國 立 交 通 大 學
應用數學系

碩 士 論 文

# 圖覆蓋的研究

# A Study of Graph Covering

研 究 生：詹　棨　丰
指導老師：傅　恆　霖　教 授

中 華 民 國 九 十 三 年 六 月

# 圖覆蓋的研究

# A Study of Graph Covering


研　究　生：詹棨丰　　　**Student：Chi-Feng Chan**

指導老師：傅恆霖教授　　**Advisor：Dr. Hung-Ling Fu**

國 立 交 通 大 學

應用數學系


碩　士　論　文

A Study of Graph Covering

National Chiao Tung University
In partial Fulfillment of Requirement
For the Degree of Master
In
Applied Mathematics
June 2004
Hsinchu, Taiwan, Republic of China

中 華 民 國 九 十 三 年 六 月

# 圖覆蓋的研究

研 究 生：詹棨丰　　　　指導老師：傅恆霖　教授

國 立 交 通 大 學

應 用 數 學 系

## 摘　　要

令 G 為一個給定的圖而 H 是 G 的子圖所形成的集合。使用最少數量 H 中的圖而能覆蓋所有 G 中的邊，而這個數量我們定義為 cov(G ,H)。這篇論文中的主要工作是證明，如果 G 是一個三連通圖或者 G 的邊連通數是 1 或 2，H 是 G 的奇子圖所形成的集合，則 cov(G ,H)≤3。

中 華 民 國 九 十 三 年 六 月

# A Study of Graph Covering

Student: Chi-Feng Chan Advisor: Hung-Lin Fu

*Department of Applied Mathematics*
*National Chiao Tung University*

**Abstract**

Let $G$ be a fixed graph and $\boldsymbol{H}$ be a class of subgraphs of $G$. Denote the minimum number of graphs in $\boldsymbol{H}$ covering the edges of $G$ by $cov(G, \boldsymbol{H})$. The main work of this thesis is to prove $cov(G, \boldsymbol{H}) \leq 3$ if $G$ is a 3-connected graph or $\kappa'(G) = 1$ or 2 and $\boldsymbol{H}$ is a class of odd subgraphs of $G$.

# 致　　謝

首先要感謝指導教授傅恆霖老師，這兩年中，老師的悉心指導，讓我受益良多。除了學識方面的增長，老師的教誨，也讓我在精神面上成長許多。

再來我要感謝交大應數提供了一個這麼好的環境讓我研習。感謝黃光明老師、黃大原老師、陳秋媛老師以及翁志文老師等，課業上給與我們指導，也不時關心我們的課外生活。也要感謝嚴志弘學長、張嘉芬學姐、郭志銘學長、許弘松學長、張飛黃學長以及陳宏賓學長的許多幫助。與啟賢、昭芳、抮君、正傑、喻培、嘉文、致維、建瑋、貴弘及文祥這些同學一同研究、打球及遊玩的回憶，是我人生中美好的兩年。

最後要感謝我的父母及女友惠敏，有著他們的關心照顧和支持，我才能無後顧之憂的專心於我的學業上，謝謝他們陪伴我走過這兩年的研究所生涯。

# Contents

**Abstract(in Chinese)** **i**

**Abstract(in English)** **ii**

**Acknowledgment** **iii**

**Contents** **iv**

**1 Introduction and Preliminaries** **1**

1.1 Motivation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1

1.2 Graph Terms . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1

1.3 The Known Results . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3

**2 The Main Results in Connectivity** **5**

**3 The Main Results in Edge-connectivity** **10**

**4 Conclusion** **16**

# 1 Introduction and Preliminaries

## 1.1 Motivation

Graph decomposition has been one of the most important topic in the study of graph theory and also combinatorial design theory. Mainly, we try to partition the edge set of a graph $G$ into sets $E_1, E_2, \ldots, E_t$ such that the edge-induced subgraph of $G < E_i >_G \in$ $\boldsymbol{H}$ where $\boldsymbol{H}$ is a collection of graphs.

If we look at graph decomposition on the other angle, we may use a collection of subgraphs of $G$ to cover $G$. This is the so-called covering. More precisely, a graph $G$ is said to be covered by a class of its subgraphs $\boldsymbol{H} = \{H_1, H_2, \ldots, H_t\}$ if the edge set of $G$, $E(G)$, is contained in the union of some $H_i$'s in $\boldsymbol{H}$. Denote the minimum number of graphs in $\boldsymbol{H}$ covering the edges of $G$ by $cov(G, H)$. In case that we require all the graphs in $\boldsymbol{H}$ we use to cover $G$ are edge-disjoint, we have a similar notion of $cov(G, H)$, the minimum number is denoted by $cov^*(G, H)$. Clearly, $cov^*(G, H) \geq cov(G, H)$. In this thesis, we shall study $cov(G, H)$, where $H$ is a collection of odd subgraphs of $G$.

