markets, such as stock markets, have been proved that the impact of 'bad news' is stronger than 'good news' on conditional variance. Therefore, we can find the papers discussing the stock markets, for example, Steven L. Heston and Saikat Nandi(2000), will usually make c is non-negative even if non-constant. Other studies also have similar settings. Giovanni Barone-Adesi, Robert Engle, and Loriano Mancini(2004) use the Asymmetric GARCH(1,1) model to describe this phenomenon by taking the parameter of the asymmetric term positive. Besides, Heston(1993) have mentioned that the leverage effect parameter will control the skewness or the asymmetry of the distribution of the log returns. In particular, if c is zero, the distribution is symmetric.

Finally, whether the process of logarithm exchange rates or volatilities, the two process contains jumps because of N_t. The jump term is followed a compound Poisson process like Duan's model(2004). Conditional on N_t, J_t will follow a Gamma distribution with two parameters $(N_t + 1)$ and λ. Especially, if we turn off the jump (i.e. $\mu = 0$) and take $\lambda = 1$, then the volatility process is similar to the model that R. Engle(2002) have mentioned and used to test the volatility of exchange rates.

2.2 Statistics Description

There are some phenomenons in financial markets that have been pointed out in many studies. For example, the distribution of returns have fat tails and non-symmetry, volatility may be not a constant, volatility has the clustering phenomenon and so on. In this subsection, we will simply discuss some basic statistics about our model and find out how those parameters in our model effect these phenomenons.

Proposition I

The conditional variance of logarithm return is

$$\mathbf{Var}^{P}_{t-1}[\ln \frac{E_t}{E_{t-1}}] = \lambda(\mu+1)\sigma_t^2$$

The conditional covariance between $\ln \frac{E_t}{E_{t-1}}$ and σ_t^2 is

$$\mathbf{Cov}^{P}_{t-1}(\ln \frac{E_t}{E_{t-1}}, \sigma_{t+1}^2) = -2\beta c\lambda(\mu+1)\sigma_t^2$$

Proof: See Appendix

From these above statistics, we can understand that the conditional variance of exchange rates is not a constant and followed by the dynamic of σ_t^2 which is belong to the family of GARCH models. Of course, GARCH models can catch lots of phenomenons about volatilities. Besides, from the conditional variance between $\ln \frac{E_t}{E_{t-1}}$ and σ_{t+1}^2, we can know that as $\beta \neq 0$ (i.e. the affection of ARCH exists), the parameter c will decide the sign of conditional variance. If the 'leverage effect' parameter, c, is non-negative, then it mean that the conditional variance between $\ln \frac{E_t}{E_{t-1}}$ and σ_{t+1}^2 will be negative. Indeed, many studies such as Steven L. Heston and Saikat Nandi(2000) and Giovanni Barone-Adesi, Robert Engle, and Loriano Mancini(2004) have pointed out that the stock market is better, the volatility is smaller and vise versa. Our model can also match this phenomenon if the exchange rates markets have. On the other hand, in addition to $\lambda c \neq 0$ the conditional variance is not a constant because of σ_t^2.

3 Option Pricing in Risk-Neutral

In the aspect of option pricing, martingale theory plays a very important role. But in the theory, there are three problems. First, when does the risk-neutral probability measure exist? Second, Is the risk-neutral probability measure unique? Finally, how do we find the risk-neutral probability measure? The first problem can be solved by the fundamental theorem of asset pricing

Theorem (Fundamental Theorem of Asset Pricing)

The market is arbitrage-free if and only if there exists an equivalent martingale measure.

Therefore, if we assume that the market is arbitrage-free, the risk-neutral probability measure will exist. Furthermore, if the market is complete (i.e. every contingent claim is attainable), then the risk-neutral probability measure is unique and we can use the technique of replications to find it. Hence, in Black-Scholes model, the market is complete and arbitrage-free, so we can find a fair and unique option price. But in our model, the market is incomplete. The problem we face is how to decide the risk-neutral probability measure. Thus, for using the argument of utility functions, we need to add some assumption.

Definition: (Utility Function)

A continuous function $U : (0, \infty) \to \Re$ that is strictly increasing, strictly concave and continuously differentiable with $lim_{x\to\infty} U'(x) = 0, lim_{x\to 0} U'(x) = \infty$ is called a utility function.

Here, if we assume that the representative agent is an expected utility maximizer and the utility function is time separable and additive. By the Euler equation from the standard expected utility maximization argument, we can get the conclusion:

$$E_{t-1} = \mathbf{E}^P_{t-1}[E_t \frac{U'(E_t)}{U'(E_{t-1})}]$$

$$\mathbf{E}^P_{t-1}[\frac{U'(E_t)}{U'(E_{t-1})}] = e^{-r_t}$$

where r_t is the "true" risk-less interest rate in the period $[t-1, t]$ in the exchange rate market.

Proposition II

Let

$$dQ = e^{\sum_{t=1}^{T} r_t} \frac{U'(E_T)}{U'(E_0)} dP$$

then Q is a probability measure and the discount process of exchange rates is a martingale process with respect to measure Q

Proof: The proof is similar to Duan(2004), See Duan(2004).

In this paper, we suppose that the above assumptions also hold in our economy. Furthermore, we assume that the utility function is

$$U(x) = \frac{x^a}{a} \qquad if \qquad a < 1$$
$$U(x) = \ln x \qquad if \qquad a = 0$$

where a is a parameter whose range is $(-\infty, 1)$

Proposition III

the "true" risk-less interest rate r_t is

$$r_t = (1-a)(r_t^B - r_t^A) - \ln \frac{2}{2 - \lambda(a-1)^2\sigma_t^2} - \frac{\mu\lambda(a-1)^2\sigma_t^2}{2 - \lambda(a-1)^2\sigma_t^2} \tag{4}$$

Proof: See Appendix.

Proposition IV

Under Q-measure, the process of the exchange rates is

if $a = 0$

$$\ln E_t = \ln E_{t-1} + (r_t^B - r_t^A) - \ln \frac{2}{2-\lambda\sigma_t^2} - \frac{\mu\lambda\sigma_t^2}{2-\lambda\sigma_t^2} \tag{5}$$

$$\sigma_t^2 = \omega + \beta c^2 + \alpha\sigma_{t-1}^2 \tag{6}$$

if $a \neq 0$

$$\ln E_t = \ln E_{t-1} + \widetilde{R}_t + \widetilde{h}_t \widetilde{X}_t \tag{7}$$

$$\widetilde{R}_t = (r_t^B - r_t^A) - \ln \frac{2}{2-\lambda(\frac{a-1}{a})^2\widetilde{\sigma}_t^2} - \frac{\mu\lambda(\frac{a-1}{a})^2\widetilde{\sigma}_t^2}{2-\lambda(\frac{a-1}{a})^2\widetilde{\sigma}_t^2} \tag{8}$$

$$\widetilde{h}_t^2 = \widetilde{\sigma}_t^2 \widetilde{J}_t$$

$$\widetilde{\sigma}_t^2 = \widetilde{\omega} + \widetilde{\alpha}\widetilde{\sigma}_{t-1}^2 + \widetilde{\beta}(\widetilde{h}_{t-1}\widetilde{X}_{t-1} - \widetilde{c})^2 \tag{9}$$

$$\widetilde{J}_t = \sum_{j=1}^{\widetilde{N}_t+1} \widetilde{\Im}_t^j$$

where

$$\widetilde{\Im}_t^j \sim Exponential(\lambda)\ (i.i.d.)$$

$$\widetilde{N}_t \sim Poisson(\mu)\ (i.i.d.)$$

$$\widetilde{X}_t \sim N(0,1)\ (i.i.d.)$$

and

$$\widetilde{\omega} = a^2\omega$$

$$\widetilde{\alpha} = \alpha$$

$$\widetilde{\beta} = \beta$$

$$\widetilde{c} = |a|c$$

$$\widetilde{\sigma}_t = |a|\sigma_t$$

Proof: See Appendix.

In the risk-neutral process, the form of the overall dynamic is almost the same as the physical process. Whether the logarithm return or the volatility, the two processes still contain the jump term. The distribution of $\widetilde{J}_t$ is unchanged, which is also followed a compound Poisson process that the parameter of $\widetilde{N}_t$ is μ and the parameter of $\widetilde{\Im}_t^j$ is λ. $\widetilde{X}_t$ remains a standard normal distribution. The drift term $\widetilde{R}_t$ becomes a determined function of $\widetilde{\sigma}_t$ if we have given the parameters μ and λ. This result is similar to many studies, like Duan(1995). Even the Black-Scholes model, the result is the same but $\widetilde{\sigma}_t$ is a constant. In the process of volatilities, $\{\sigma_t^2\}$, the equation is still the same except the parameters. the parameters, α and β, is unchanged. This implies that the affections of ARCH and GARCH are invariable. But ω changes to $\widetilde{\omega} = (1-a)^2\omega$ and c changes to $\widetilde{c} = (1-a)c$. For our restriction, a is smaller than 1. The sign of the two parameters is not change. If $0 \leq a < 1$, then the two risk-neutral parameters, $\widetilde{\omega}$ and $\widetilde{c}$ will be smaller and if $a < 0$, the result is opposite. As we mentioned above, $\widetilde{c}$ is represented as the 'leverage effect' in the risk-neutral process. In our model, we can understand that a will effect to $\widetilde{c}$. It implies that in finance, the investor's risk version will impact the leverage effect.

