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離散時間型競爭系統的全局一致性

Global Consensus for Discrete-time Competitive System

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摘　　　要

本論文討論一個離散時間型競爭系統,我們的興趣在於這個系統如何達到全局一致性。不用 Lyapunov 函數,而使用分析的討論去推論出隨著時間趨近於無限,每個解的軌跡收斂到某個定值 。

Global Consensus for Discrete-time Competitive System

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ABSTRACT

A discrete-time competitive system is studied. We are interested in how the dynamics of the system reach global consensus. Analytical arguments are developed to conclude that every orbit converges to a point as time tends to infinity, without knowing a Lyapunov function.

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目　錄

Global Consensus for Discrete-time Competitive System

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Abstract

A discrete-time competitive system is studied. We are interested in how the dynamics of the system reach global consensus. Analytical arguments are developed to conclude that every orbit converges to a point as time tends to infinity, without knowing a Lyapunov function.

1 Introduction

One of the commonest ways to guarantee convergence of dynamics is to find a Lyapunov function for the system, that is, a continuous real valued function V on state space, which is nonincreasing along trajectories of the system. One then applies the LaSalle's invariance principle to conclude the convergence. For example, Cohen and Grossberg (1983) [1] proved one convergence theorem for neural network systems of the form

$$\dot{x}_i = a_i(\mathbf{x})[b_i(x_i) - \sum_{j=1}^{n} \omega_{ij} g_j(x_j)], \quad i = 1, \cdots, n, \tag{1.1}$$

where $a_i \geq 0$, the matrix $[\omega_{ij}]$ of coupling weights is symmetric, and $g_j^{'} \geq 0$ for all j. There exists a Lyapunov function

$$V(\mathbf{x}) = -\sum_{i=1}^{n} \int_0^{x_i} b_i(\xi) g_i^{'}(\xi) d\xi + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij} g_i(x_i) g_j(x_j).$$

They showed that if $a_i > 0$ and $g_i^{'} > 0$ for every i, then V is a strict Liapunov function and therefore the system is quasi-convergent, see also [9]. Forti et. al. (1995) [2] proved global stability of Hopfield-type neural network of the form

$$\dot{x}_i = -d_i x_i + \sum_{j=1}^{n} T_{ij} g_j(x_j) + I_i, \tag{1.2}$$

where $d_i > 0$, g_j is nondecreasing function. Again, the results obtained therein employed a Lyapunov function of the so-called generalized Lur'e-Postnikov type. However, it is not always easy to find a suitable Lyapunov function when considering convergent dynamics. Grossberg (1978) [3] proved a convergence theorem for a class of "competitive systems" for which no Lyapunov functions are known. He considered systems of the form

$$\dot{x}_i = a_i(\mathbf{x})[b_i(x_i) - c(\mathbf{x})], \tag{1.3}$$

where $a_i \geq 0$, $\frac{\partial c}{\partial x_i} \geq 0$, for $i = 1, \cdots, n$. Herein, each b_i is a function of only one variable x_i, and the function c does not depend on i. In this kind of system, population x_i at neuron i competes indirectly with other x_j through a scalar $c(\mathbf{x})$, i.e., the interaction among neurons are through function $c(\mathbf{x})$. Worth noticed, it is difficult to find a suitable Lyapunov function for (1.3). In fact, systems (1.1), (1.2) both can be written in the form

$$\dot{x}_i = a_i(\mathbf{x})[b_i(x_i) - c_i(\mathbf{x})],$$

which has a crucial difference from (1.3).

The "competition" for (1.3) by Grossberg means $a_i \geq 0, \frac{\partial c}{\partial x_j} \geq 0$, for all i, j and therefore has a little different sense from the commonly used one. Usually , a system $\dot{x}_i = G_i(x_1, x_2, \cdots, x_n)$ is competitive if $\frac{\partial G_i}{\partial x_j} \leq 0$, for $i \neq j$. The sense of competition in Grossberg's paper can be seen if we consider functions a_i as positive constants. The assumption on a_i for the studied dynamics is more general though.

Let us give more details to Grossberg's model. In (1.3), n is any integer greater than 1, $\mathbf{x}(t) = (x_1(t), x_2(t), \cdots, \cdots, x_n(t)) \in \mathbb{R}^n$. Such a system can have any number of competing populations, any interpopulation signal functions $b_i(x_i)$, any mean competition function, or adaptation level $c(\mathbf{x})$, and any state-dependent amplifications $a_i(\mathbf{x})$ of the competitive balance. That work in [3] proved that any initial value $\mathbf{x}(0) \geq 0$ (i.e. $x_i(0) \geq 0$, for any i) generates a limiting pattern $\mathbf{x}(\infty) = (x_1(\infty), x_2(\infty), \cdots, x_n(\infty))$ with $0 \leq x_i(\infty) := \lim_{t\to\infty} x_i(t) < \infty$, under some conditions on a_i, b_i, c. We shall summarize the main ideas of Grossberg's work in Section 4.

Recently, discrete-time systems have attracted much scientific interests, cf. [5], [6], [8]. In this study, we consider the following discrete-time version of Grossberg's model

$$x_i(k+1) = x_i(k) + \beta a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))], \tag{1.4}$$

where $i = 1, 2, \cdots, n$, $k \in \mathbb{N}_0 := \{0\} \bigcup \mathbb{N}$. Viewing from the δ-operator, (1.3) can be approximated by

$$x_i((k+1)\delta) = x_i(k\delta) + \delta a_i(\mathbf{x}(k\delta))[b_i(x_i(k\delta)) - c(\mathbf{x}(k\delta))]. \tag{1.5}$$

One usually takes $x_i[k]_\delta := x_i(k\delta)$ as the k-th iteration of x_i and $\mathbf{x}[k]_\delta := (x_1[k]_\delta, x_2[k]_\delta, \cdots, x_n[k]_\delta)$ as the k-th iteration of $\mathbf{x}$.

In this presentation, we mainly consider (1.4) with $\beta = 1$, i.e.

$$x_i(k+1) = x_i(k) + a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))]. \tag{1.6}$$

We define $\Delta x_i(k) := x_i(k+1) - x_i(k)$, hence system (1.6) can be rewritten in the form

$$\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))]. \tag{1.7}$$

The main purpose of this investigation is to find out under what conditions on functions a_i, b_i, and c, systems (1.4) or (1.6) possesses a global limiting pattern $\mathbf{x}(\infty) := (x_1(\infty), x_2(\infty), \cdots, x_n(\infty))$ with $-\infty < x_i(\infty) := \lim_{t\to\infty} x_i(t) < \infty$ for every i, given any initial value $\mathbf{x}(0)$.

Below, in Section 2, we state the main results of this presentation. In Section 3, we prove three key lemmas for our main result Theorem 1. In Section 4, we summarize the work of Grossberg [3] and make a generalization. A comparison of the analysis in deriving the global consensus for the continuous-time and the discrete-time competitive systems is also made.

2 Main Results

Definition 2.1. ***(Global Consensus)*** *A discrete-time competitive system is said to achieve global consensus (or global pattern information) if, given any initial value* $\mathbf{x}(\mathbf{0}) \in \mathbb{R}^n$, *the limit* $x_i(\infty) := \lim_{k\to\infty} x_i(k)$ *exist, for all* $i = 1, 2, \cdots, n$.

The main results require the following conditions :

Condition (I): Each $a_i(\mathbf{x})$ is continuous, and

$$0 < a_i(\mathbf{x}) \leq 1, \text{ for all } \mathbf{x} \in \mathbb{R}^n, i = 1, \cdots, n. \tag{2.1}$$

Condition (I)$'$: Each $a_i(\mathbf{x})$ is continuous, and

$$0 < a_i(\mathbf{x}) \leq A, \text{ for all } \mathbf{x} \in \mathbb{R}^n, i = 1, \cdots, n. \tag{2.2}$$

Condition (II): $c(\mathbf{x})$ is bounded and continuously differentiable with bounded derivatives; namely, there exist constants M_1, M_2, r_j such that

$$M_1 \leq c(\mathbf{x}) \leq M_2, \tag{2.3}$$

$$0 \leq \frac{\partial c}{\partial x_j}(\mathbf{x}) \leq r_j, \tag{2.4}$$

for all $\mathbf{x} \in \mathbb{R}^n$, and $j = 1, 2, \cdots, n$.

