國 立 交 通 大 學
應用數學系
碩 士 論 文

距離正則圖的特徵值譜刻劃

Spectral Characterizations of Distance-Regular Graphs

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中 華 民 國 九 十 三 年 六 月

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碩　士　論　文

A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
In partial Fulfillment of Requirement
For the Degree of Master
In
Applied Mathematics
June 2004
Hsinchu, Taiwan, Republic of China

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摘　　要

令 G 是一個距離正則圖，假設 G’是一與 G 有相同特徵值譜的圖，而且對任一固定距離 t 而言，在 G 及 G’中與一點距離為 t 的平均點數是相同的。則 G’也是一個距離正則圖，且 G’與 G 有相同的相交參數。

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Abstract

Let G be a distance-regular graph. Suppose G' is a graph with the same spectrum of G. Suppose for each nonnegative integer t, G and G' have the same average number in the t-subconstituent with respect to a vertex. Then G' is a distance-regular graph with the same intersection parameters of G.

誌　謝

首先，感謝指導教授翁志文老師的盡心指導和論文的修正。在交大應數所的這兩年來，可以感受到老師對於研究的認真與熱忱，老師樂觀的人生態度及對學生的關懷更是深深影響了我。

同時要感謝交大應數所的所有老師們，在學業和生活各方面都提供了我很多關懷和協助，整個系所由上而下就像是一家人，在嚴謹的學術氣氛中處處可見溫馨。特別感謝組合組的黃光明老師、黃大原老師、傅恆霖老師、陳秋媛老師，老師們不只在課業上提供了很多幫忙，對學術研究的精神及學者的風範更是給了我許多啟發。

其次，謝謝我的每一個同學，抮君、正傑、昭芳、宏嘉、縈丰、嘉文、致維、建瑋、貴弘、啟賢、文祥，有緣相聚的這兩年來，大家一起討論課業、一起生活、一起出遊的點點滴滴，都將是我人生中最美好的回憶。

最後要謝謝我的家人，一直以來默默地支持我、陪伴我，讓我能順利、充實地走過這兩年的研究所生涯。

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1 Introduction

Spectral characterization is an important subject in algebraic graph theory. Some properties of a graph can be recognized from the spectrum of it. For example, a graph is bipartite if and only if its eigenvalues are located symmetrically to the origin [1, Theorem 8.8.2]. Another well-known result is that a graph is regular if and only if its largest eigenvalue equal to its average valency [2, Lemma 3.2.1]. See Lemma 2.2 below. Spectral characterization of strongly regular graphs can also be done [3, Theorem 8.6.37]. Here we are interested in the question: Is a graph with the same spectrum of a distance-regular graph distance-regular? That is, for a graph, is the distance-regularity determined by its spectrum? The answer is no and there exist many counterexamples. See Remark 4.4. But furthermore, by imposing some restrictions on the conditions of the distance-regular graphs we consider, such as distance-regular graphs with some special given intersection parameters, the answer can be yes. In [8], by assuming the odd cycles in a given distance-regular graph have length greater than 2 times its diameter, Huang and Liu proved a graph with the same spectrum of it is a distance-regular graph too. In [5][6], the result is generalized that for two graphs with the same spectrum, if the girth g satisfies $g \geq 2d-1$ in the known distance-regular one, then the other is distance-regular too where d is the diameter of G. In [4], by assuming $c_{d-1} = 1$ in one graph, the other cospectral graph is shown to be distance-regular and have the same intersection parameters. In this thesis, we present a uniform way to this line of study. As a consequence, we can reprove the above mentioned results [3, Theorem 8.6.37],[4],[5],[6]. See Theorem 4.1, Corollary 4.2. Furthermore, we show that if two cospectral graphs have the same average number of vertices in the t-subconstituent with respect to a vertex for each t, then one is distance-regular implies the other is distance-regualr. See Theorem 4.3 for details.