4 Empirical Research

Given any contingent claim with payoff function $\psi(S_T)$ and the maturity at time T. For example, the payoff function of the European call option with the maturity at time T is $(S_T - K)^+ = max(S_T - K, 0)$ where S_T is the price of the underlying asset and K is the exercise price. Under the martingale theory, we can get the result

$$\widehat{p}_t = e^{-\sum_{k=t+1}^{T} r_k} \mathbf{E}_t^Q[\psi(S_T)]$$

where r_t is the risk-less interest rate from time t-1 to time t and $\widehat{p}_t$ is the value of the contingent claim at time t. In simple conditions like the Black-Scholes model, $\widehat{p}_t$ has a closed form. Steven L. Heston and Saikat Nandi(2000) also derived a closed form about some GARCH model. But in many more complex models, it is very difficult to find the closed form, even the closed form may be not exist. The alternative way to find $\widehat{p}_t$ is the Monte Carlo simulation.

In the above section, we have derived the risk-neutral process by the argument of utility maximizers. Thus, we can use the Monte Carlo simulation to find the approximately value of $\widehat{p}_t$ where the underlying asset of the contingent claim is the exchange rate. In this section, we will present the performance about our model and analyze it. But before we going on, the first work we need to do is to estimate the parameters about our model.

4.1 Estimation About Parameters

About parameters model, the method to estimate the model parameters in most studies is the Maximum Likelihood Estimation (MLE). Here, we will also use the method to estimate our parameters.

Let

$$Y_t = \ln E_t - \ln E_{t-1} - R_t = h_t X_t = \sigma_t \sqrt{J_t} X_t$$

where σ_t^2 is a function of σ_{t-1}^2 and y_{t-1} as the parameters is given.

Proposition IV

The conditional probability density function, $l_{Y_t|t-1}(x)$, of Y_t is

$$l_{Y_t|t-1}(x) = \sum_{k=0}^{\infty} e^{-\mu} \frac{\mu^k}{k!} \int_0^{\infty} \frac{1}{\sigma_t \sqrt{s}} f_{X_t}\left(\frac{x}{\sigma_t \sqrt{s}}\right) f_{G_t^k}(s) ds \quad (10)$$

where $f_{X_t}(x)$ is the density function of X_t that $X_t \sim N(0,1)$ and $f_{G_t^k}(x)$ is the density function of G_t^k that $G_t^k \sim Gamma(k+1, \lambda)$.
Proof: See Appendix.

The log-likelihood function of the sample is

$$L(\theta; y_1, y_2, \cdots, y_T) \sum_{t=2}^{T} \ln l_{Y_t|t-1}(y_t)$$

and all we to do is to find $\theta*$ where

$$\theta* = \{\omega*, \alpha*, \beta*, c*, \mu*, \lambda*\}$$

such that $\theta*$ will maximize the above log-likelihood function.

4.2 Data Description

The market data we use in this study contains JPY/USD, GBP/USD, DEM/USD, EUR/USD, and NTD/USD. The sources of data mainly come from the Bloomberg database and the central bank of each region. The period of each market is at least ten years, except EUR/USD. This is because Euro starts at 1999/1/1. JPY/USD, and NTD/USD are from 1993/1/1 to 2004/12/31. DEM/USD is from 1983/10/6 to 1997/12/31. EUR/USD is from 1999/1/1 to 2004/12/31. Data in a week we consider don't contain Saturday and Sunday. Besides, if no trading occurs on other days, we will fill the data of the day before the non-trading day. Therefore, the number of the data in a week we have is five. As we gather these data, we can estimate the parameters of our model. Furthermore, we can test the prediction of our model by the market price of the options whose maturity is after the above time periods.

4.3 Empirical Analysis

In this part, we will not only observe the performances of the GARCH models with jumps in the exchange rates markets we have mentioned above, but also compare with other models. These models are as following:

Case I (h_t is precipitable)

$$\ln E_t = \ln E_{t-1} + R_t + \sigma_t X_t$$

Case II (h_t is adapted without jumps)

$$\ln E_t = \ln E_{t-1} + R_t + h_t X_t$$

$$h_t^2 = \sigma_t^2 \Im_t$$

where $\Im_t \sim Exponential(\lambda)$ (i.i.d.)

Regardless of cases, we all assume that the dynamics of σ_t follow the equation (2). By the same techniques, we can get the risk-neutral processes of the two cases as following, respectively:

Case I:

If $a = 0$

$$\ln E_t = \ln E_{t-1} + (r_t^B - r_t^A) - \frac{1}{2}\sigma_t^2$$

If $a \neq 0$

$$\ln E_t = \ln E_{t-1} + (r_t^B - r_t^A) - \frac{1}{2}(\frac{1-a}{a})^2 \widetilde{\sigma}_t^2 + \widetilde{\sigma}_t \widetilde{X}_t$$

Case II:

If $a = 0$

$$\ln E_t = \ln E_{t-1} + (r_t^B - r_t^A) - \ln \frac{2}{2 - \lambda\sigma_t^2}$$

If $a \neq 0$

$$\ln E_t = \ln E_{t-1} + (r_t^B - r_t^A) - \ln \frac{2}{2 - \lambda(\frac{1-a}{a})^2 \widetilde{\sigma}_t^2} + \widetilde{h}_t \widetilde{X}_t$$

$$\widetilde{h}_t^2 = \widetilde{\sigma}_t^2 \widetilde{\Im}_t$$

where $\widetilde{\Im}_t \sim Exponential(\lambda)$ (i.i.d.)
And $\{\sigma_t\}$ follows the equation (6) if $a = 0$ and if $a \neq 0$, $\{\widetilde{\sigma}_t\}$ follows the equation (9).

Exchange Rate and Risk Version

As using MLE to find the appreciate parameters, this does not mean that we can examine the performance of our model because of the parameter, a. In our assumption, a is represented as the degree of the representative agent's risk version. a is smaller, the change rate of the utility function is greater. The graphs from figure(1) to figure(4) are the exchange markets of JPY/USD, DEM/USD, EUR/USD, and TWD/USD. The x axis is the value of a and the y axis is represented as the value of exchange rate we simulate as a is given. Each point is the mean of 5000 simulations. As we see, a and the exchange rate have the monotone relation, except case I. The relation is very natural. From case I, we understand that the assumption of the utility function is indeed too strong such that in the simple model like case I, the relation is default. Compare Case II and Jump case, we can find the effect of the negative relation is not apparent as we add the jump term. In addition, the simulation values will diverge as a is removed from 0. The reason is because the value of a will effect the volatility. As $|a|$ is larger, the variance of the stochastic term will be greater. In particular, if $a = 0$, then the stochastic term will disappear.

Mean and Numbers of Simulations

By laws of large numbers, we understand that as we simulate more, the average value will more approximate to the expectation. But how many times of simulations are enough? The graphs from figure(5) to figure(7) are the error with the average value of 100000 simulations, and the graphs form figure(8) to figure(11) are represented as the relative error. The order of markets is the same as above. The x axis is the numbers of simulations. As we see, regardless of cases, the convergence rate will have the exponential decay phenomenons. Besides, the jump case is faster than others. This means that if reader cares about the time of computations very much, then the jump case is very efficient. To understand this, we can calculate the variance of each case as the following:

Jump Case:

$$\mathbf{Var}_{t-1}[\ln \frac{E_t}{E_{t-1}}] = \lambda(\mu + 1)\sigma_t^2$$

Case I:

$$\mathbf{Var}_{t-1}[\ln \frac{E_t}{E_{t-1}}] = \sigma_t^2$$

Case II:

$$\mathbf{Var}_{t-1}[\ln \frac{E_t}{E_{t-1}}] = \lambda\sigma_t^2$$

If reader concerns the risk-neutral process, then using $\widetilde{\sigma}_t$ to substitute σ_t.

Therefore, we can know if the variance is smaller, the convergence rate should be faster. Hence, the parameter we estimate will play an important role in the aspect of the convergence.

Prediction

Finally, we will examine the prediction of the three models. The first graph of figure(12) is the prediction of the three models in the Yen/USD market. The x axis is the time to maturity of the option price. y axis is the relative error. From the figure, we can compare the performance of the three models. Jump case is better than case II, and case II is better than case I. The second graph of figure(12) is to show the average error with different strike prices. The x axis is the strike price, and the y axis is the average error. From the figure, we can find adapted cases (i.e. jump case and case II) have better performances than predictable case. In addition, although jump case is not superior than case II absolutely, in overall, the jump case still have better than case II.

5 Conclusion

In this study, we mainly use a GARCH model with jumps described by a compounded Poisson random variable to examine exchange rates markets. Furthermore, depending on the fundamental theorem of asset pricing and the argument of the utility maximization that the utility function we set is the most common function, we can calculate the error of the option price of simulations and discuss the efficiency of the models containing jumps. In addition, we also compare the model including jumps with models without jumps and models that the volatility is predictable. From the empirical researches, we can find adapted cases are better than predictable case. The result tells us that although in two cases, the volatilities are both stochastic, not constants, the performance will be very different. In addition, in two adapted cases, the model with jump is better than the model without jump in overall. Maybe reader believe that the price path is continuous, but at least, in the aspect of predictions which is the main interesting in modelling, the jump case indeed have better results than the model without jump in the markets we consider.

Besides the empirical results, the contribution of this paper mainly presents to the importance of jumps and provide a standard procedure to price the derivatives. In fact, we also have many problems that readers can research. First, what is the 'true' dynamic of jumps? Compounded Poisson processes may suit but may not be correct. Second, what is the 'true' utility function in the market? Even reader believes that the utility function is belong to the family we assume in the exchange rate market, is the utility function the same in others? These questions in our model are only included in our assumption. We believe that if the questions can be solved, then the results will be ideal.