Condition (III): $b_i(\xi)$ is continuously differentiable, strictly decreasing and there exist $d_i > 0$, $l_i \in \mathbb{R}$, $u_i \in \mathbb{R}$ such that

$$-d_i \leq b_i'(\xi) < 0, \text{ for all } \xi \in \mathbb{R}, \tag{2.5}$$

and

$$b_i(\xi) > M_2, \text{ for } \xi \leq l_i, \tag{2.6}$$

$$b_i(\xi) < M_1, \text{ for } \xi \geq u_i. \tag{2.7}$$

Condition (IV): For $i = 1, \cdots, n$,

$$0 < d_i \leq 1 - \sum_{j=1}^{n} r_j < 1. \tag{2.8}$$

Condition (IV)$'$: For $i = 1, \cdots, n$,

$$0 < d_i \leq \frac{1}{\beta} - \sum_{j=1}^{n} r_j < \frac{1}{\beta}. \tag{2.9}$$

Condition (IV)$''$: For $i = 1, \cdots, n$,

$$0 < d_i \leq \frac{1}{A\beta} - \sum_{j=1}^{n} r_j < \frac{1}{A\beta}. \tag{2.10}$$

Set

$$d := min\{d_i : i = 1, 2, \cdots, n\}, \tag{2.11}$$

$$M := max\{|M_1|, |M_2|\}. \tag{2.12}$$

Theorem 1. *System* (1.6) *with functions* a_i,b_i,*and* c *satisfying Conditions (I), (II), (III), and (IV) achieves global consensus.*

The proof of Theorem 1 consists of three lemmas stated below. For system (1.4), the following corollary can be derived.

Corollary 2. *System* (1.4) *with functions* a_i, b_i, *and* c *satisfying Conditions (I),(II), (III), and (IV)′ achieves global consensus.*

In fact, we only need that function a_i is continuous, positive and bounded above by some real number, say A, for all i, instead of Condition (I). It is due to that (1.6) can be rewritten as

$$x_i(k+1) = x_i(k) + \frac{a_i(\mathbf{x}(k))}{A}[Ab_i(x_i(k)) - Ac(\mathbf{x}(k))].$$

We thus derive the following Corollary.

Corollary 3. *System* (1.4) *whose functions* a_i, b_i,*and* c *satisfy Condition (I)′, (II), (III), and (IV)″ achieves global consensus.*

Remark 2.1. *From Corollary 3, we find that the smaller* β *in* (1.4) *(*δ *in* (1.5)*) is, the weaker restrictions on functions* a_1, b_i, c *are. In other words, when we consider* (1.4) *in stead of* (1.3)*, and want to have the global consensus proposition, we must choose sufficiently small* β *in* (1.4)*, basically.*

In order to state the key lemmas for our main result, Theorem 1, we introduce some notations and definition as follows:

Notation 2.2.

$$\begin{aligned}
g_i(k) &= b_i(x_i(k)) - c(\mathbf{x}(k)), \\
\Delta g_i(k) &= g_i(k+1) - g_i(k), \\
\hat{g}(k) &= \max\{g_i(k) : i = 1, 2, \cdots, n\}, \\
\check{g}(k) &= \min\{g_i(k) : i = 1, 2, \cdots, n\}, \\
I(k) &= \min\{i : g_i(k) = \hat{g}(k)\}, \\
J(k) &= \min\{i : g_i(k) = \check{g}(k)\}, \\
\hat{x}(k) &= x_{I(k)}(k), \\
\check{x}(k) &= x_{J(k)}(k), \\
\hat{b}(k) &= b_{I(k)}(\hat{x}(k)), \\
\check{b}(k) &= b_{J(k)}(\check{x}(k)), \\
\Delta\hat{b}(k) &= \hat{b}(k+1) - \hat{b}(k), \\
\Delta\check{b}(k) &= \check{b}(k+1) - \check{b}(k), \\
\Delta b_i(x_i(k)) &= b_i(x_i(k+1)) - b_i(x_i(k)).
\end{aligned}$$

Definition 2.3. *(i) A jump of type-1 is said to occur from i to j at k-th iteration if $I(k) = i$, I(k+1)=j, (ii) A jump of type-2 is said to occur from i to j at k-th iteration if $J(k) = i, J(k+1) = j$.*

Lemma 1. *Consider system* (1.6) *with a_i, b_i, and c satisfying* (2.1), (2.3), (2.5), (2.6) *and* (2.7). *Given any initial value $\mathbf{x}(0) \in \mathbb{R}^n$, $\{\mathbf{x}(k)\}$ will be attracted to some compact set contained in $\mathbb{R}^n$. Hence sequence $\{x_i(k) \mid k \in \mathbb{N}_0\}$ are bounded above and below for all $i = 1, 2, \cdots, n$.*

If Lemma 1 is valid, consider an arbitrary orbit $\{\mathbf{x}(k)\}$. Then $\{|a_i(\mathbf{x}(k))| \mid k \in \mathbb{N}_0\}$ is bounded below by some positive number, say $0 < \rho_i \le |a_i(\mathbf{x}(k))|$ for all $k \in \mathbb{N}_0$ and $\{b_i'(x_i(k)) \mid k \in \mathbb{N}_0\}$ are bounded above by some negative number, say $b_i'(x_i(k)) \le -\epsilon_i < 0$ for all $k \in \mathbb{N}_0$. We define

$$\rho := \min\{\rho_i : i = 1, 2, \cdots, n\}, \tag{2.13}$$

$$\epsilon := \min\{\epsilon_i : i = 1, 2, \cdots, n\}. \tag{2.14}$$

Lemma 2. *Consider system* (1.6) *with a_i, b_i, and c satisfying* (2.1), (2.4), (2.5) *and* (2.8). *Then*

(I) for function $\hat{g}$*, either case* $(\hat{g}$-(i)) *or case* $(\hat{g}$-(ii)) *holds, where*

$(\hat{g}$-(i)): $\hat{g}(k) < 0$*, for all* $k \in \mathbb{N}_0$*,*

$(\hat{g}$-(ii)): $\hat{g}(k) \geq 0$*, for all* $k \geq K_1$*, for some* $K_1 \in \mathbb{N}_0$*;*

(II) for function $\check{g}$*, either case* $(\check{g}$-(i)) *or case* $(\check{g}$-(ii)) *holds, where*

$(\check{g}$-(i)): $\check{g}(k) > 0$*, for all* $k \in \mathbb{N}_0$*,*

$(\check{g}$-(ii)): $\check{g}(k) \leq 0$*, for all* $k \geq K_2$*, for some* $K_2 \in \mathbb{N}_0$*.*

If Lemma 2 is valid, there are only four possibilities to consider.

case (i): Both $(\hat{g}$-(i)) and $(\check{g}$-(i)) hold. This case is impossible from our definition of $\hat{g}$ and $\check{g}$.

case (ii): Both $(\hat{g}$-(i)) and $(\check{g}$-(ii)) hold, then sequence $\{x_i(k)\}$ will always decrease as k increases, for all $i = 1, 2, \cdots, n$. By Lemma 1, $\{x_i(k)\}$ are bounded below for every i, hence the limit $x_i(\infty)$ exists, for every $i = 1, 2, \cdots, n$.

case (iii): Both $(\hat{g}$-(ii)) and $(\check{g}$-(i)) hold, then sequence $\{x_i(k)\}$ will always increase as k increases, for all $i = 1, 2, \cdots, n$. By Lemma 1, $\{x_i(k)\}$ are bounded above for every i, hence the limit $x_i(\infty)$ exists, for every $i = 1, 2, \cdots, n$.

case (iv): Both $(\hat{g}$-(ii)) and $(\check{g}$-(ii)) hold.

Accordingly, we are left with the case **case (iv)** only, for the conclusion of global consensus for (1.6). We thus assume that $\hat{g}(0) \geq 0$, $\check{g}(0) \leq 0$, without loss of generality.