2 Preliminary

Let $G = (X, R)$ be a finite undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path length distance function δ and diameter $d := \max\{\delta(x, y)|x, y \in X\}$. Sometimes we write diam(G) to denote the diameter of G. By a subgraph of G, we mean a graph (Δ, Ξ), where Δ is a non-empty subset of X and $\Xi = \{\{xy\}|x, y \in \Delta, \{xy\} \in R\}$. We refer to (Δ, Ξ) as the subgraph induced on Δ and, by abuse of notation, we refer to this subgraph as Δ. For any $x \in X$ and any integer i, set

$$G_i(x) := \{y|\ y \in X, \delta(x, y) = i\}.$$

The *valency* $k(x)$ of a vertex is the cardinality of $G_1(x)$. The graph G is said to be *regular* with valency k whenever each vertex in X has valency k. For any $x \in X$, for any integer i, and for any $y \in G_i(x)$, set

$$\begin{aligned} B(x,y) &:= G_1(y) \cap G_{i+1}(x), \\ A(x,y) &:= G_1(y) \cap G_i(x), \\ C(x,y) &:= G_1(y) \cap G_{i-1}(x). \end{aligned}$$

For all $x, y \in X$ with $\delta(x, y) = i$, the numbers

$$c_i := |C(x,y)|, \quad a_i := |A(x,y)|, \quad b_i := |B(x,y)|.$$

are said to be *well-defined* if they are independent of x and y. G is said to be *t-distance-regular* whenever for all integers i $(0 \le i \le t)$, a_{i-1}, b_{i-1}, c_i are all well-defined. A d-distance-regular graph is also called a *distance-regular graph*. The constants c_i, a_i and b_i $(0 \le i \le d)$ are known as the *intersection numbers* or *intersection parameters* of G. Note that the valency $k = b_0,\ c_0 = 0,\ c_1 = 1,\ b_d = 0$ and

$$k = c_i + a_i + b_i \quad (0 \le i \le d).$$

Let $k_i(x)$ denotes the cardinality of $G_i(x)$. For a distance-regular graph G, we know that $k_i(x)$ is a constant for any $x \in G$ for all i. We denote this constant by k_i. A graph is said to be *strongly regular with parameters* (n, k, a, c) if it is regular with valency k, every pair of adjacent vertices has a common neighbors, and every pair of distinct nonadjacent vertices has c

common neighbors, where $c > 0$. We see that a strongly regular graph is a distance-regular graph with diameter 2 with intersection parameters

$$\begin{array}{lll} c_0 = 0, & a_0 = 0, & b_0 = k, \\ c_1 = 1, & a_1 = a, & b_1 = k - a - 1, \\ c_2 = c, & a_2 = k - c, & b_2 = 0. \end{array}$$

For the *adjacency matrix* A of a graph G, we mean a symmetric $(0,1)-$matrix determined by G with rows and columns indexed by the vertices of G, and with entries

$$A_{xy} = \begin{cases} 1, & \text{if } x, y \text{ is adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Since the adjacency matrix A of a graph G is a real symmetric matrix, we have that the eigenvalues of A are all real numbers. We represent the distinct eigenvalues of A with their corresponding multiplicities by an array as follows:

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$. Note $m_0 + m_1 + \cdots + m_d = v$ where v is the number of vertices in G. This array is said to be the *spectrum* of G. Two graphs are said to be *cospectral* if they have the same spectrum.

The following Lemma follows immediately from linear algebra.

Lemma 2.1. *$Tr(A^n) = \sum_{i=0}^{d} m_i \theta_i^n$ for any $n \in \mathbb{N}$.*

Lemma 2.2. *Let $G = (X, R)$ be a graph with v vertices and have average valency $\overline{k} = \frac{1}{v} \sum_{x \in X} k(x)$. Let A be the adjacency matrix of G with eigenvalues $\theta_0 \geq \theta_1 \geq \cdots \geq \theta_v$. We have $\theta_0 \geq \overline{k}$, with equality if and only if G is regular.*

Proof. Let $\beta = \{u_1, u_2, \cdots, u_v\}$ be an orthonormal basis of $\mathbb{R}^v$ which are all eigenvectors of A, and let θ_i be the corresponding eigenvalue of u_i. Consider the all-1 vector $\mathbf{1}$ in $\mathbb{R}^v$. We can express $\mathbf{1}$ in terms of a linear combination of β,that is, $\mathbf{1} = \sum_{i=1}^{v} a_i u_i$. We have that $v = \mathbf{1}^t \mathbf{1} = \sum_{i=1}^{v} a_i^2$, and $\sum_{x \in X} k(x) =$

$v\overline{k} = \mathbf{1}^t A\mathbf{1} = \sum_{i=1}^{v} a_i^2\theta_i \leq \sum_{i=1}^{v} a_i^2\theta_0 = v\theta_0$, so $\overline{k} \leq \theta_0$. The equality holds if and only if $a_i(\theta_0 - \theta_i) = 0$ for all i $(o \leq i \leq v)$, that is, $A\mathbf{1} = \theta_0\mathbf{1}$, i.e.,$G$ is regular with valency θ_0. □

Theorem 2.3. *Let $G = (X, R)$ be a graph with v vertices and has spectrum*

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}$$

where $\theta_0 > \theta_1 > \cdots > \theta_d$. Then the following (i)-(ii) are equivalent.