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Appendices

Lemma I

$$
\begin{aligned}
\mathbf{M}_{J_t}(\theta) &= \mathbf{E}_{t-1}^{P}[e^{\theta J_t}] = (\frac{1}{1-\lambda\theta})\exp\{\frac{\mu\lambda\theta}{1-\lambda\theta}\} \quad if \quad \theta < \frac{1}{\lambda} \\
\mathbf{M}_{\sqrt{J_t}X_t}(\theta) &= \mathbf{E}_{t-1}^{P}[e^{\theta\sqrt{J_t}X_t}] = \frac{2}{2-\lambda\theta^2}\exp\{\frac{\mu\lambda\theta^2}{2-\lambda\theta^2}\} \quad if \quad |\,\theta\,| < \sqrt{\frac{2}{\lambda}}
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\mathbf{M}_{J_t}(\theta) &= \mathbf{E}_{t-1}^{P}[e^{\theta J_t}] = \mathbf{E}_{t-1}^{P}[e^{\theta\sum_{j=1}^{N_t+1}\Im_t^j}] \\
&= \mathbf{E}_{t-1}^{P}[\mathbf{E}_{t-1}^{P}[e^{\theta\sum_{j=1}^{N_t+1}\Im_t^j} \mid N_t]] = \mathbf{E}_{t-1}^{P}[(\frac{1}{1-\lambda\theta})^{N_t+1}] \\
&= (\frac{1}{1-\lambda\theta})\mathbf{E}_{t-1}^{P}[e^{N_t\ln(\frac{1}{1-\lambda\theta})}] \\
&= (\frac{1}{1-\lambda\theta})\exp\{\mu(e^{\ln(\frac{1}{1-\lambda\theta})}-1)\} \\
&= (\frac{1}{1-\lambda\theta})\exp\{\mu(\frac{1}{1-\lambda\theta}-1)\}
\end{aligned}
$$

Corollary

$$\mathbf{E}_{t-1}^{P}[J_t] = \lambda(\mu+1)$$

$$\mathbf{Var}_{t-1}^{P}(J_t) = \lambda^2(2\mu+1)$$

$$\mathbf{E}_{t-1}^{P}[\sqrt{J_t}X_t] = 0$$

$$\mathbf{Var}_{t-1}^{P}(\sqrt{J_t}X_t) = \lambda(\mu+1)$$

Proof of Proposition I

$$
\begin{aligned}
\mathbf{Var}_{t-1}^{P}(\ln\frac{E_t}{E_{t-1}}) &= \mathbf{Var}_{t-1}^{P}(h_tX_t) = \sigma_t^2\mathbf{Var}_{t-1}^{P}(\sqrt{J_t}X_t) \\
&= \lambda(\mu+1)\sigma_t^2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Cov}^P_{t-1}(\sigma^2_{t+1}, \ln\frac{E_t}{E_{t-1}}) &= \mathbf{Cov}^P_{t-1}(\omega + \alpha\sigma^2_t + \beta(h_tX_t - c)^2, R_t + h_tX_t) \\
&= \beta\mathbf{Cov}^P_{t-1}((h_tX_t - c)^2, h_tX_t) \\
&= \beta[\mathbf{Cov}^P_{t-1}(h^2_tX^2_t, h_tX_t)] - 2c\mathbf{Cov}^P_{t-1}(h_tX_t, h_tX_t)] \\
&= \beta\{\mathbf{E}^P_{t-1}[h^3_tX^3_t] - \mathbf{E}^P_{t-1}[h^2_tX^2_t]\mathbf{E}^P_{t-1}[h_tX_t] - 2c[\mathbf{E}^P_{t-1}[h^2_tX^2_t] - (\mathbf{E}^P_{t-1}[h_tX_t])^2]\} \\
&= -2\beta c\mathbf{E}^P_{t-1}[h^2_tX^2_t] = -2\beta c\mathbf{E}^P_{t-1}[h^2_t] = -2\beta c\sigma^2_t\mathbf{E}^P_{t-1}[J_t] \\
&= -2\beta c\lambda(\mu+1)\sigma^2_t
\end{aligned}
$$

Proof of Proposition III

$$
\begin{aligned}
\mathbf{E}^P_{t-1}[\frac{U'(E_t)}{U'(E_{t-1})}] &= \mathbf{E}^P_{t-1}[\frac{(E_t)^{a-1}}{(E_{t-1})^{a-1}}] \\
&= \mathbf{E}^P_{t-1}[(e^{(r^B_t - r^A_t)+h_tX_t})^{a-1}] \\
&= e^{(a-1)(r^B_t - r^A_t)}\mathbf{E}^P_{t-1}[e^{(a-1)\sigma_t\sqrt{J_t}X_t}] \\
&= e^{(a-1)(r^B_t - r^A_t)}\frac{2}{2-\lambda(a-1)^2\sigma^2_t}e^{\frac{\mu\lambda(a-1)^2\sigma^2_t}{2-\lambda(a-1)^2\sigma^2_t}} = e^{-r_t} \\
\text{Hence} & \\
r_t &= (1-a)(r^B_t - r^A_t) - \ln\frac{2}{2-\lambda(a-1)^2\sigma^2_t} - \frac{\mu\lambda(a-1)^2\sigma^2_t}{2-\lambda(a-1)^2\sigma^2_t}
\end{aligned}
$$

Proof of Proposition IV

$$
\begin{aligned}
\mathbf{E}^Q_{t-1}[E_t] &= \mathbf{E}^P_{t-1}[E_t \frac{U'(E_t)}{U'(E_{t-1})} e^{r_t}] \\
&= \mathbf{E}^P_{t-1}[E_t (\frac{E_t}{E_{t-1}})^{a-1} e^{r_t}] \\
&= \mathbf{E}^P_{t-1}[E_t (e^{R_t + h_t X_t})^{a-1} e^{r_t}] \\
&= \mathbf{E}^P_{t-1}[E_{t-1} (e^{R_t + h_t X_t})^{a} e^{r_t}] \\
& \quad \text{if } a = 0 \text{ then the risk-neutral process is} \\
\ln E_t &= \ln E_{t-1} + r_t \\
&= \ln E_{t-1} + (r_t^B - r_t^A) - \ln \frac{2}{2 - \lambda \sigma_t^2} - \frac{\mu \lambda \sigma_t^2}{2 - \lambda \sigma_t^2} \\
& \quad \text{because the residual term is zero, the GARCH model becomes} \\
\sigma_t^2 &= \omega + \beta c^2 + \alpha \sigma_{t-1}^2 \\
& \quad \text{if } a \neq 0 \\
&= \mathbf{E}^P_{t-1}[E_{t-1} e^{a R_t + a h_t X_t + r_t}] \\
&= \mathbf{E}^P_{t-1}[E_{t-1} e^{a(r_t^B - r_t^A) + (1-a)(r_t^B - r_t^A) - \ln \frac{2}{2-\lambda(a-1)^2\sigma_t^2} - \frac{\mu\lambda(a-1)^2\sigma_t^2}{2-\lambda(a-1)^2\sigma_t^2} + a h_t X_t}] \\
&= \mathbf{E}^P_{t-1}[E_{t-1} e^{(r_t^B - r_t^A) - \ln \frac{2}{2-\lambda(a-1)^2\sigma_t^2} - \frac{\mu\lambda(a-1)^2\sigma_t^2}{2-\lambda(a-1)^2\sigma_t^2} + |a| h_t sgn(a) X_t}] \\
&= \mathbf{E}^P_{t-1}[E_{t-1} e^{(r_t^B - r_t^A) - \ln \frac{2}{2-\lambda(a-1)^2\sigma_t^2} - \frac{\mu\lambda(a-1)^2\sigma_t^2}{2-\lambda(a-1)^2\sigma_t^2} + |a| h_t \widetilde{X}_t}] \\
&= \mathbf{E}^P_{t-1}[E_{t-1} e^{(r_t^B - r_t^A) - \ln \frac{2}{2-\lambda(\frac{a-1}{a})^2(|a|\sigma_t)^2} - \frac{\mu\lambda(\frac{a-1}{a})^2(|a|\sigma_t)^2}{2-\lambda(\frac{a-1}{a})^2(|a|\sigma_t)^2} + (|a|\sigma_t)\sqrt{J_t}\widetilde{X}_t}] \\
&= \mathbf{E}^P_{t-1}[E_{t-1} e^{\widetilde{R}_t + \widetilde{h}_t \widetilde{X}_t}]
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{X}_t &= sgn(a) X_t \sim N(0,1) \ , sgn(a) = \begin{cases} 1 & if \ a > 0 \\ -1 & if \ a < 0 \end{cases} \\
\widetilde{R}_t &= (r_t^B - r_t^A) - \ln \frac{2}{2 - \lambda(\frac{a-1}{a})^2 \widetilde{\sigma}_t^2} - \frac{\mu\lambda(\frac{a-1}{a})^2 \widetilde{\sigma}_t^2}{2 - \lambda(\frac{a-1}{a})^2 \widetilde{\sigma}_t^2} \\
\widetilde{\sigma}_t &= |a| \sigma_t \\
\widetilde{h}_t^2 &= \widetilde{\sigma}_t^2 J_t
\end{aligned}
$$

and

$$\sigma_t^2 = \omega + \alpha\sigma_{t-1}^2 + \beta(h_{t-1}X_{t-1} - c)^2$$

implies

$$\widetilde{\sigma}_t^2 = \widetilde{\omega} + \widetilde{\alpha}\widetilde{\sigma}_{t-1}^2 + \widetilde{\beta}(\widetilde{h}_{t-1}\widetilde{X}_{t-1} - \widetilde{c})^2$$

where

$$\begin{aligned}
\widetilde{\omega} &= a^2\omega \\
\widetilde{\alpha} &= \alpha \\
\widetilde{\beta} &= \beta \\
\widetilde{c} &= |a|c \\
\widetilde{\sigma}_t &= |a|\sigma_t
\end{aligned}$$