Lemma 3. *Consider system (1.6) with* a_i*,* b_i*, and* c *satisfying Conditions (I), (II), (III), and (IV) then,*

(i) $\lim_{k\to\infty} \hat{b}(k)$ *exists, denoted by* $\hat{B}$*, and* $\lim_{k\to\infty} c(\mathbf{x}(k)) = \hat{B}$*,*

(ii) $\lim_{k\to\infty} \check{b}(k)$ *exists, denoted by* $\check{B}$*, and* $\lim_{k\to\infty} c(\mathbf{x}(k)) = \check{B}$*.*

If Lemma 3 holds, we find that

$$\lim_{k\to\infty} \hat{b}(k) = \lim_{k\to\infty} \check{b}(k) =: \bar{B}, \tag{2.15}$$

since $\lim_{k\to\infty} c(\mathbf{x}(k))=\hat{B}=\check{B}$. For any $i = 1, 2, \cdots, n$, $\check{g}(k) \leq g_i(k) \leq \hat{g}(k)$, for all $k \in \mathbb{N}_0$. Equivalently,

$$\check{b}(k) - c(\mathbf{x}(k)) \leq b_i(x_i(k)) - c(\mathbf{x}(k)) \leq \hat{b}(k) - c(\mathbf{x}(k)),$$

for all $k \in \mathbb{N}_0$. Thus, $\check{b}(k) \leq b_i(x_i(k)) \leq \hat{b}(k)$, for all $k \in \mathbb{N}_0$. Therefore

$$\lim_{k\to\infty} \check{b}(k) \leq \lim_{k\to\infty} b_i(x_i(k)) \leq \lim_{k\to\infty} \hat{b}(k).$$

We obtain

$$\lim_{k\to\infty} \hat{b}(k) = \lim_{k\to\infty} b_i(x_i(k)) = \lim_{k\to\infty} \check{b}(k) = \bar{B},$$

by (2.15). Therefore we conclude that

$$\lim_{k\to\infty} b_i(x_i(k)) = \bar{B}, \text{for all } i = 1, 2, \cdots, n. \tag{2.16}$$

Moreover, $\lim_{k\to\infty} x_i(k)$ exists, for every $i = 1, 2, \cdots, n$, by (2.5) and (2.16). Hence, global consensus of system (1.6) is achieved, if functions a_i, b_i, and c satisfy Conditions (I), (II), (III), (IV).

3 Proofs of Lemmas

Proof of Lemma 1 : For any initial vale $\mathbf{x}(0)$, we consider the iteration sequence $\{x_i(k)\}$ and their components $x_i(k)$. We divide the proof into several steps.
(i) By (2.3) and(2.7),

$$b_i(x_i) - c(\mathbf{x}) < 0, \tag{3.1}$$

for all $x_i \geq u_i$. Therefore

$$\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))] < 0, \tag{3.2}$$

if $x_i(k) \geq u_i$. Similarly, By (2.3) and (2.6),

$$b_i(x_i) - c(\mathbf{x}) > 0, \tag{3.3}$$

for all $x_i \leq l_i$. Therefore

$$\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))] > 0, \tag{3.4}$$

if $x_i(k) \leq l_i$. We claim that for all $k \in \mathbb{N}_0$,

$$|b_i(x_i(k))| \leq d_i|x_i(k)| + |b_i(0)|. \tag{3.5}$$

This follows from

$$b_i(x_i(k)) - b_i(0) = b_i'(\cdot)x_i(k),$$

where "$\cdot$" means some real number between $x_i(k)$ and 0. Thus, by (2.5),

$$\begin{aligned} |b_i(x_i(k))| &= |b_i(0) + b_i'(\cdot)x_i(k)| \\ &\leq |b_i(0)| + |b_i'(\cdot)x_i(k)| \\ &\leq |b_i(0)| + d_i|x_i(k)|. \end{aligned}$$

(ii) Next, we show that for fixed constant L_i, there exist some constants $u_i^{'}$ and $d_i^{'}$, where $u_i^{'} > 0$, $0 < d_i < d_i^{'} < 1$ such that

$$d_i|x_i| + L_i < d_i^{'}|x_i|, \text{ if } |x_i| \geq u_i^{'}. \tag{3.6}$$

Let us verify this. Notably,

$$\frac{d_i|x_i| + L_i}{|x_i|} = d_i + \frac{L_i}{|x_i|} \to d_i < 1,$$

as $|x_i| \to \infty$. Therefore, there exist some $u_i^{'}$ and $d_i^{'}$, where $u_i^{'} > 0$, $0 < d_i < d_i^{'} < 1$ such that $(d_i|x_i| + L_i)/|x_i| < d_i^{'}$, if $|x_i| \geq u_i^{'}$.
(iii)

$$\begin{aligned}
|\Delta x_i(k)| &= |a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))]| \\
&\leq |b_i(x_i(k)) - c(\mathbf{x}(k))| \text{ (by (2.1))} \\
&\leq |b_i(x_i(k))| + |c(\mathbf{x}(k))| \\
&\leq d_i|x_i(k)| + |b_i(0)| + |c(\mathbf{x}(k))| \text{ (by (3.5))} \\
&\leq d_i|x_i(k)| + |b_i(0)| + M \text{ (by (2.3), (2.12))}.
\end{aligned}$$

Hence, by (3.6), we choose $|b_i(0)| + M = L_i$, there exist some constants $u_i^{'}$ and $d_i^{'}$ where $u_i^{'} > 0$, $0 < d_i < d_i^{'} < 1$ such that

$$|\Delta x_i(k)| < d_i^{'}|x_i(k)| < |x_i(k)|, \text{ if } |x_i(k)| \geq u_i^{'}. \tag{3.7}$$

(iv) Set, for each i,

$$q_i^{'} := \max\{|u_i|, |l_i|, u_i^{'}\}. \tag{3.8}$$

Let $Q' := [-q_1', q_1'] \times \cdots \times [-q_n', q_n']$. Q' is a compact set, hence $|a_i(\mathbf{x})[b_i(x_i) - c(\mathbf{x})]|$ is bounded on Q', say

$$|a_i(\mathbf{x})[b_i(x_i) - c(\mathbf{x})]| \leq K, \tag{3.9}$$

for all $\mathbf{x} \in Q^{'}$, for all i. Set

$$\begin{aligned}
q_i &:= q_i^{'} + K, && (3.10) \\
Q &:= [-q_1, q_1] \times \cdots \times [-q_n, q_n]. && (3.11)
\end{aligned}$$

We shall utilize (3.2), (3.4), (3.7), (3.8), (3.9), (3.10) in the following discussions.
(v) If $-q_i \leq x_i(0) \leq q_i$, then $-q_i < x_i(k) < q_i$, for all $k \in \mathbb{N}_0$.

case (a): If $x_i(0) \in [-q_i, -q_i^{'}]$, then $\Delta x_i(0) > 0$, due to $x_i(0) \le -q_i^{'} \le l_i$, and $|\Delta x_i(0)| < |x_i(0)|$, due to $x_i(0) \le -u_i^{'}$, hence $x_i(1)$ still stays in $(-q_i, -q_i^{'}]$, or moves into $(-q_i^{'}, q_i^{'})$. If the former case occurs, we consider $x_i(1)$ as case (a) again. If the latter case occurs, we consider $x_i(1)$ as in the following case (b).

case (b): If $x_i(0) \in (-q_i^{'}, q_i^{'})$, then $|\Delta x_i(0)| < K$, by (3.9), hence $x_i(1)$ will stay in $[-q_i, -q_i^{'}]$ or $(-q_i^{'}, q_i^{'})$ or $[q_i^{'}, q_i]$. Then we can still consider $x_i(1)$ as in case (a), case (b), and case (c), respectively.

case (c): If $x_i(0) \in [q_i^{'}, q_i]$, then $\Delta x_i(0) < 0$, by $x_i(0) \ge q_i^{'} \ge u_i$, and $|\Delta x_i(0)| < |x_i(0)|$, by $x_i(0) \ge u_i^{'}$, hence $x_i(1)$ still stays in $[q_i^{'}, q_i)$, or moves into $(-q_i^{'}, q_i^{'})$. If the former case occurs, we consider $x_i(1)$ as in case (c) again. If the latter case occurs, we consider $x_i(1)$ as in case (b). From the above arguments, we find that if $-q_i \le x_i(0) \le q_i$, then $-q_i < x_i(1) < q_i$, and we can prove that $-q_i < x_i(k) < q_i$, for all $k \ge 2$, by induction.