(i) $\sum_{i=0}^{d} m_i\theta_i^2 = v\theta_0$

(ii) G is regular with valency θ_0.

Proof. Observe $(A^2)_{xx} = k(x)$ for all $x \in X$. Hence we have that $\sum_{i=1}^{d} m_i\theta_i^2 = Tr(A^2) = \sum_{x\in X} k(x) = v\overline{k}$.Then simply applying Lemma 2.2, we have that (i)-(ii) are equivalent. □

We quote a Theorem from [1, Lemma 8.12.1].

Theorem 2.4. *If G is a graph with diameter d, then $A(G)$ has at least $d+1$ distinct eigenvalues.*

3 t-distance-regular graphs

Let $G = (X, E)$ and $G' = (X', E')$ be two connected graphs with the same spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}.$$

Let $t \leq d$ be a positive integer. Suppose G is t-distance-regular. That is in G the parameters a_i, b_i, and $c_{i+1}, (0 \leq i \leq t-1)$ are well-defined. Hence k_i $(0 \leq i \leq t)$ is well defined. Suppose G' is $(t-1)$-distance-regular , the parameters a'_i, b'_i, and $c'_{i+1}, (0 \leq i \leq t-2)$ are well-defined. Furthermore assume these parameters are the same as the corresponding intersection parameters of G. Hence $k'_i = k_i$ $(0 \leq i \leq t-1)$. Let A, A' denote the adjacency matrices of G, G' respectively.

Lemma 3.1. $\sum_{x \in X'} \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y) = v k_{t-1} a_{t-1}$ *, where* $v = |X| = \sum_{i=0}^{d} m_i$.

Proof. The number of closed walks of length $2t-1$ through x in G' is $(A'^{2t-1})_{xx}$. These closed walks divide into 2 parts. One contains an edge in the induced subgraph $G'_{t-1}(x)$ and the other does not. The number of the first part is

$$\sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y)(c^2_{t-1} c^2_{t-2} \cdots c^2_2).$$

Let K denote the number of remaining closed walks. Hence

$$K + \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y)(c^2_{t-1} c^2_{t-2} \ldots c^2_2) = (A'^{2t-1})_{xx}. \tag{3.1}$$

Note K can be expressed in terms of the known (and well-defined) intersection parameters. Then we know that

$$\begin{aligned} Tr(A'^{2t-1}) &= \sum_{x \in X'} (A'^{2t-1})_{xx} \\ &= vK + \sum_{x \in X'} \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x, y)(c^2_{t-1} c^2_{t-2} \cdots c^2_2). \end{aligned} \tag{3.2}$$

Similarly,

$$
\begin{aligned}
Tr(A^{2t-1}) &= vK + \sum_{x \in X} \sum_{y \in G_{t-1}(x)} a_{t-1}(c_{t-1}^2 c_{t-2}^2 \dots c_2^2) \\
&= vK + vk_{t-1}a_{t-1}(c_{t-1}^2 c_{t-2}^2 \dots c_2^2). \qquad (3.3)
\end{aligned}
$$

Since A and A' have the same spectrum, we know that

$$Tr({A'}^{2t-1}) = Tr(A^{2t-1}).$$

Thus by (3.2)-(3.3)

$$\sum_{x \in X'} \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x,y) = vk_{t-1}a_{t-1}.$$

□

Corollary 3.2. *Suppose either* $a'_{t+1}(x,y) \geq a_{t-1}$ *or* $a'_{t-1}(x,y) \leq a_{t-1}$ *for any* $x \in X', y \in G'_{t-1}(x)$. *Then* $a'_{t-1} = a_{t-1}$.