Proof of Proposition V

$$\begin{aligned}
F_{Y_t|t-1}(x) &= P_{t-1}(Y_t \le x) = P_{t-1}(\sigma_t\sqrt{J_t}X_t \le x) = P_{t-1}(\sigma_t\sqrt{\sum_{j=1}^{N_t+1}\Im_t^j}X_t \le x) \\
&= \sum_{k=0}^{\infty} P_{t-1}(\sigma_t\sqrt{\sum_{j=1}^{k+1}\Im_t^j}X_t \le x)P_{t-1}(N_t = k) \\
&= \sum_{k=0}^{\infty} e^{-\mu}\frac{\mu^k}{k!}P_{t-1}(\sigma_t\sqrt{G_t^k}X_t \le x) \qquad G_t^k \sim Gamma(k+1, \lambda) \\
&= \sum_{k=0}^{\infty} e^{-\mu}\frac{\mu^k}{k!}\int_0^{\infty} P_{t-1}(X_t \le \frac{x}{\sigma_t\sqrt{s}})f_{G_t^k}(s)ds \\
&= \sum_{k=0}^{\infty} e^{-\mu}\frac{\mu^k}{k!}\int_0^{\infty} F_{X_t}(\frac{x}{\sigma_t\sqrt{s}})f_{G_t^k}(s)ds
\end{aligned}$$

where $F_{X_t}(x)$ is the distribution function of X_t and $f_{G_t^k}(x)$ is the density function of G_t^k

$$\begin{aligned}
l_{Y_t|t-1}(x) &= \frac{d}{dx}F_{Y_t|t-1}(x) \\
&= \sum_{k=0}^{\infty} e^{-\mu}\frac{\mu^k}{k!}\int_0^{\infty} \frac{1}{\sigma_t\sqrt{s}}f_{X_t}(\frac{x}{\sigma_t\sqrt{s}})f_{G_t^k}(s)ds
\end{aligned}$$

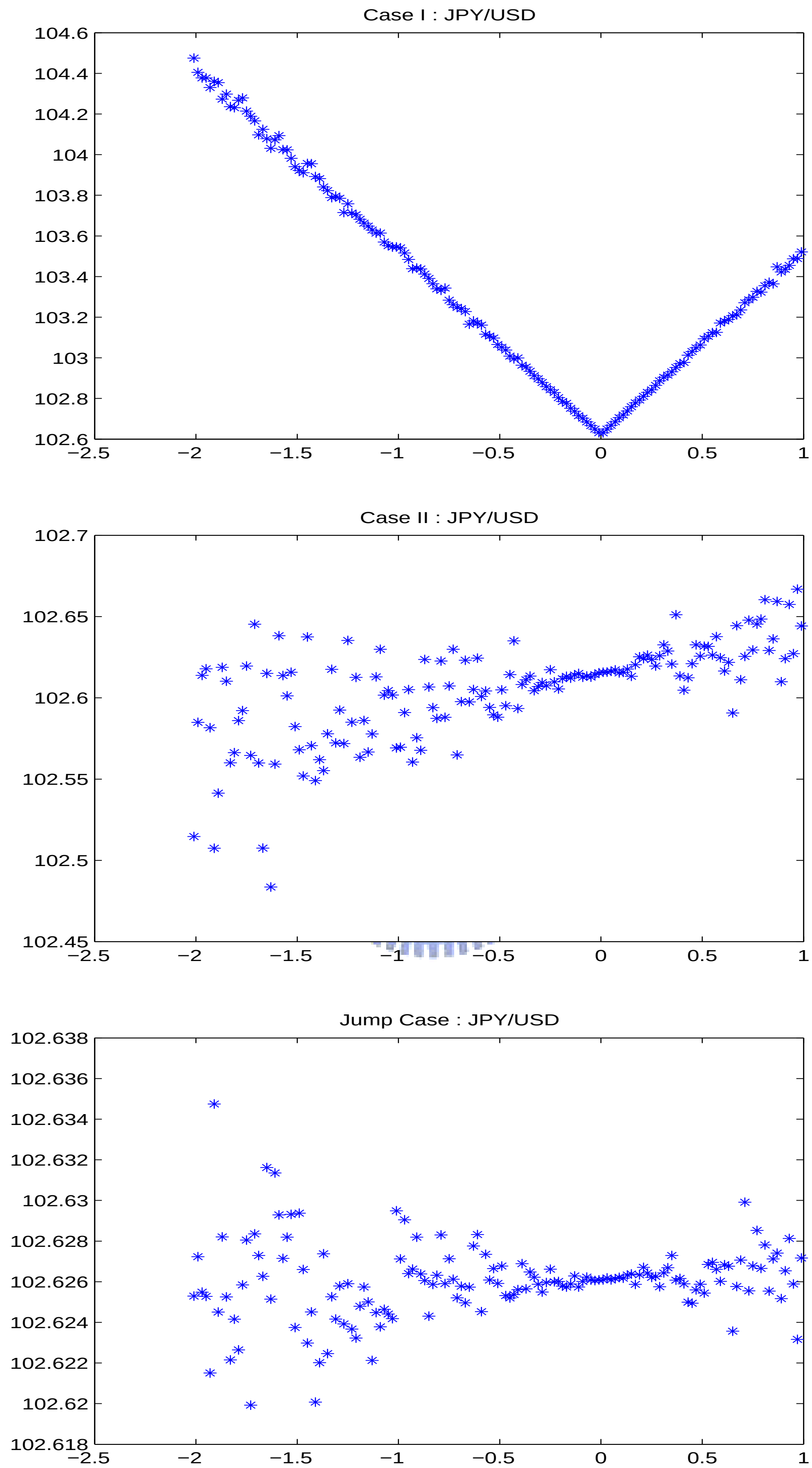

Figure 1:
Relation between a and exchange rate in Yen/USD (simulation:5000)

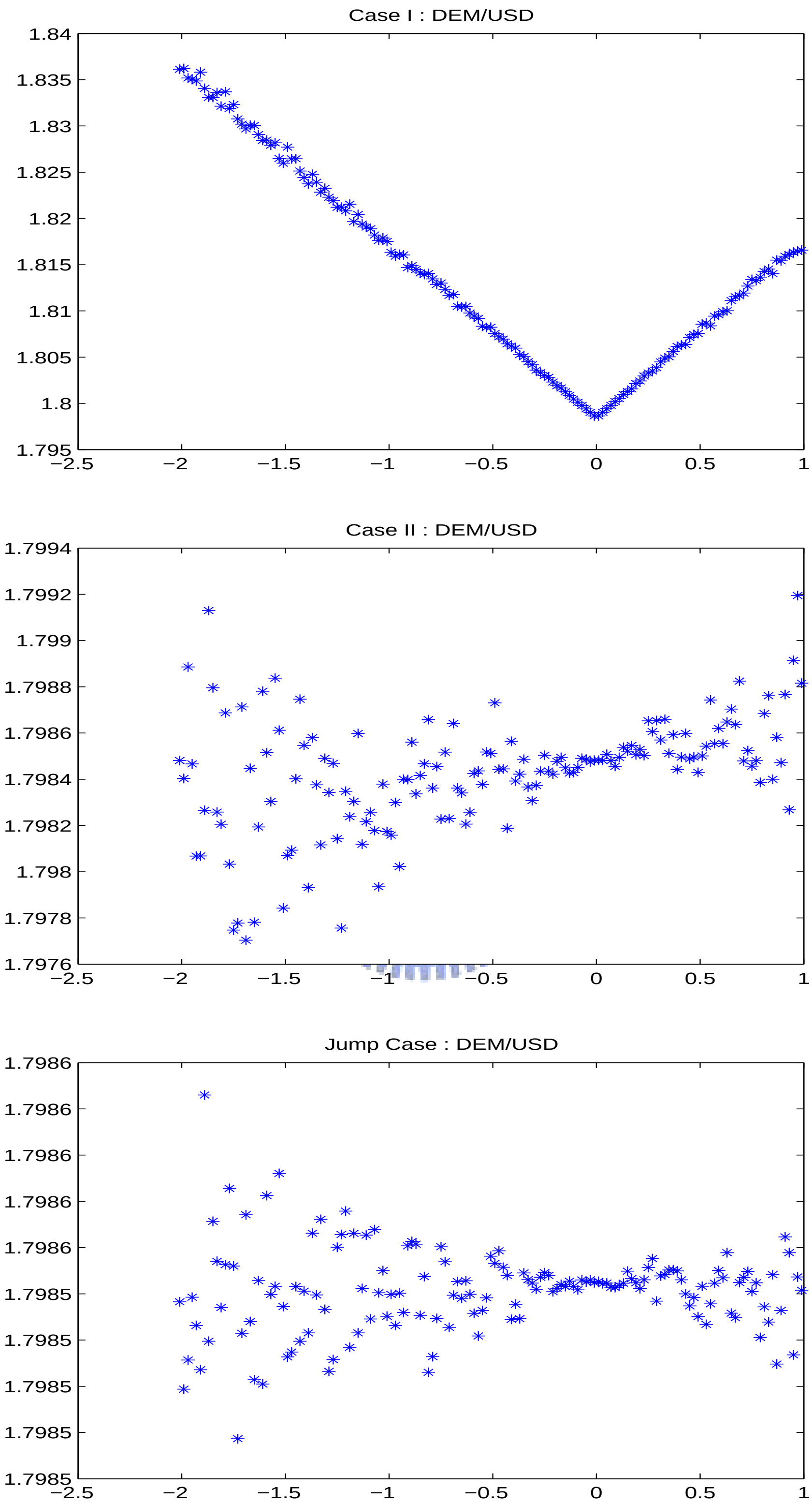

Figure 2:
Relation between a and exchange rate in DEM/USD (simulation:5000)