## 1.2 Graph Terms

Let $G$ be a graph. For each vertex $v$ in a graph $G$, the number of edges incident to $v$ is the **degree** of $v$, denoted by $deg(v)$. A **graph** $G$ consists of a finite non-empty set $V(G)$ of **vertices** and a finite set $E(G)$ of distinct unordered pairs of distinct vertices called **edges**. The number of vertices of $G$ is called the **order** of $G$ and denoted by $|V(G)|$. The number of edges of $G$ is called the **size** of $G$ and denoted by $|E(G)|$. A relation that associates with each edge two vertices called its **endpoints**. Two or more edges joining the same pair of vertices are called **multiple edges**. A **loop** is an edge whose endpoints are equal. A graph is simple if it has no loops and multiple edges. Throughout of this thesis we consider only simple graphs.

If $e = uv$ or $(u, v)$ is an edge of $G$, then $e$ is said to join the vertices $u$ and $v$, and these vertices $u$ and $v$ are then said to be **adjacent**, denoted by $u \sim v$. We also say that $e$ is **incident** (or **joined**) to $u$ and $v$, . The maximum and minimum degrees in $G$ are denoted by $\triangle(G)$ and $\delta(G)$ respectively. A vertex of degree 0 is called an **isolated vertex**, and a vertex of degree 1 is called an **end-vertex**. If all vertices of $G$ have the same degree, then $G$ is a **regular graph**; if each degree is k, then $G$ is a **k-regular** graph. A 0-regular graph (that is, one with no edges) is a **null graph**. A graph is an **odd**(respectively **even**) graph if each vertex of the graph is of odd degree(respectively even degree). A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph. Therefore, an **odd forest** is an odd graph which is a forest.

The **neighborhood** $N_G(u)$ of $u$ is the set of all vertices of $G$ adjacent to $u$, the **closed neighborhood** $N_G[u]$ of $u$ is the union of $N_G(u)$ and $u$. Two edges incident to the same vertex are **adjacent edges**. A **matching** in $G$ is a set of edges no two of which are adjacent. Two graphs are **isomorphic** if there is a one-to-one correspondence between their vertex-sets which preserves the adjacency of vertices.

A **subgraph** of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, denoted by $H \subseteq G$. If $V(H) = V(G)$, then $H$ is called a **spanning subgraph** of $G$. A **spanning tree** of a graph $G$ is a graph $T$ that $T$ is a spanning subgraph of $G$ and $T$ is a tree. If $W$ is any set of vertices in $G$, then the subgraph **induced** by $W$ is the subgraph of $G$ obtained by joining those pairs of vertices in $W$ which are joined in $G$. Any induced subgraph $G[W]$ of $G$ is a subgraph induced by the subset $W$ of $V(G)$.

If $e$ is an edge of $G$, then the **edge-deleted subgraph** $G - e$ is the graph obtained from $G$ by removing the edge $e$. Similarly, if $v$ is a vertex of $G$, then the **vertex-deleted subgraph** $G - v$ is the graph obtained from $G$ by removing the vertex $v$ together with all its incident edges.

A graph $G$ is **connected** if each pair of vertices in $G$ belongs to a path; otherwise, $G$ is **disconnected**. A **separating set** or **vertex cut** of a graph $G$ is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The **connectivity** of $G$, written $\kappa(G)$, is the minimum size of a vertex set $S$ such that $G - S$ is disconnected or has only one vertex. A graph $G$ is **k-connected** if its connectivity is at least $k$. A **disconnecting set** (**edge cut**) of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. A graph is **k-edge-connected** if every disconnecting set has at least $k$ edges. The **edge-connectivity** of $G$, written by $\kappa'(G)$, is the minimum size of a disconnecting set.

Throughout of this thetis, all the graphs we consider are simple graphs, i.e., multiple edges and loops are not allowed. For all the terminologies we use in this thesis, we refer to the textbook by West [4].

## 1.3 The Known Results

The following theorem is one of the well-known results in the study of graph covering.

**Theorem 1.3.1** [1]. Any graph of order $n$ can be covered by at most $\lfloor \frac{n^2}{4} \rfloor$ $K_3$'s or $K_2$'s.

Note that Theorem 1.3.1 also give a bound for the number of elements when represent a graph by distinct sets, see [1]. So far, quite a few beautiful works in covering have been obtained, see [2] for a survey. Our study is motivated by the following result.

**Theorem 1.3.2** [3, 5] Every bridgeless graph can be covered by at most three even graphs.

As a counterpart of the above theorem, the following problem was posed by Pyber[2] after showing that every graph $G$ can be covered by at most 4 disjoint odd subgraphs.