Proof. It's trivial by Lemma 3.1. □

Lemma 3.3. $\sum_{x \in X'} \sum_{z \in G'_t(x)} c'_t(x,z) = vk_{t-1}b_{t-1} = vk_t c_t.$

Proof. For each $x \in X'$, by counting the number of edges between $G'_{t-1}(x)$ and $G'_t(x)$ in two ways and Lemma 3.1,

$$
\begin{aligned}
\sum_{x \in X'} \sum_{z \in G'_t(x)} c'_t(x,z) &= \sum_{x \in X} \sum_{y \in G'_{t-1}(x)} b'_{t-1}(x,y) \\
&= \sum_{x \in X'} \sum_{y \in G'_{t-1}(x)} (k_1 - c_{t-1} - a'_{t-1}(x,y)) \\
&= vk_{t-1}(k_1 - c_{t-1}) - \sum_{x \in X} \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x,y) \\
&= vk_{t-1}(k_1 - c_{t-1}) - vk_{t-1}a_{t-1} \\
&= vk_{t-1}b_{t-1} \\
&= vk_t c_t.
\end{aligned}
$$

□

Corollary 3.4. *Let $\overline{k'_t}$ denotes $\dfrac{1}{v}$ times the cardinality of the set $\{(x,z)|\ x,z \in G',\ d(x,z)=t\}$. Suppose either $\overline{k'_t} \geq k_t$ and $c'_t(x,z) \geq c_t$, or $\overline{k'_t} \leq k_t$ and $c'_t(x,z) \leq c_t$ for any $x \in X, z \in G'_t(x)$. Then $\overline{k'_t} = k_t$ and $c'_t = c_t$.*

Proof. It's trivial by Lemma 3.3. □

Lemma 3.5.

$$\begin{aligned} Tr(A'^{2t}) = vC \ &+ \sum_{x\in X'} \sum_{y \in G'_{t-1}(x)} a'_{t-1}(x,y)^2 (c^2_{t-1}c^2_{t-2}\cdots c^2_2) \\ &+ \sum_{x\in X'} \sum_{z \in G'_{t}(x)} c'_{t}(x,z)^2 (c^2_{t-1}c^2_{t-2}\cdots c^2_2) \\ &+ \ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\cdots c^2_2)(a_{t-2}+\cdots+a_1) \end{aligned}$$

for some constant C determined by a_i, b_i, c_{i+1} $(0 \leq i \leq t-2)$.

Proof. For any vertex x of G', we count the number of closed walks $x = x_0, x_1, \cdots, x_{2t} = x$. There are 4 cases.

Case 1 : $x_{t-1}, x_t, x_{t+1} \in G'_{t-1}(x)$. The number of closed walks in this case can be expressed as

$$\sum_{x_t \in G'_{t-1}(x)} a'_{t-1}(x,x_t)^2(c^2_{t-1}c^2_{t-2}\cdots c^2_2).$$

Case 2 : $x_t \in G'_t(x)$. The number of closed walks in this case can be expressed as

$$\sum_{x_t \in G'_{t}(x)} c'_{t}(x,x_t)^2(c^2_{t-1}c^2_{t-2}\ldots c^2_2).$$

Case 3 : $x_t \in G'_{t-1}(x), |\{x_{t-1}, x_{t+1}\} \cap G'_{t-1}(x)| = 1$. The number of closed walks in this case can be expressed as

$$\sum_{x_t \in G'_{t-1}(x)} a'_{t-1}(x,x_t)(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1).$$

By simply apply Lemma 3.1, we know that this term equals

$$vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1).$$

Case 4 : The remaining cases. The number of closed walks in this case can be expressed as a known constant C.

As before, we know the number of the closed walks of length $2t$ is $Tr(A'^{2t})$. Hence

$$\begin{aligned}Tr(A'^{2t}) = vC &+ \sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2)\\ &+ \sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2)\\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\cdots c_2^2)(a_{t-2}+\cdots+a_1).\end{aligned}$$

□

Corollary 3.6. *Let C be as in Lemma 3.5. Then*

$$\begin{aligned}Tr(A^{2t}) = vC &+ vk_{t-1}a_{t-1}^2(c_{t-1}^2c_{t-2}^2\ldots c_2^2)\\ &+ vk_tc_t^2(c_{t-1}^2c_{t-2}^2\ldots c_2^2)\\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\ldots c_2^2)(a_{t-2}+\ldots+a_1).\end{aligned}$$