(vi): If $x_i(0) < -q_i$, then

case (d): $\{x_i(k)\}$ either increases as k increases and remains bounded above by $-q_i$, or

case (e): $\{x_i(k)\}$ enter $[-q_i, q_i]$ at some iteration, and never leaves $[-q_i, q_i]$ again.

(vii) if $x_i(0) > q_i$, then

case (f): $\{x_i(k)\}$ either decreases as k increases and remains bounded below by q_i, or

case (g): $\{x_i(k)\}$ enters $[-q_i, q_i]$ at some iteration, and never leaves $[-q_i, q_i]$ again.

We find that no matter which case above occurs, $\{x_i(k)\}$ are bounded above and below for all i. Therefore, $\{|a_i(\mathbf{x}(k))|\}$ are bounded below by some positive number, say $0 < \rho_i^{'} \le |a_i(\mathbf{x}(k))|$, and $\{b_i^{'}(x_i(k))\}$ are bounded above by some negative number, say $b_i^{'}(x_i(k)) \le -\epsilon_i^{'} < 0$. In fact, it is impossible for the above case (d) and case (f) to occur. This is due to that if case (d) occurs, then

$$\begin{aligned} b_i(x_i(k)) - c(\mathbf{x}(k)) &= b_i(x_i(k)) - b_i(l_i) + b_i(l_i) - c(\mathbf{x}(k)) \\ &> b_i(x_i(k)) - b_i(l_i) \\ &= b_i^{'}(\cdot)[x_i(k) - l_i] \\ &\ge \epsilon_i^{'} K, \end{aligned}$$

for all $x_i(k) \le -q_i \le l_i - K$, by (2.5), (3.3), where "$\cdot$" means some real number between $x_i(k)$ and l_i. Therefore $\Delta x_i(k) = a_i(\mathbf{x}(k))[b_i(x_i(k)) - c(\mathbf{x}(k))] > \epsilon_i^{'} K \rho_i^{'}$.

Hence $\{x_i(k)\}$ will increase unboundedly, and this yields a contradiction. Therefore case (d) never occurs. Similarly, case (f) never occurs, either. By the arguments above, we can find that given any initial value $\mathbf{x}(0)$, $\{\mathbf{x}(k)\}$ will be attracted by Q.

Proof of Lemma 2:

For function $\hat{g}$, if $\hat{g}(k) \geq 0$ for some k, say $I(k) = i$, then $g_j(k) \leq g_i(k)$, for all $j \neq i$. Consider two possibilities $|\Delta g_i(k)| \leq g_i(k)$, and $|\Delta g_i(k)| > g_i(k)$.
case (i) $|\Delta g_i(k)| \leq g_i(k)$: It follows that

$$\hat{g}(k+1) \geq g_i(k+1) = g_i(k) + \Delta g_i(k) \geq 0.$$

case (ii) $|\Delta g_i(k)| > g_i(k)$: Let us elaborate.

$$\begin{aligned}
\Delta g_i(k) &= g_i(k+1) - g_i(k) \\
&= b_i(x_i(k+1)) - c(\mathbf{x}(k+1)) - [b_i(x_i(k)) - c(\mathbf{x}(k))] \\
&= b_i(x_i(k+1)) - b_i(x_i(k)) - [c(\mathbf{x}(k+1)) - c(\mathbf{x}(k))] \\
&= b_i'(\cdot)[x_i(k+1) - x_i(k)] - \sum_{j=1}^{n} \frac{\partial c}{\partial x_j}(\bullet)[x_j(k+1) - x_j(k)],
\end{aligned}$$

where "$\cdot$" means some real number between $x_i(k+1)$ and $x_i(k)$, "$\bullet$" means some vector between $\mathbf{x}(k+1)$ and $\mathbf{x}(k)$. Thus,

$$\begin{aligned}
\Delta g_i(k) &= b_i'(\cdot)a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} \frac{\partial c}{\partial x_j}(\bullet)a_j(\mathbf{x}(k))g_j(k) \\
&\geq -d_i a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} r_j a_j(\mathbf{x}(k))g_i(k) \\
&\quad \text{(by (2.3), (2.5) and } g_j(k) \leq g_i(k) \geq 0) \\
&\geq -d_i g_i(k) - \sum_{j=1}^{n} r_j g_i(k) \text{ (by (2.1))} \\
&= (-d_i - \sum_{j=1}^{n} r_j) g_i(k) \\
&\geq -g_i(k) \text{ (by (2.8))}.
\end{aligned}$$

Hence $\Delta g_i(k) > 0$, since $|\Delta g_i(k)| > g_i(k)$ and $\Delta g_i(k) \geq -g_i(k)$. Therefore, $\hat{g}(k+1) \geq g_i(k+1) = g_i(k) + \Delta g_i(k) > 0$.

For function $\check{g}$, if $\check{g}(k) \leq 0$ for some k, say $J(k) = i$. Then $g_j(k) \geq g_i(k)$, for all $j \neq i$. Then either $|\Delta g_i(k)| \leq -g_i(k)$ or $|\Delta g_i(k)| > -g_i(k)$ holds.

case (i) $|\Delta g_i(k)| \le -g_i(k)$: It follows that $\check{g}(k+1) \le g_i(k+1) = g_i(k) + \Delta g_i(k) \le 0$.
case(ii) $|\Delta g_i(k)| > -g_i(k)$:

$$\begin{aligned}\Delta g_i(k) &= g_i(k+1) - g_i(k) \\ &= b_i(x_i(k+1)) - c(\mathbf{x}(k+1)) - [b_i(x_i(k)) - c(\mathbf{x}(k))] \\ &= b_i(x_i(k+1)) - b_i(x_i(k)) - [c(\mathbf{x}(k+1)) - c(\mathbf{x}(k))] \\ &= b_i^{'}(\cdot)[x_i(k+1) - x_i(k)] - \sum_{j=1}^{n} \frac{\partial c}{\partial x_j}(\bullet)[x_j(k+1) - x_j(k)],\end{aligned}$$

where "$\cdot$" means some real number between $x_i(k+1)$ and $x_i(k)$, "$\bullet$" means some vector between $\mathbf{x}(k+1)$ and $\mathbf{x}(k)$. Thus

$$\begin{aligned}|\Delta g_i(k)| &= b_i^{'}(\cdot)a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} \frac{\partial c}{\partial x_j}(\bullet)a_j(\mathbf{x}(k))g_j(k) \\ &\le -d_i a_i(\mathbf{x}(k))g_i(k) - \sum_{j=1}^{n} r_j a_j(\mathbf{x}(k))g_i(k) \\ &\quad \text{(by (2.3), (2.5) and } g_j(k) \ge g_i(k) \le 0) \\ &\le -d_i g_i(k) - \sum_{j=1}^{n} r_j g_i(k) \text{ (by (2.1))} \\ &= (d_i + \sum_{j=1}^{n} r_j)(-g_i(k)) \\ &\le -g_i(k) \text{ (by (2.8))}.\end{aligned}$$

Hence $\Delta g_i(k) < 0$, since $|\Delta g_i(k)| > -g_i(k)$ and $\Delta g_i(k) \le -g_i(k)$. So, $\check{g}(k+1) \le g_i(k+1) = g_i(k) + \Delta g_i(k) < 0$.

From the above arguments, we find that function $\hat{g}$ may keep negative at all iterations. But once it becomes nonnegative at some iteration, it will always remain nonnegative after this iteration. Similarly, $\check{g}$ may keep positive at all iterations. But once it get nonpositive at some iteration, it will always be nonpositive after this iteration. This completes the proof of Lemma 2. With Lemma 2, we assume that $\hat{g}(0) \ge 0$, $\check{g}(0) \le 0$, without loss of generality.

Proof for Lemma 3:
We assert that $\lim_{k\to\infty} \hat{b}(k)$ exists, and denote it by $\hat{B}$; moreover, $\lim_{k\to\infty} c(\mathbf{x}(k))=\hat{B}$.

Case (i): There exist finitely many jumps of type-1.
In this case, there exist some $K_3 \in \mathbb{N}$, some i, say 1, such that $\hat{g}(k) = g_1(k) \ge 0$, for all $k \ge K_3$. Hence $\{x_1(k)\}$ will be non-decreasing as k increases. By Lemma 1,

$\{x_1(k)\}$ are bounded above. Therefore, $\lim_{k\to\infty} x_1(k)$ exists, hence $\lim_{k\to\infty} b_1(x_1(k))$ exists, denoted by $\hat{B}$. Restated, $\lim_{k\to\infty} \hat{b}(k) = \hat{B}$.