**Problem** Is it true that every graph can be covered by at most three odd graphs?

In fact, Pyber has proved the following result.

**Proposition 1.3.3** [2] Let $G$ be a connected graph of even order. Then $G$ can be covered by at most three odd graphs. Moreover, $G$ can be covered by a forest $F$ and an odd graph $G'$ which are subgraphs of $G$.

Therefore, it remains to show that a graph of odd order can be covered by at most three odd graphs. So far, we are not able to solve the problem in general. But we manage to prove that every 3-connected graph can be covered by at most three odd graphs in this thesis. Note that Seymour's 6-flow Theorem implies that every 3-connected graph can be covered by at most three even graphs[3]. Furthermore, for those graphs with low edge-connectivities, the problem of Pyber can also be solved.

## 2 The Main Results in Connectivity

In this section, We study the covering of a 3-connected graph of $G$ by odd subgraphs. The following lemmas are essential to the proof of the first result.

**Lemma 2.1.** Let $G$ be $n$-connected, $n \geq 2$, and $v_1, v_2, \ldots, v_k$ be $k$ vertices in $G$ where $k \leq n-1$. Then there exists a spanning tree $T$ of $G$ such that $deg_T(v_i) = 1, i = 1, 2, \ldots, k$.

**Proof.** Since $G' = G - \{v_1, v_2, \ldots, v_k\}$ is connected, $G'$ has a spanning tree $T'$. By the fact that $G$ is $n$-connected, each $v_i$, $i = 1, 2, \ldots, k$, is adjacent to a vertex $u_i$ in $V(T')$. Now, let $T$ be the subgraph of $G$ obtained by joining $v_i$ to $u_i$ in $T'$ (not necessarily distinct) for $i = 1, 2, \ldots, k$. Then, $T$ is a spanning tree of $G$ satisfying the condition $deg_T(v_i) = 1$ for $i = 1, 2, \ldots, k$. This concludes the proof. □

**Corollary 2.2.** Let $G$ be a 2-connected graph. Then for each vertex $v$ in $G$, there exists a spanning tree $T$ of $G$ such that $deg_T(v) = 1$.

**Proof.** A direct consequence of Lemma 2.1. □

**Lemma 2.3.** Let $G$ be a forest. Then $G$ can be covered by at most two disjoint odd forests.

**Proof.** It suffices to show that a tree $T$ can be covered by at most two odd forests. Starting from a vertex $v_1$(root), if $deg(v_1)$ is odd, we color all edges incident to $v_1$ by black. If $deg(v_1)$ is even, we color an odd number of edges which are incident to $v_1$ by black and the rest of edges by white. Next, we consider a vertex $v_2 \in N(v_1) = \{v_2, v_3, \ldots, v_t\}$. W.L.O.G. let $e_1 = v_1v_2$ be colored by black. If $deg(v_2)$ is even, we color an odd number of edges incident to $v_2$ (not including $e_1$) by white and others by black. If $deg(v_2)$ is odd, we color all edges incident to $v_2$ by black. Continuing the above process, we can color all the edges incident to $v_3, v_4, \ldots, and\ v_t$ respectively. Since $T$ is a tree, we move on to another level(start from root $v_1$) while all edges are colored. Now, it's easy to check that the edges of each color induces an odd forest. □

The following result was obtained in [2]. Since knowing the idea of his proof is very helpful in understanding the technique we use in proving the main theorem, we present its proof here.

**Proposition 2.4.** [2] Every connected graph $G$ can be covered by at most four odd subgraphs of $G$.

**Proof.** First, we consider the case where $|V(G)|$ is even. If $G$ itself is an odd graph, then there is nothing to prove. On the other hand, let $v_1, v_2, \ldots, v_{2k-1}, v_{2k}$ be the vertices in $G$ with even degree and $T$ be a spanning tree of $G$. By the property of $T$, $v_{2i-1}$ and $v_{2i}$ can be connected by a unique path on $T$, let it be $P_i$, $i = 1, 2, \ldots, k$. Now, take the modulo 2 sum of $P_1, P_2, \ldots,$ and $P_k$, we obtain a graph $G'$($e$ is in $E(G')$ if and only if $e$ is in an odd number of $P_i$'s). Clearly, $G'$ is a subgraph of $T$ which is a forest. Moreover, $G - G'$ is an odd graph by the way we construct $G'$. Thus, by Lemma 2.3, $G$ can be covered by at most three odd graphs. It is left to consider the case when $G$ is of odd order. First, let $T'$ be a spanning tree of $G$ and $v$ be a vertex of $G$ such that $deg_{T'}(v) = 1$. Since $v$ is an end point in $T$, then $G - v$ is connected. By the even case, $G - v$ can be covered by an odd graph $\tilde{G}$ and a forest $F$. The proof follows easily if $deg_G(v)$ is odd. On the other hand, if $deg_G(v)$ is even, let $G''$ be a star with center $v$ and $deg_G(v) - 1$ edges, and $F'$ be a forest obtained from the union of $F$ and an edge incident to $v$ which is not in $G''$. Now, $G$ can be covered by $G''$, $F'$ and $\tilde{G}$ and the proof follows by covering $F'$ with two odd forests. □