Proof. We express $Tr(A^{2t})$ by the way in Lemma 3.5.

$$\begin{aligned}Tr(A^{2t}) = vC &+ \sum_{x\in X}\sum_{y\in G_{t-1}(x)} a_{t-1}(x,y)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2)\\ &+ \sum_{x\in X}\sum_{z\in G_t(x)} c_t(x,z)^2(c_{t-1}^2c_{t-2}^2\cdots c_2^2)\\ &+ \sum_{x\in X}\sum_{y\in G_{t-1}(x)} a_{t-1}(x,y)(c_{t-1}^2c_{t-2}^2\cdots c_2^2)(a_{t-2}+\cdots+a_1).\end{aligned}$$

Since the intersection parameters a_{t-1}, c_t are well-defined and known in G, we simply substitute the parameters in and get the result. □

Lemma 3.7.

$$\begin{aligned}Tr(A'^{2t}) \geq vC &+ vk_{t-1}a_{t-1}^2(c_{t-1}^2c_{t-2}^2\ldots c_2^2)\\ &+ vk_tc_t(c_{t-1}^2c_{t-2}^2\ldots c_2^2)\\ &+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\ldots c_2^2)(a_{t-2}+\ldots+a_1). \qquad (3.4)\end{aligned}$$

Furthermore, the following (i)-(ii) are equivalent.

(i) Equality holds in (3.4).

(ii) $a'_{t-1}(x,y) = a_{t-1}, c'_t(x,z) = 1$ for any $x \in X', y \in G'_{t-1}(x), z \in G'_t(x)$.

Proof. Applying Cauchy's inequality and $c'_t(x,z)^2 \geq c'_t(x,z)$ on the expression of $Tr(A'^{2t})$ in Lemma 3.5, it follows that

$$
\begin{aligned}
Tr(A'^{2t}) \geq vC &+ \frac{1}{vk_{t-1}}(\sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y)(c_{t-1}c_{t-2}\ldots c_2))^2 \\
&+ \sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z)(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1) \\
= vC &+ \frac{1}{vk_{t-1}}(vk_{t-1}a_{t-1}(c_{t-1}c_{t-2}\ldots c_2))^2 \\
&+ vk_{t-1}b_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1) \\
= vC &+ vk_{t-1}a^2_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}b_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2) \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1).
\end{aligned}
$$

The above equality holds if and only if $a'_{t-1}(x,y) = a_{t-1}$ and $c'_t(x,z) = 1$ for any x, y, z with $\delta(x,y) = t-1$ and $\delta(x,z) = t$.
The equivalence of (i)-(ii) is clear. □

Lemma 3.8.

$$
\begin{aligned}
Tr(A'^{2t}) \geq vC &+ vk_{t-1}a^2_{t-1}(c^2_{t-1}\cdots c^2_2) \\
&+ \frac{(vk_tc_tc_{t-1}\cdots c_2)^2}{v\overline{k'_t}} \\
&+ vk_{t-1}a_{t-1}(c^2_{t-1}c^2_{t-2}\ldots c^2_2)(a_{t-2}+\ldots+a_1). \quad (3.5)
\end{aligned}
$$

Furthermore, the following (i)-(ii) are equivalent.

(i) Equality holds in (3.5), $\overline{k'_t} = k_t$.

(ii) $a'_{t-1}(x,y) = a_{t-1},\ c'_t(x,z) = c_t$ for any $x \in X, y \in G'_{t-1}(x), z \in G'_t(x)$.

Proof. Applying Cauchy's inequality on the expression of $Tr(A'^{2t})$ in Lemma 3.5, it follows

$$\begin{aligned}
Tr(A'^{2t}) \geq vC &+ \frac{1}{vk_{t-1}}(\sum_{x\in X'}\sum_{y\in G'_{t-1}(x)} a'_{t-1}(x,y)c_{t-1}c_{t-2}\ldots c_2)^2 \\
&+ \frac{1}{v\overline{k'_t}}(\sum_{x\in X'}\sum_{z\in G'_t(x)} c'_t(x,z)c_{t-1}c_{t-2}\ldots c_2)^2 \\
&+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\ldots c_2^2)(a_{t-2}+\ldots+a_1). \\
= vC &+ \frac{1}{vk_{t-1}}(vk_{t-1}a_{t-1}c_{t-1}c_{t-2}\ldots c_2)^2 \\
&+ \frac{(vk_tc_tc_{t-1}\cdots c_2)^2}{v\overline{k'_t}} \\
&+ vk_{t-1}a_{t-1}(c_{t-1}^2c_{t-2}^2\ldots c_2^2)(a_{t-2}+\ldots+a_1).
\end{aligned}$$