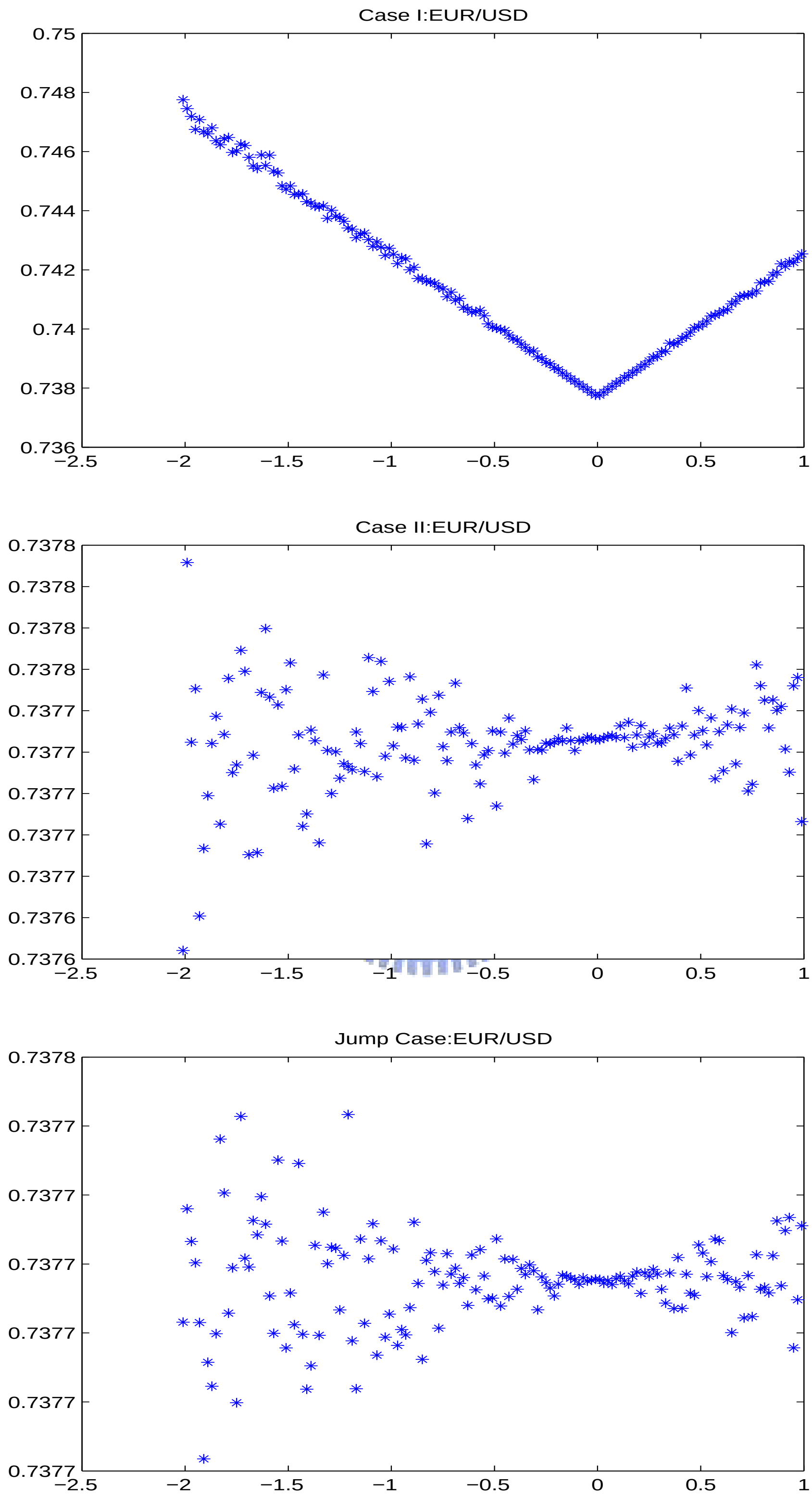

Figure 3:
Relation between a and exchange rate in EUR/USD (simulation:5000)

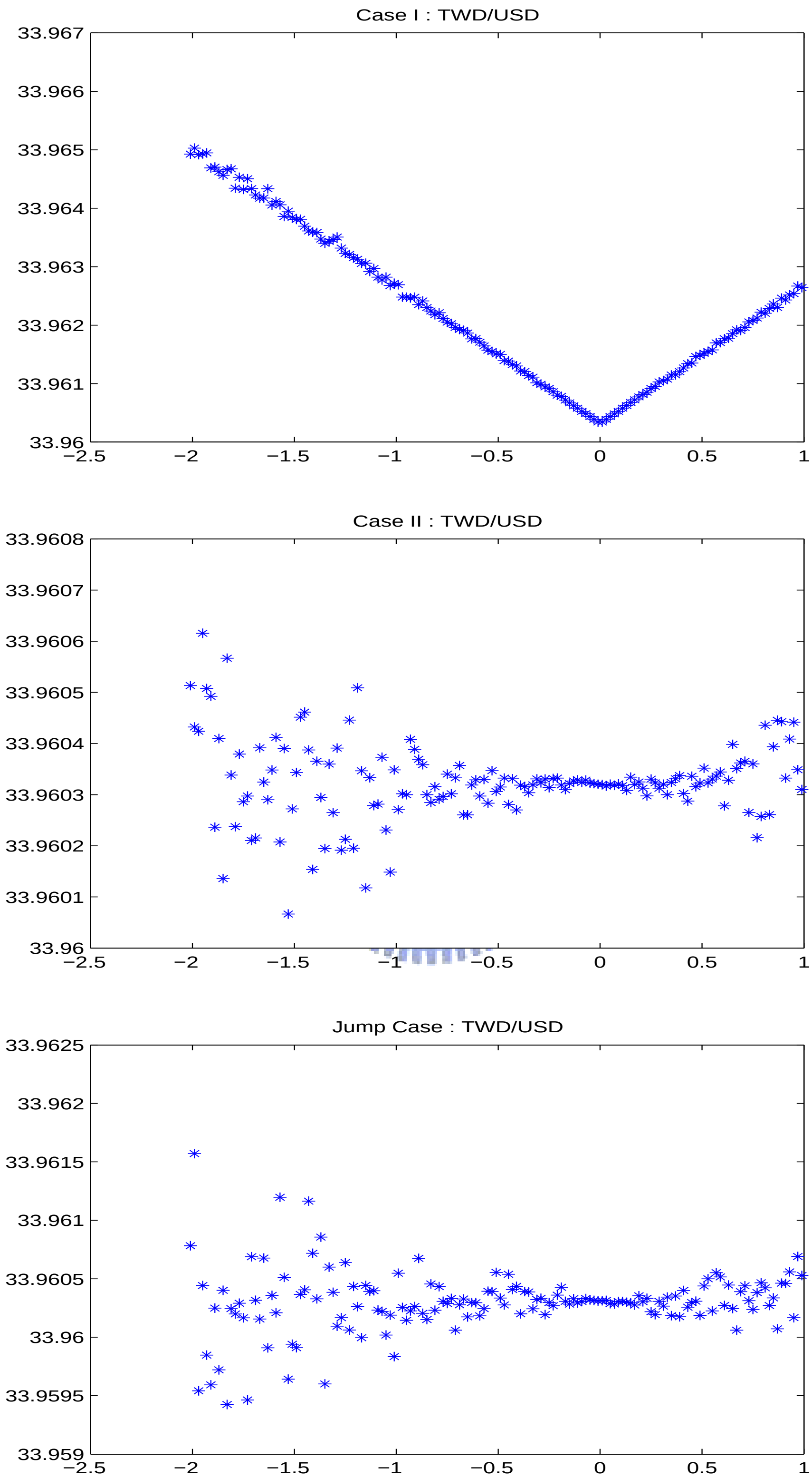

Figure 4:
Relation between a and exchange rate in NTD/USD (simulation:5000)