Next, we justify that $\lim_{k\to\infty} c(\mathbf{x}(k))=\hat{B}$. Assume otherwise, $\lim_{k\to\infty} c(\mathbf{x}(k)) \neq \hat{B}$. It follows from $\hat{g}(k) = g_1(k) \geq 0$, for all $k \geq K_3$, that $b_1(x_1(k)) \geq c(\mathbf{x}(k))$, for all $k \geq K_3$. There exists some $\varepsilon > 0$, and subsequence $\{k_l\}_{l=1}^{\infty}$ of positive integer numbers with $k_1 > K_3$ such that $|c(\mathbf{x}(k_l)) - \hat{B}| > \varepsilon$, for all $l \in \mathbb{N}$. Because $\lim_{k\to\infty} b_1(x_1(k)) = \hat{B}$, for such ε, there exists $K_4 \in \mathbb{N}$, such that $|b_1(x_1(k)) - \hat{B}| \leq \frac{\varepsilon}{2}$, for all $k \geq K_4$. Therefore $g_1(k_l) = b_1(x_1(k_l)) - c(\mathbf{x}(k_l)) > \frac{\varepsilon}{2}$, for all $k_l \geq K_4$. We find that $\{x_1(k)\}$ is always increasing after K_4-th iteration. In fact,

$$\Delta x_1(k_l) = a_1(\mathbf{x}(k_l))[b_1(x_1(k_l) - c(\mathbf{x}(k_l))] > \rho\frac{\varepsilon}{2},$$

if $k_l \geq K_4$. Hence $\{x_1(k)\}$ will increase unboundedly, and yields a contradiction to Lemma 1.

Case (ii): There exist infinitely many jumps of type-1.
We shall justify that $\{\hat{b}(k)\}$ decreases as $\{k\} \uparrow \infty$. Consider a fixed $k \in \mathbb{N}_0$.

Subcase (ii-a): no jump of type-1 occurs at k-th iteration.
Suppose $I(k) = I(k+1) = i$, then $g_i(k) \geq 0, g_i(k+1) \geq 0$. In addition,

$$\begin{aligned}\hat{b}(k+1) &= b_i(x_i(k+1)) \\ &\leq b_i(x_i(k)) \\ &= \hat{b}(k),\end{aligned}$$

thank to (2.5), and $\Delta x_i(k) = a_i(\mathbf{x}(k))g_i(k) \geq 0$. Thus $\{\hat{b}(k)\}$ decreases as k increases.

Subcase (ii-b): jump of type-1 occurs at k-th iteration and $g_i(k) \geq 0, g_j(k) \geq 0$, where $I(k) = i \neq I(k+1) = j$.
It follows that

$$\begin{aligned}\hat{b}(k+1) &= b_j(x_j(k+1)) \\ &\leq b_j(x_j(k)) \\ &\leq b_i(x_i(k)) \\ &= \hat{b}(k),\end{aligned}$$

due to (2.5), $\Delta x_j((k)) = a_j(\mathbf{x}(k))g_j(k) \geq 0$, and by $I(k) = i \neq j$.

Subcase (ii-c): jump of type-1 occurs at k-th iteration and $g_i(k) \geq 0, g_j(k) < 0$, where $I(k) = i \neq I(k+1) = j$.

Notably, we still have $g_j(k+1) \geq 0$. We claim that

$$b_j(x_j(k+1)) - b_j(x_j(k)) \leq b_i(x_i(k)) - b_j(x_j(k)). \tag{3.12}$$

Indeed,

$$\begin{aligned}
LHS &= b_j^{'}(\cdot)\Delta x_j(k) \\
&= b_j^{'}(\cdot)a_j(\mathbf{x}(k))g_j(k) \\
&\leq b_j^{'}(\cdot)g_j(k) \text{ (by (2.1))} \\
&\leq -d_j g_j(k) \text{ (by (2.5), and } g_j(k) < 0)) \\
&\leq g_i(k) - g_j(k) \text{ (by } (1-d_j)g_j(k) < 0 \leq g_i(k)) \\
&= b_i(x_i(k)) - b_j(x_j(k)) \\
&= RHS.
\end{aligned}$$

Herein, "$\cdot$" is defined as before. Hence, $\hat{b}(k+1) = b_j(x_j(k+1)) \leq b_i(x_i(k)) = \hat{b}(k)$. All these cases indicate that $\{\hat{b}(k)\}$ decreases as $\{k\}$ increases. By Lemma 1, $\{\mathbf{x}(k)\}$ are attracted into some compact set Q contained in $\mathbb{R}^n$. Therefore, $\{b_i(x_i(k))\}$ are bounded below, and so are $\{\hat{b}(k)\}$. Hence $\{\hat{b}(k)\}$ decreases and converges to some number $\hat{B}$ as k tends to infinity (denoted by $\{\hat{b}(k)\} \downarrow \hat{B}$).

Next, we verify that $\lim_{k\to\infty} c(\mathbf{x}(k)) = \hat{B}$. Assume otherwise: $\lim_{k\to\infty} c(\mathbf{x}(k)) \neq \hat{B}$. There exist some positive μ, subsequence $\{k_l\}_{l=1}^{\infty}$ of positive integers, such that

$$|c(\mathbf{x}(k_l)) - \hat{B}| > \frac{\mu}{\epsilon\rho}, \tag{3.13}$$

Where ϵ, ρ are defined in (2.13) and (2.14). Because $\{\hat{b}(k)\} \downarrow \hat{B}$, for $\mu' := \min\{\frac{\mu}{\epsilon\rho}, \mu\} > 0$, there exists $L \in \mathbb{N}$ such that

$$\hat{B} \leq b_{I(k)}(x_{I(k)}(k)) \leq \hat{B} + \mu', \tag{3.14}$$

for all $k \geq L$. Moreover

$$\hat{g}(\ell) = b_{I(\ell)}(x_{I(\ell)}(\ell) - c(\mathbf{x}(\ell)) \geq 0, \tag{3.15}$$

for all $\ell \in \mathbb{N}$. Consider the k_L-th iteration. Notably, $k_L > L$. By (3.13), (3.14), and (3.15), we have

$$\hat{g}(k_L) = b_1(x_1(k_L)) - c(\mathbf{x}(k_L)) > \frac{\mu}{\epsilon\rho},$$

where, for convenience, we set $I(k_L)$=1 without loss of generality. There are two possibilities at k_L-th iteration, either jump of type-1 occurs or not. If it does not occur, then

$$
\begin{aligned}
|\Delta\hat{b}(k_L)| &= |\hat{b}(k_L+1)-\hat{b}(k_L)| \\
&= |b_1(x_1(k_L+1))-b_1(x_1(k_L))| \\
&= |b_1'(\cdot)||x_1(k_L+1)-x_1(k_L)| \\
&= |b_1'(\cdot)||a_1(\mathbf{x}(k_L))||g_1(k_L)| \\
&= |b_1'(\cdot)||a_1(\mathbf{x}(k_L))||\hat{g}(k_L)| \\
&> \epsilon\rho\frac{\mu}{\epsilon\rho} \\
&= \mu.
\end{aligned}
$$

But it is impossible, because of (3.14).

If jump of type-1 occurs at k_L-th iteration. Assume that $I(k_L+1)$=2. Below we consider three different cases for $b_2(x_2(k_L))$:

Case (a): $\hat{B} \le b_2(x_2(k_L)) < b_1(x_1(k_L))$. Then $g_2(k_L) > \frac{\mu}{\epsilon\rho}$, and $|\Delta b_2(x_2(k_L))| = |b_2'(\cdot)||a_2(\mathbf{x}(k_L))||g_2(k_L)| > \epsilon\rho\frac{\mu}{\epsilon\rho} = \mu$. It is impossible, due to (3.14).