(i)⇒(ii) is clear. (ii)⇒(i) is from the observation that the last term in the above equation is $Tr(A^{2t})$ which is equal to $Tr(A'^{2t})$. □

Lemma 3.9. *Suppose $c_t = 1$. Then a'_{t-1}, b'_{t-1}, c'_t are well-defined, and are the same as the corresponding ones in G .*

Proof. Comparing to Collary 3.6 and using $c_t = 1$, we find the equality in Lemma 3.7 holds. Hence a'_{t-1}, c'_t are well-defined. Note $b'_{t-1} = b_0 - c_{t-1} - a'_{t-1}$. □

4 Applications

Theorem 4.1. [4, Theorem 1] Let $G = (X, E)$ and $G' = (X', E')$ be two connected graphs with the same spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_d \\ m_0 & m_1 & \cdots & m_d \end{pmatrix}.$$

Suppose that G is distance-regular with intersection parameters a_i, b_i, c_i for $0 \leq i \leq d$. Suppose $c_j = 1$ for $1 \leq j \leq d-1$. Then G' is a distance-regular graph with the same intersection parameters of G.

Proof. We first show $a'_i = a_i, b'_i = b_i, c'_{i+1} = c_{i+1} = 1 \quad (0 \leq i \leq d-2)$ by induction on i. $a'_0 = 0 = a_0$, $c'_1 = 1 = c_1$ are clear. $b'_0 = b_0$ is from Theorem 2.3. Hence we have the case $i = 0$. Suppose this is true for $i \leq t-2$. The case $i = t-1$ is true from Lemma 3.9. So we have $a'_i = a_i, b'_i = b_i, c'_{i+1} = c_{i+1} = 1 \quad (0 \leq i \leq d-2)$. For the remaining parameters, we know $k'_i = k_i$ is well-defined for each $0 \leq i \leq d-1$. Note the diameter of G' is at most d by Lemma 2.4. Hence $k'_d = v - k_0 - k_1 \cdots - k_{d-1}$ is well-defined. Then the equality in Lemma 3.8 (iii) holds for $t = d$, so by Lemma 3.8 (ii) we have $a'_{d-1} = a_{d-1}, c'_d = c_d$. Note $a'_d = b_0 - c_d = a_d$. □

Corollary 4.2. *Let G be a strongly regular graph. Suppose that G' is a graph with the same spectrum of G. Then G' is a strongly regular graph with the same intersection parameters of G.*

Proof. This is immediate from Theorem 4.1 since $c_1 = 1$. □

Theorem 4.3. *Let G be a distance-regular graph. Suppose G' is a graph with the same spectrum of G. Furthermore, with refering to Corollary 3.4, suppose $\overline{k'_t} = k_t$. Then G' is a distance-regular graph with the same intersection parameters.*

Proof. We show $a'_i = a_i, b'_i = b_i,\ c'_{i+1} = c_{i+1} \quad (0 \leq i \leq d-1)$ by induction on i. $a'_0 = a_0$, $c'_1 = 1 = c_1$ are clear. $b'_0 = b_0$ is from Theorem 2.3. Hence we have the case $i = 0$. Suppose this is true for $i \leq t-2$. Since Lemma 3.8 (iii) holds, we have Lemma 3.8 (ii). Then $a'_{t-1} = a_{t-1}$ and $c'_t = c_t$. Note $b'_{t-1} = b_0 - c_{t-1} - a_{t-1}$. □

Remark 4.4. [6, Example 2.] The Gosset graph Γ is the unique distance-regular graph on 56 vertices with intersection array $\{27, 10, 1; 1, 10, 27\}$. Notice that in Γ, $k_0 = 1,\ k_1 = 27,\ k_2 = 27,\ k_3 = 1$. We have a graph Γ' with diameter 2 which is obtained by taking some special kind of switching on Γ such that in Γ', $k'_0 = 1, k'_1 = 27, k'_2 = 28$ where Γ and Γ' are cospectral.

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