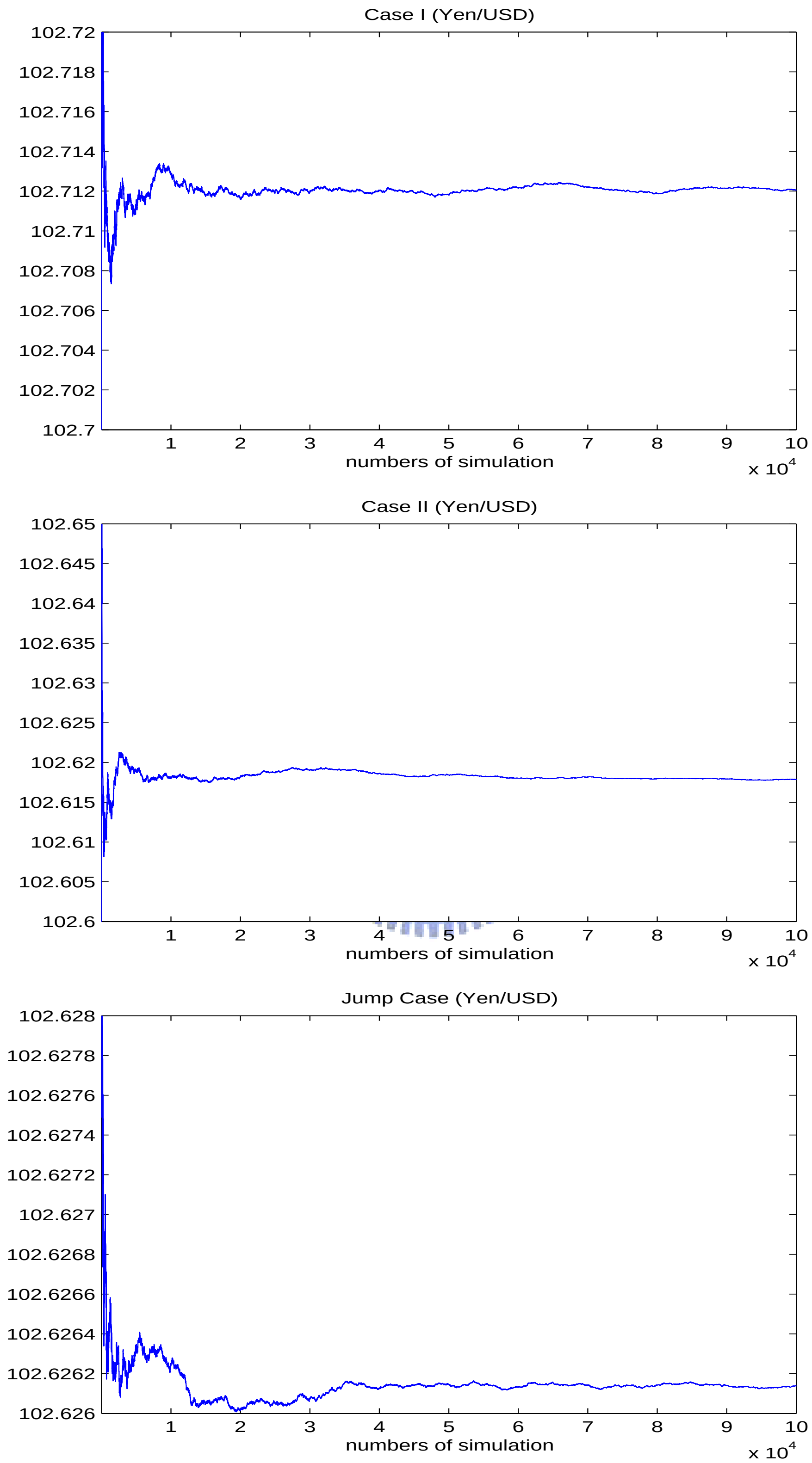

Figure 5:
Expected exchange rates of simulations in Yen/USD

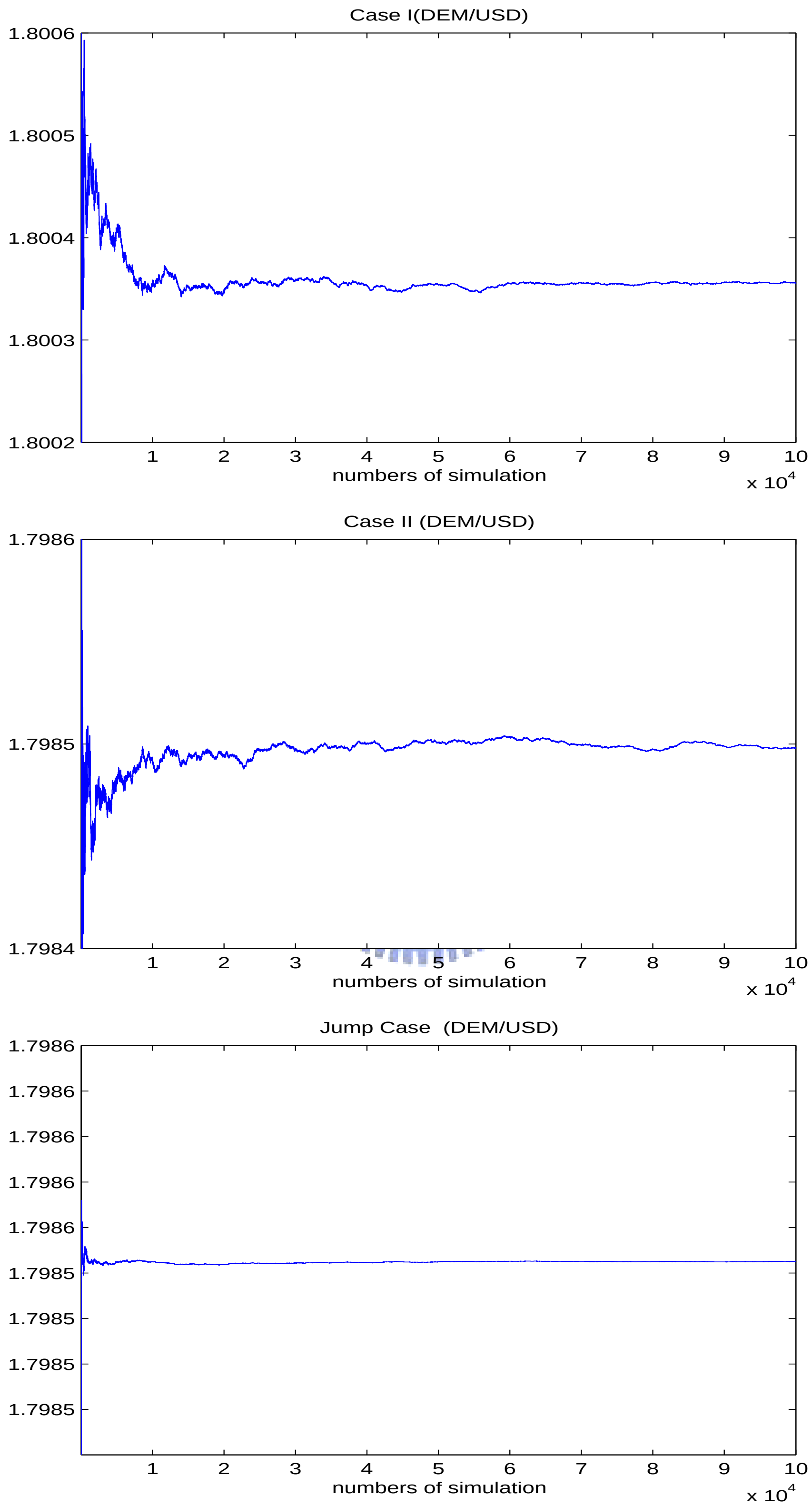

Figure 6:
Expected exchange rates of simulations in DEM/USD

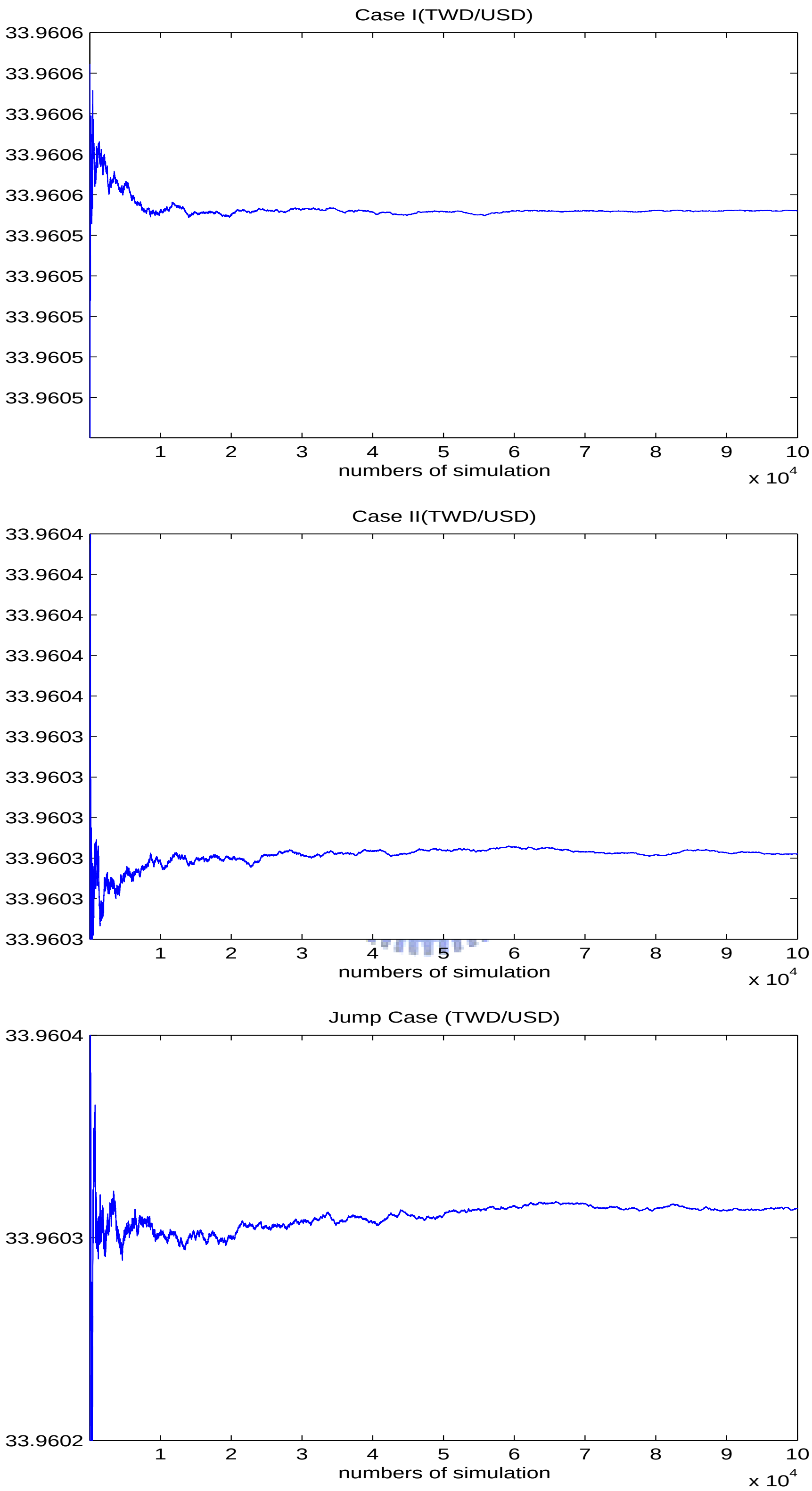

Figure 7:
Expected exchange rates of simulations in NTD/USD