Case (b): $\hat{B} > b_2(x_2(k_L)) \ge c(\mathbf{x}(k_L))$. Then $g_2(k_L) \ge 0$, and $x_2(k_L+1) \ge x_2(k_L)$. Thus,

$$
\begin{aligned}
\hat{b}(k_L+1) &= b_2(x_2(k_L+1)) \\
&\le b_2(x_2(k_L)) \\
&< \hat{B}.
\end{aligned}
$$

It is impossible, since $\{\hat{b}(k)\} \downarrow \hat{B}$.

Case (c): $b_2(x_2(k_L)) < c(\mathbf{x}(k_L))$. Then $g_2(k_L) < 0$, and

$$
\begin{aligned}
\Delta b_2(x_2(k_L)) &= b_2(x_2(k_L+1)) - b_2(x_2(k_L)) \\
&= b_2'(\cdot)a_2(\mathbf{x}(k_L))g_2(k_L) \\
&\le -d_2 g_2(k_L) \\
&< -g_2(k_L).
\end{aligned}
$$

Thus, $b_2(x_2(k_{L+1})) = b_2(x_2(k_L)) + \Delta b_2(x_2(k_L)) < b_2(x_2(k_L)) - g_2(k_L) = c(\mathbf{x}(k_L))$. Hence $\hat{b}(k_L+1) = b_2(x_2(k_L+1)) < c(\mathbf{x}(k_L)) < \hat{B}$. It is impossible, since $\{\hat{b}(k)\} \downarrow \hat{B}$.

From the above discussions, we conclude that $\lim_{k\to\infty} c(\mathbf{x}(k)) = \hat{B}$.

The second part of the lemma asserts that $\lim_{k\to\infty} \check{b}(k)$ exists, denoted by $\check{B}$, and $\lim_{k\to\infty} c(\mathbf{x}(k))=\check{B}$. The proof for the assertion resembles the first part. Let us elaborate.

Case (i): There exist finitely many jumps of type-2.

In this case, there exists some $K_5 \in \mathbb{N}$, some i, say 1, such that $\check{g}(k) = g_1(k) \le 0$, for all $k \ge K_5$. Hence $\{x_1(k)\}$ will be non-increasing as k increases. By Lemma 1, $\{x_1(k)\}$ are bounded below. Therefore, $\lim_{k\to\infty} x_1(k)$ exists, hence $\lim_{k\to\infty} b_1(x_1(k))$ exists, denoted by $\check{B}$. Restated, $\lim_{k\to\infty} \check{b}(k) = \check{B}$.

Next,we justify that $\lim_{k\to\infty} c(\mathbf{x}(k))=\check{B}$. Assume otherwise, $\lim_{k\to\infty} c(\mathbf{x}(k)) \neq \check{B}$. It follows from $\check{g}(k) = g_1(k) \le 0$, for all $k \ge K_5$, $b_1(x_1(k)) \le c(\mathbf{x}(k))$, for all $k \ge K_5$. There exists some $\varepsilon > 0$, and subsequence $\{k_l\}_{l=1}^{\infty}$ of positive integer numbers with $k_1 > K_5$ such that $|c(\mathbf{x}(k_l)) - \check{B}| > \varepsilon$, for all $l \in \mathbb{N}$. Because $\lim_{k\to\infty} b_1(x_1(k)) = \check{B}$, for such ε, there exists $K_6 \in \mathbb{N}$, such that $|b_1(x_1(k)) - \check{B}| \le \frac{\varepsilon}{2}$, for all $k \ge K_6$. Therefore $g_1(k_l) = b_1(x_1(k_l)) - c(\mathbf{x}(k_l)) < -\frac{\varepsilon}{2}$, for all $k_l \ge K_6$. We find that $\{x_1(k)\}$ are always decreasing after $K_6 - th$ iteration. In fact,

$$\Delta x_1(k_l) = a_1(\mathbf{x}(k_l))[b_1(x_1(k_l)) - c(\mathbf{x}(k_l))] < -\rho\frac{\varepsilon}{2},$$

if $k_l \ge K_6$. Hence, $\{x_1(k)\}$ will decrease unboundedly, and yields a contradiction to Lemma 1.

Case (ii): There exist infinitely many jumps of type-2.

We shall justify that $\{\check{b}(k)\}$ increases as $\{k\} \uparrow \infty$. Consider a fixed $k \in \mathbb{N}_0$,

Subcase (ii-a): no jump of type-2 occurs at k-th iteration. Suppose $J(k) = J(k+1) = i$, then $g_i(k) \le 0, g_i(k+1) \le 0$. In addition,

$$\begin{aligned} \check{b}(k+1) &= b_i(x_i(k+1)) \\ &\ge b_i(x_i(k)) \\ &= \check{b}(k) \end{aligned}$$

thank to (2.5), and $\Delta x_i((k)) = a_i(\mathbf{x}(k))g_i(k) \le 0$. Thus $\{\check{b}(k)\}$ increases as $\{k\}$ increases.

Subcase (ii-b): jump of type-2 occurs at k-th iteration and $g_i(k) \le 0, g_j(k) \le 0$, where $J(k) = i \neq J(k+1) = j$.

It follows that

$$\begin{aligned}\check{b}(k+1) &= b_j(x_j(k+1)) \\ &\geq b_j(x_j(k)) \\ &\geq b_i(x_i(k)) \\ &= \check{b}(k)\end{aligned}$$

due to (2.5), $\Delta x_j((k)) = a_j(\mathbf{x}(k))g_j(k) \leq 0$ and $J(k) = i \neq j$.

Subcase (ii-c): jump of type-2 occurs at k-th iteration and $g_i(k) \leq 0, g_j(k) > 0$, where $J(k) = i \neq J(k+1) = j$.

Notably, we still have $g_j(k+1) \leq 0$. We claim that

$$b_j(x_j(k+1)) - b_j(x_j(k)) \geq b_i(x_i(k)) - b_j(x_j(k)). \tag{3.16}$$

Indeed,

$$\begin{aligned}LHS &= b_j^{'}(\cdot)\Delta x_j(k) \\ &= b_j^{'}(\cdot)a_j(\mathbf{x}(k))g_j(k) \\ &\geq b_j^{'}(\cdot)g_j(k) \text{ (by (2.1))} \\ &\geq -d_j g_j(k) \text{ (by (2.5), and } g_j(k) > 0)) \\ &\geq g_i(k) - g_j(k) \text{ (by } (1-d_j)g_j(k) > 0 \geq g_i(k)) \\ &= b_i(x_i(k)) - b_j(x_j(k)) \\ &= RHS.\end{aligned}$$

Herein, "$\cdot$" is defined as before. Hence, $\check{b}(k+1) = b_j(x_j(k+1)) \geq b_i(x_i(k)) = \check{b}(k)$. All these cases indicate that $\{\check{b}(k)\}$ increase as $\{k\}$ increases. By Lemma 1, $\{\mathbf{x}(k)\}$ are attracted into some compact set Q contained in $\mathbb{R}^n$. Therefore, $\{b_i(x_i(k))\}$ are bounded above, and so are $\{\check{b}(k)\}$. Hence $\{\check{b}(k)\}$ increase and converge to some number, say $\check{B}$ as $\{k\}$ tend to infinity (denoted by $\check{b}(k)\} \uparrow \check{B}$).

Next, we verify that $\lim_{k\to\infty} c(\mathbf{x}(k))=\check{B}$. Assume otherwise: $\lim_{k\to\infty} c(\mathbf{x}(k)) \neq \check{B}$. There exist some positive μ, subsequence $\{k_l\}_{l=1}^{\infty}$ of positive integers, such that

$$|c(\mathbf{x}(k_l)) - \check{B}| > \frac{\mu}{\epsilon\rho}. \tag{3.17}$$

Where ϵ, ρ are defined in (2.13) and (2.14). Because $\check{b}(k)\} \uparrow \check{B}$, for $\mu^{'} := \min\{\frac{\mu}{\epsilon\rho}, \mu\} > 0$, there exists $L \in \mathbb{N}$, such that

$$\check{B} \geq b_{J(k)}(x_{J(k)}(k)) \geq \check{B} - \mu^{'}, \tag{3.18}$$

for all $k \geq L$. Moreover

$$\breve{g}(\ell) = b_{J(\ell)}(x_{J(\ell)}(\ell) - c(\mathbf{x}(\ell)) \leq 0, \tag{3.19}$$

for all $\ell \in \mathbb{N}$. Consider the k_L-th iteration. Notably, $k_L > L$. By (3.17), (3.18),and (3.19), we have

$$\breve{g}(k_L) = b_1(x_1(k_L)) - c(\mathbf{x}(k_L)) < -\frac{\mu}{\epsilon\rho},$$

where, for convenience, we set $J(k_L)$=1 without loss of generality. There are two possibilities at $k_L - th$ iteration, either jump of type-2 occurs or not. If it dose not occur, then

$$\begin{aligned}
|\Delta\breve{b}(k_L)| &= |\breve{b}(k_L+1) - \breve{b}(k_L)| \\
&= |b_1(x_1(k_L+1)) - b_1(x_1(k_L))| \\
&= |b_1^{'}(\cdot)||x_1(k_L+1)) - (x_1(k_L))| \\
&= |b_1^{'}(\cdot)||a_1(\mathbf{x}(k_L))||g_1(k_L)| \\
&= |b_1^{'}(\cdot)||a_1(\mathbf{x}(k_L))||\breve{g}(k_L)| \\
&> \epsilon\rho\frac{\mu}{\epsilon\rho} \\
&= \mu.
\end{aligned}$$

But it is impossible, because (3.18).