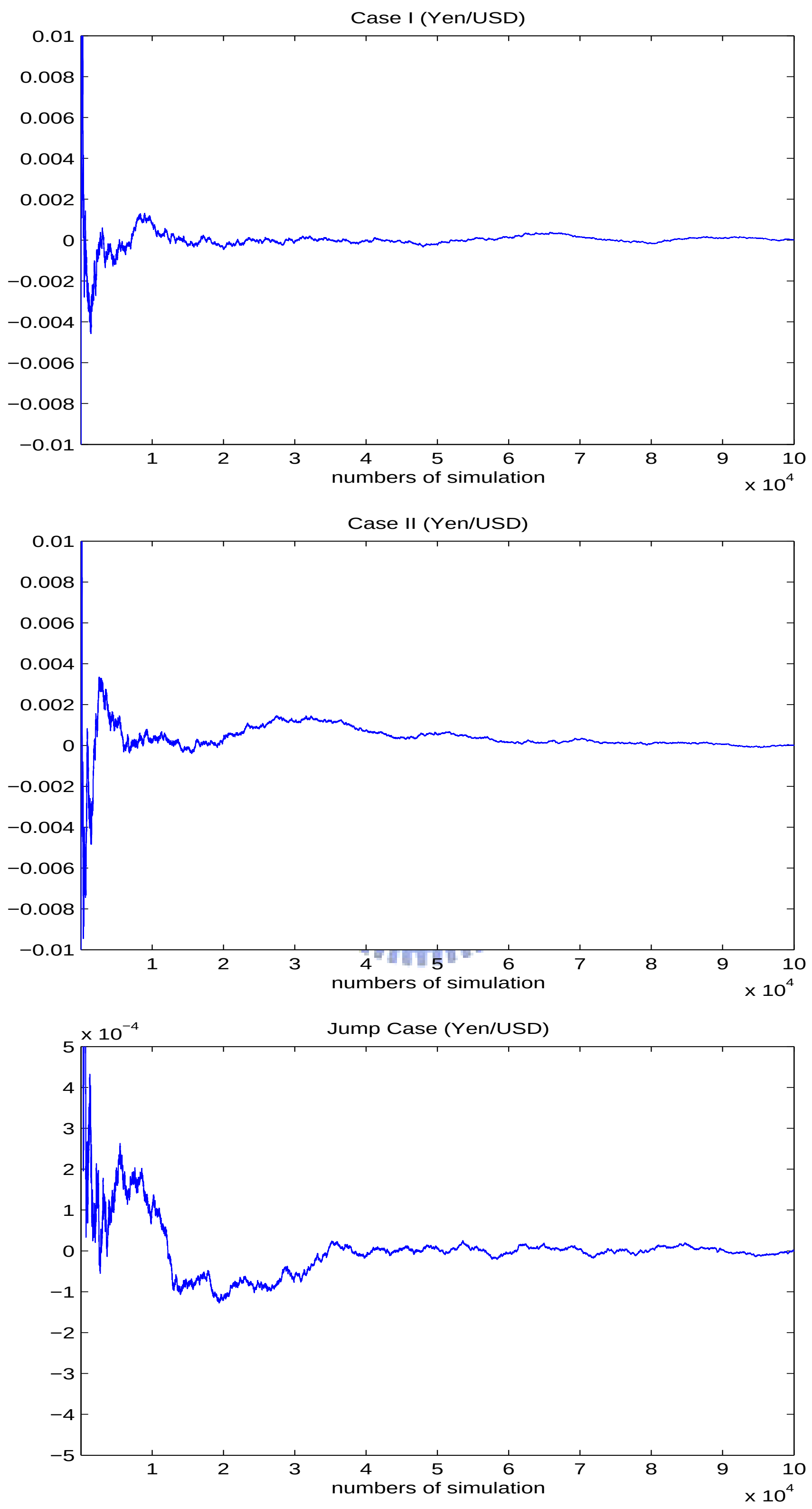

Figure 8:
Relative error between expected exchange rates of simulations and market(Yen/USD)

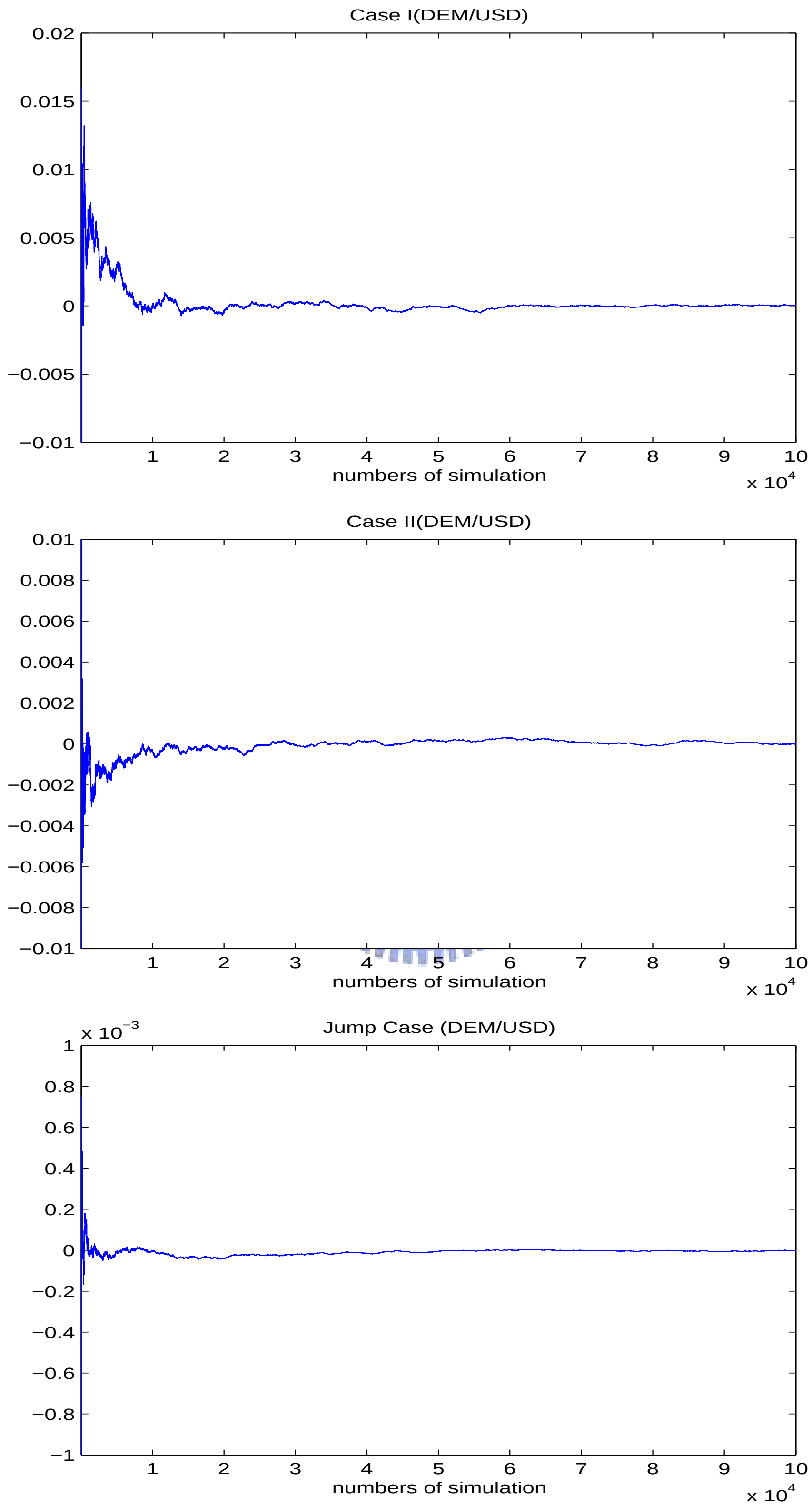

Figure 9:
Relative error between expected exchange rates of simulations and market(DEM/USD)