If jump of type-2 occurs at $k_L - th$ iteration. Assume that $J(k_L+1)$=2. Below we consider three different cases for $b_2(x_2(k_L))$:

Case (a): $\breve{B} \geq b_2(x_2(k_L)) > b_1(x_1(k_L))$. Then $g_2(k_L) < -\frac{\mu}{\epsilon\rho}$, and $|\Delta b_2(x_2(k_L))| = |b_2^{'}(\cdot)||a_2(\mathbf{x}(k_L))||g_2(k_L)| > \epsilon\rho\frac{\mu}{\epsilon\rho} = \mu$. It is impossible, due to (3.18).

Case (b): $\breve{B} < b_2(x_2(k_L)) \leq c(\mathbf{x}(k_L))$. Then $g_2(k_L) \leq 0$, and $x_2(k_L+1) \leq x_2(k_L)$. Thus

$$\begin{aligned}
\breve{b}(k_L+1) &= b_2(x_2(k_L+1)) \\
&\geq b_2(x_2(k_L)) \\
&> \breve{B}.
\end{aligned}$$

It is impossible, since $\{\breve{b}(k)\} \uparrow \breve{B}$.

Case (c): $b_2(x_2(k_L)) > c(\mathbf{x}(k_L))$. Then $g_2(k_L) > 0$, and

$$\begin{aligned}\Delta b_2(x_2(k_L)) &= b_2(x_2(k_L+1)) - b_2(x_2(k_L)) \\ &= b_2'(\cdot)a_2(\mathbf{x}(k_L))g_2(k_L) \\ &\geq -d_2 g_2(k_L) \\ &> -g_2(k_L).\end{aligned}$$

Thus, $b_2(x_2(k_{L+1})) = b_2(x_2(k_L)) + \Delta b_2(x_2(k_L)) > b_2(x_2(k_L)) - g_2(k_L) = c(\mathbf{x}(k_L))$. Hence $\check{b}(k_L+1) = b_2(x_2(k_L+1)) > c(\mathbf{x}(k_L)) > \check{B}$. It is impossible, since $\{\check{b}(k)\} \uparrow \check{B}$.

From the above discussions, we conclude that $\lim_{k\to\infty} c(\mathbf{x}(k)) = \check{B}$.

4 A Comparison between Continuous-time and Discrete-time Models

We first introduce the results for (1.3) stated in Grossberg's paper [4].

Definition 4.1. *A competitive system is said to achieve weak global consensus (or weak global pattern formation), if given any initial value* $\mathbf{x}(\mathbf{0}) \geq 0$, *all the limits* $b_i(x_i(\infty)) := \lim_{t\to\infty} b_i(x_i(t))$ *exist, for all* $i = 1, 2, \cdots, n$.

Definition 4.2. *A competitive system is said to achieve strong global consensus (or strong global pattern formation) if, given any initial value* $\mathbf{x}(\mathbf{0}) \geq 0$, *all the limits* $x_i(\infty) := \lim_{t\to\infty} x_i(t)$ *exist, for all* $i = 1, 2, \cdots, n$.

The following conditions are needed for the main results in Grossberg's paper [4].

Condition (G1):

(a): $a_i(\mathbf{x})$ is continuous for $\mathbf{x} \geq 0$,

(b): $b_i(x_i)$ is either continuous with piecewise derivative for $x_i \geq 0$, or is continuous with piecewise derivative for $x_i > 0$ and $b_i(0) = \infty$,

(c): $c(\mathbf{x})$ is continuous with piecewise derivative for $\mathbf{x} \geq 0$.

Condition (G2):

$a_i(x) > 0$ if $x_i > 0$ and $x_j \geq 0$, $j \neq i$,and $a_i(x) = 0$ if $x_i = 0$ and $x_j \geq 0$, $j \neq i$. Moreover, there exist a function $\bar{a}_i(x_i)$ such that, for sufficiently small $\lambda > 0$, $\bar{a}_i(x_i) \geq a_i(x_i)$ if $\mathbf{x} \in [0, \lambda]^n$ and

$$\int_0^\lambda \frac{d\omega}{\bar{a}_i(\omega)} = \infty \tag{4.1}$$

Condition (G3): $\limsup_{\omega\to\infty} b_i(\omega) < c(0,0,\cdots,\infty,\cdots,0)$, where "$\infty$" occurs in the ith entry, $i = 1,2,\cdots,n$.

Condition (G4): $\frac{\partial c}{\partial x_j} \geq 0$, $j = 1,2,\cdots,n$

Theorem 4 (Grossberg). *Any system of the form* (1.3) *satisfying Conditions (G1), (G2), (G3) and(G4) achieves weak global consensus. Moreover,* $b_i(x_i(\infty)) = c(\mathbf{x}(\infty))$*, for every* i*.*

Similar to proof of Theorem 1, the one of Theorem 4 consists of three main parts which we describe as follows:

First, the theorem will be proved for the case that $b_i \equiv b$, then this proof can be adapted to the case of i-dependent b_i.

Part (I): (This part works as Lemma 1)

By Conditions (G1) and (G2), if $x_i(0) > 0$, then $x_i(t) > 0$ for $t \geq 0$. If $x_i(0) = 0$, then component x_i can be deleted from the network without loss of generality [4]. By (4.1) and Condition (G3), there exist a B such that $x_i(t) \in [0, B]$ for all $i = 1,2,\cdots,n$, $t \geq 0$. Hence our attention is restricted to positive initial values. It is then derived that $\mathbf{x}(t)$ stays in some compact subset in $\mathbb{R}^n$, for all time $t \geq 0$.

Part (II): (This part works as Lemma 2)

Define

$$\begin{aligned} g_i(t) &= b(x_i(t)) - c(\mathbf{x}(t)) & (4.2)\\ \hat{g}(t) &= max\{g_j(t) : j = 1,2,\cdots,n\} & (4.3) \end{aligned}$$

Then either $\hat{g}(t) < 0$, for all $t \geq 0$, or there exists $t = T$ such that $\hat{g}(T) \geq 0$ implies $\hat{g}(t) \geq 0$, for $t \geq T$. This is due to Condition (G4). If at any time $t = s$, $\hat{g}(s) = 0$, say $\hat{g}(s) = g_i(s)$, then

$$\lim_{t\to 0^+} \frac{\hat{g}(s+t) - \hat{g}(s)}{t} \geq \dot{g}_i(s) = b'(x_i(s))\dot{x}_i(s) - \sum_{j=1}^{n} \frac{\partial c}{\partial x_j}(\mathbf{x}(s))\dot{x}_j(s) \geq 0. \quad (4.4)$$

If $\hat{g}(t) < 0$ for all $t \geq 0$, it is a trivial case. Hence $\hat{g}(t) \geq 0$ are assumed below without loss of generality

Part (III): (This part works as Lemma 3)

To state this part, we introduce the following definitions.

Definition 4.3. *(i): A jump is said to occur from i to j at time $t = T$, if there exists time s and u such that $\hat{g}(t) = g_i(t)$, for $s \leq t \leq T$, and $\hat{g}(t) = g_j(t)$, for $T \leq t \leq u$. (ii): $I(t)$=min$\{i : \hat{g}(t) = g_i(t)\}$. (iii): $\hat{b}(t)$=$b(x_{I(t)}(t))$.*

If Part (I)and (II) are valid, then we have the following three conclusions.
(i): $\hat{b}(t)$ is monotone at all large time, hence $\lim_{t\to\infty} \hat{b}(t)$ exists , and denoted by $\hat{B}$, (ii): $\lim_{t\to\infty} c(\mathbf{x}(t))$=$\hat{B}$, (iii): $\lim_{t\to\infty} b(x_i(t))$=$\hat{B}$, for all $i = 1, 2, \cdots, n$. Hence, weak global consensus is achieved.

Corollary 5 (Grossberg). *Any system of the form* (1.3) *whose functions satisfy Condition(G1)-(G4) and whose b_i possess finitely many local maxima, or intervals of local maxima ,within the range of x_i, achieves strong global consensus.*

From the process of the proof above, we can find that Part (I) and (II) play very dominant roles for Theorem 4. From this view point, we can extend Theorem 4 to Theorem 6, i.e. the phase space for (1.3) can be extended to $\mathbb{R}^n$.

We need some conditions for the theorem.
Condition (A): (a): $a_i(\mathbf{x})$ is continuous and positive for all $\mathbf{x} \in \mathbb{R}^n$,
(b): $b_i(\xi)$ is continuously differentiable for $\xi \in \mathbb{R}$,
(c): $c(\mathbf{x})$ is continuously differentiable for $\mathbf{x} \in \mathbb{R}^n$.
Condition (B): Given any initial value $\mathbf{x}(0)$, $\mathbf{x}(t)$ will be attracted by some compact set contained in $\mathbb{R}^n$.
Condition $(B)'$: (a): $\lim_{\xi\to\infty} b_i(\xi) = -\infty$, $\lim_{\xi\to-\infty} b_i(\xi) = \infty$,$i = 1, 2, \cdots, n$, (b): $c(\mathbf{x})$ is bounded below.
Condition (C):

$$\frac{\partial c}{\partial x_k} \geq 0 \ , \ k = 1, 2, \cdots, n$$

Theorem 6. *Any system of form* (1.3) *whose functions satisfy Condition (A), (B), (C) achieves weak global consensus (herein, I mean that given any initial value $\mathbf{x}(0) \in \mathbb{R}^n$, all the limits $b_i(x_i(\infty)) := \lim_{t\to\infty} b_i(x_i(t))$ exist , for all $i = 1, 2, \cdots, n$.). Moreover, each $b_i(x_i(\infty)) = c(\mathbf{x}(\infty))$.*

The proof of the Theorem 6 is mainly because that Condition (B) works as Part (I) in the proof of Theorem 4, Condition (C) works as Part (III) in the proof of Theorem 4 (because of (4.4), mainly). Therefore the work as Part (III) in the proof of Theorem 4 will also be done, and weak global consensus will be achieved. By the arguments above, for the purpose of comparing the difference of convergence theorems for continuous-time and discrete-time competitive network (details in Section 4), we can see that the proof of Theorem 6 can also be completed by the parallel three parts, just as in Theorem 4.

Corollary 7. *Any system of form* (1.3) *whose functions satisfies Condition (A), (B), (C) and whose b_i possess finitely many local maxima, or intervals of local maxima, within the range of x_i, achieves strong global consensus.*

Remark 4.1. *In Theorem 6, Condition (B) is a more abstract condition, and it can be achieved by the more concrete one as Condition (B').*

Proof. By Condition $(B)'$ and (C). For each i, there exist $p_i,\ q_i \in \mathbb{R}$, such that $\dot{x}_i(t) = a_i(\mathbf{x}(t))[b_i(x_i(t)) - c(\mathbf{x}(t))] < 0$, if $x_i(t) \geq q_i$, and $\dot{x}_i(t) = a_i(\mathbf{x}(t))[b_i(x_i(t)) - c(\mathbf{x}(t))] > 0$, if $x_i(t) \leq p_i$. Hence, given initial valve $\mathbf{x}(0)$, $\mathbf{x}(t)$ will be bounded, for all t. Then both $|a_i(\mathbf{x}(t))|$, $|[b_i(x_i(t)) - c(\mathbf{x}(t))]|$ are bounded below from some positive number, and so is $|\dot{x}_i(t)|$. Therefore if $x_i(t) > q_i$ at some time, say S_i, then $x_i(t)$ must be decreasing until $x_i(t)$ enters $[p_1, q_i]$ and never leave it again, as time goes by after S_i. If $x_i(t) < p_i$ at some time, say T_i, then $x_i(t)$ must be increasing until $x_i(t)$ enters $[p_i, q_i]$ and never leave it again, as time goes by after T_i. Hence $\mathbf{x}(t)$ will be attracted by $[p_1, q_1] \times \cdots \times [p_n, q_n]$. □

Below, let us compare the difference of convergence theorems for continuous-time and discrete-time competitive network via Theorem 1 and Theorem 6 (with Condition $(B)'$).

In the process of proving Theorem 1, we can find that different from (1.3) with "continuous solution", the behavior of solution$\{\mathbf{x}(k)\}$ is much unpredictable. Hence we have to control $\Delta x_i(k)$ at each iteration. Details are shown as follows:

(I): Different from Part (I) for Theorem 6 (just as Remark 4.1 for Theorem 6), we need more conditions as those function a_i, c ,b_i' must be bounded; namely, "$0 < (\mathbf{x}) \leq 1$", "$|b_i'(\xi)| \leq d_i$ " and "$M_1 \leq c(\mathbf{x}) \leq M_2$" to achieve Lemma 1 ,

(II): Different from Part (II) for Theorem 6 with continuous $\hat{g}(t)$, $\{\check{g}(k)\}$, $\{\check{g}(k)\}$ in Lemma 2 are sequences. We must control functions b_i, c in addition to make the same work as Part (II). Hence we need more conditions ,"$b_i'(\xi) \geq -d_i$ and $0 < d_i \leq 1 - \sum_{i=1}^{n} r_i < 1$" to achieve Lemma 2,

(III): Different from Part (III) for Theorem 6, Lemma 1 and Lemma 2 are "not sufficient" for Theorem 1. To achieve the "monotonicity" of $\{\hat{b}(k)\}$ and $\{\check{b}(k)\}$, we need function b_i to be decreasing; namely $0 \geq b_i'(\xi) \geq -d_i > -1$. For the purpose "$\lim_{k\to\infty} c(\mathbf{x}(k)) = \hat{B} = \check{B}$", we demand function b_i to be strictly decreasing; namely $0 > b_i'(\xi) \geq -d_i > -1$.

References

[1] Cohen, A. M. & Grossberg, S. *Absolute Stability of Global Patttern Formation and Parallel Memory Storage by Competitive Neural Networks*, IEEE Transations on Systems on Man anf Cybernetics, SMC-13(1983), pp. 815-826.

[2] Forti, M. & Tesi, A. *New Conditions for Global Stability of Neural Networks with Application to Linear and Quadratic Programming Problems*, IEEE Transations on Circuit and Systems-I:Fundament Theory and Application, Vol. 42, No. 7(1995), pp. 354-366

[3] Grossberg, S. *Competition, decision, consensus.* J. Math. Anal. Appl. 66, (1978), pp. 470-493

[4] Grossberg, S. *Preference order competition implies global limits in n-dimension competition systems.* 1977, preprint.

[5] Hänggi, M., Reddy, H. C. & Moschytz, G. S. *Unifying results in cnn theory using delta operator*, IEEE International Symposium on Circuits and Systems, 3(1999), pp. 547–550.

[6] Harrer, H. & Nossek, J. A. *An analog implementation of discrete-time cnns*, IEEE Transactions on Neural Networks, 3(1992), pp. 466–476.

[7] Hirsh, M. *Convergent Activation Dynamics in Continuous Time Networks*, Neural Netwok, Vol. 2 (1989), pp. 331-349.

[8] Sbitnev, V. I. & Chua, L. O. *Local activity for discrete-map CNN* Int. J. Bifur. Chaos, 12 (6) (2002), pp. 1227–1272.

[9] Shih, C.-W. & Weng, C.-W. *Cycle-symmetric matrices and convergent neural networks*, Physica D 146 (2000), 213-220.