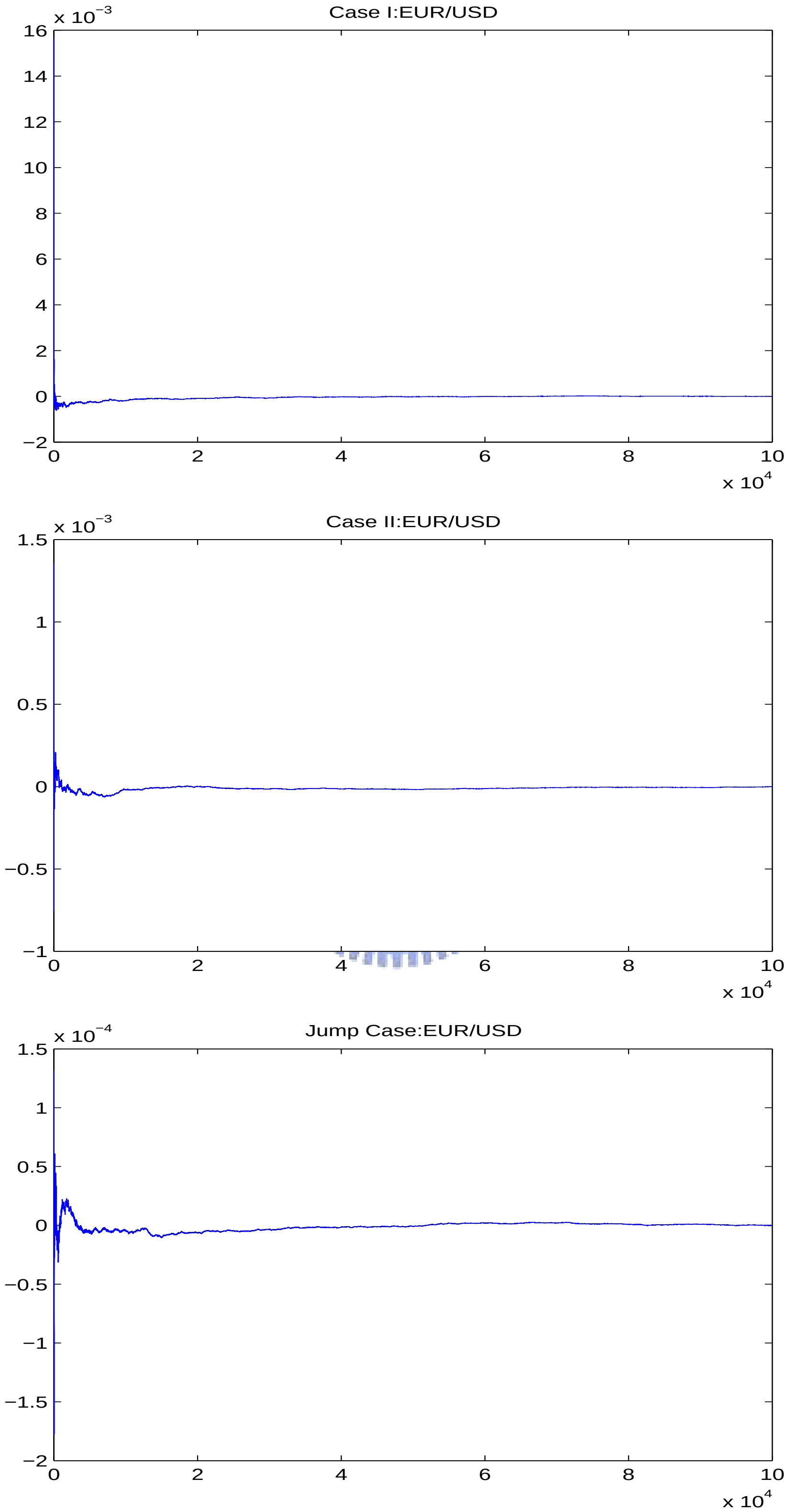

Figure 10:
Relative error between expected exchange rates of simulations and market(EUR/USD)

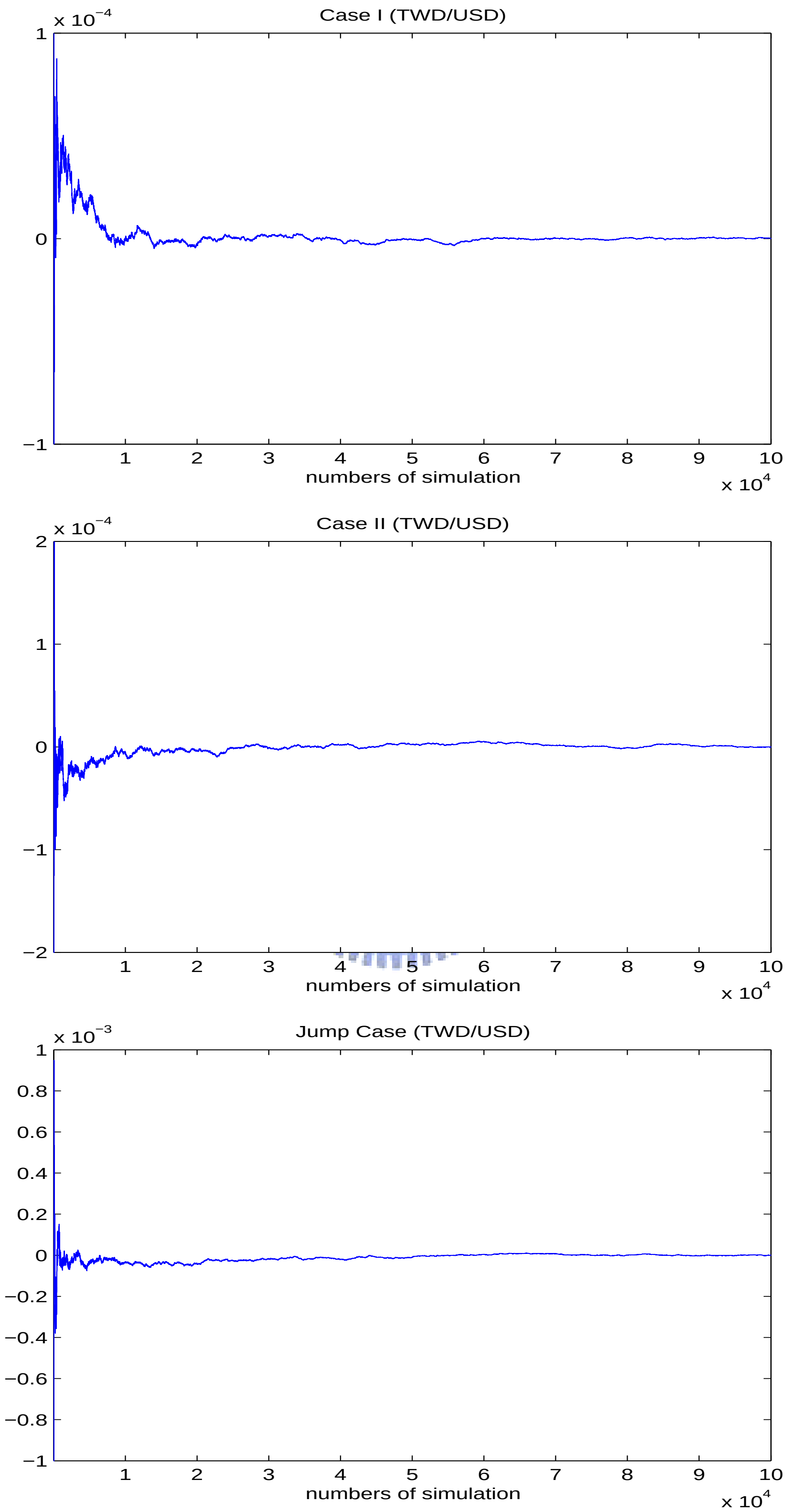

Figure 11:
Relative error between expected exchange rates of simulations and market(NTD/USD)

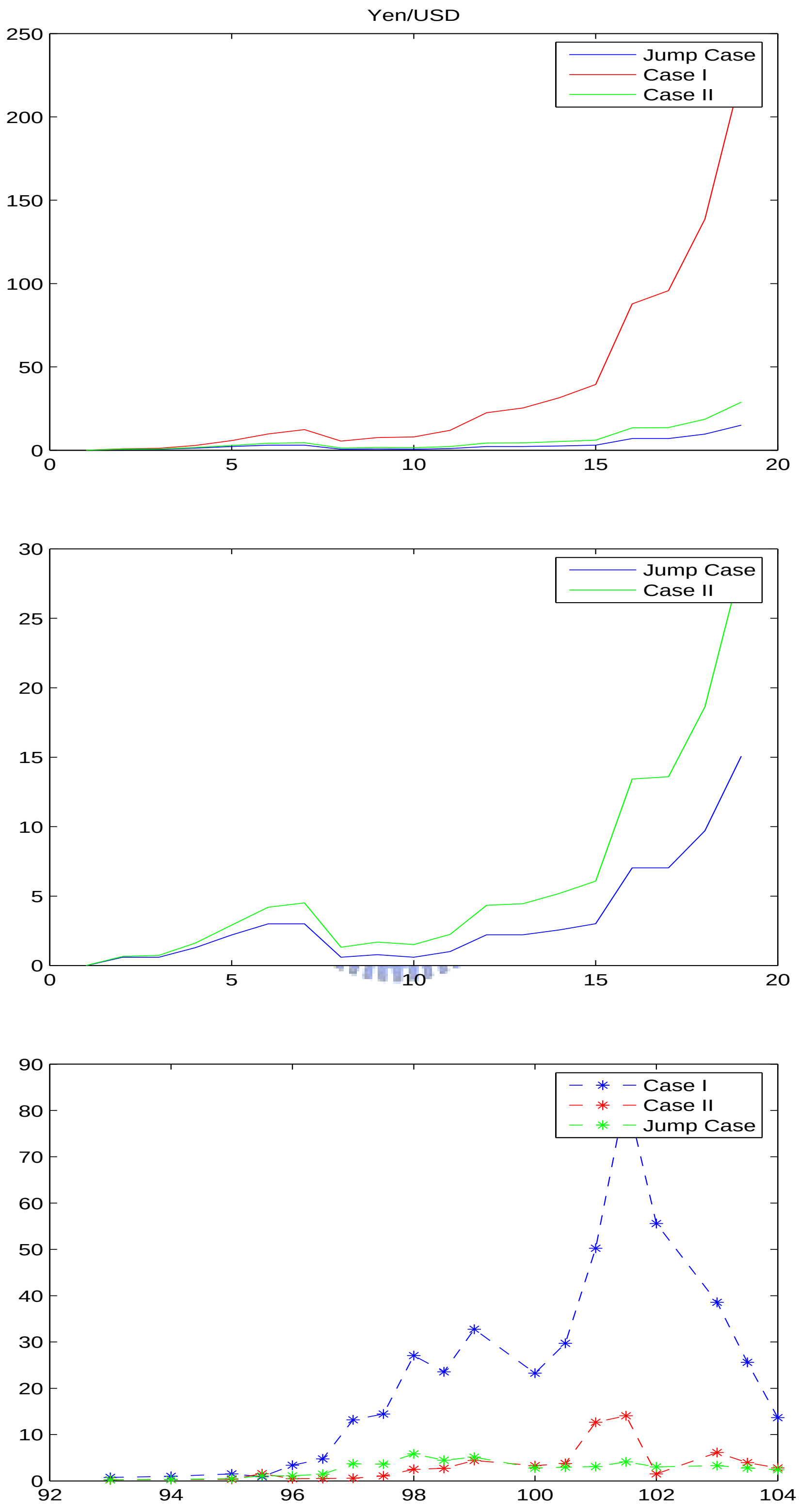

Figure 12:
Prediction of models in Yen/USD
First: error between option price of simulations and markets
that the strike price is 103 Yen/USD.
Second:the magnification of the first graph.
Third: the average error between option price of simulations and markets in different strike prices.