**Corollary 2.5.** $G$ is a graph then $cov(G,\{\text{even subgraph or odd subgraph}\}) \leq 3$.

**Proof.** We can prove this corollary by Theorem 1.3.2, Proposition 1.3.3 and Lemma 2.12. But, we shall use a similar idea as in Proposition 2.4. If $|V(G)|$ is even, there is nothing to prove. On the other hand, let $v_1, v_2, \ldots, v_{2k-1}, v_{2k}$ be the vertices in $G$ with odd degree and $T$ be a spanning tree of $G$. By the property of $T$, $v_{2i-1}$ and $v_{2i}$ can be connected by a unique path on $T$, let it be $P_i$, $i = 1, 2, \ldots, k$. Now, take the modulo 2 sum of $P_1, P_2, \ldots,$ and $P_k$ we obtain a graph $G'$ ($e$ is in $E(G')$ if and only if $e$ is in an odd number of $P_i$'s). Clearly, $G'$ is a subgraph of $T$ which is a forest. Moreover, $G - G'$ is an even graph by the way we construct $G'$. Thus by Lemma 2.3, $G$ can be covered by at most three odd or even graphs. □

Note here that the odd forests in Lemma 2.3 are disjoint and an odd subgraph $G'$ is a spanning odd subgraph of $G$ in Proposition 1.3.3. Also, the idea of attaching edges to a forest using in Proposition 2.4 plays an important role in the proof of the main theorem.

**Theorem 2.6.** Let $G$ be a 3-connected graph. Then $G$ can be covered by at most three odd subgraphs.

**Proof.** By Proposition 1.3.3, it suffices to consider the case when $G$ is of odd order. Therefore, $G$ contains a vertex $v$ of even degree. For simplicity, we split the proof into three cases.

**Case 1.** There exists a vertex $u_1 \in N(v)$ such that $deg_G(u_1)$ is even.

Clearly, $G - v$ is 2-connected and $G - v$ has a spanning tree $T$ such that $deg_T(u_1) = 1$(by Lemma 2.2). Now, if $G - v$ is an odd graph, then the proof follows by decomposing the star with center $v$ into two odd subgraphs. On the other hand, $G - v$ is not an odd graph. Then, by Proposition 1.3.3, $G - v$ can be covered by an odd graph $G'$ and a forest $F$ such that $deg_F(u_1) = 0$, since $deg_{G-v}(u_1)$ is odd and $deg_T(u_1) = 1$. Thus, if we can cover $G - G'$ by at most two odd subgraphs, then we are done. For this purpose, we first delete $v$ and replace $v$ with $|N(v)| = t$ vertices $v_1, v_2, \ldots, v_t$ where $N(v) = \{u_1, u_2, \ldots, u_t\}$ and obtain a new forest $F^*$ by adding $(v_i, u_i)$, $i = 1, 2, \ldots, t$, to $F$. Note that $deg_F(u_1) = 0$. Now, by Proposition 1.3.3, $F^*$ can be covered by at most two odd forests $F_1^*$ and $F_2^*$ such that $(v_1, u_1)$ is in $F_1^*$. Since $deg_G(v)$ is even, either each odd forest contains an odd number of edges incident to $v_i$'s or each odd forest contains an even number of edges incident to $v_i$'s. In the front case, we replace each $(v_i, u_i)$ with $(v, u_i)$ to obtain $F_1$ and $F_2$ respectively. Then $G$ is covered by the odd subgraphs $G'$, $F_1$ and $F_2$. Otherwise, since $(v_1, u_1)$ is a component in $F^*$, by moving $(v_1, u_1)$ to $F_2^*$, the proof follows as above process. This concludes the proof of Case 1.

**Case 2.** For each $u \in N(v)$, $deg_G(u)$ is odd and $N(v) = V(G - v)$.

Let $N(v) = \{u_1, u_2, \ldots, u_t\}$ and $G_1$ be the star of size $|N(u_1)|$ with center $u_1$. Clearly, $G_1$ is an odd graph which has $|N(u_1)|$ edges. Therefore, $|N(u_1) \bigcap N(v)|$ is even, let $N(u_1) \bigcap N(v) = \{u_2, u_3, \ldots, u_{2s-1}\}$ if it is not empty. Now, let $G_2$ be the graph induced by the edge set $A \bigcup B$ where $A = \{(v, u_i)|i = 2s, 2s + 1, \ldots, t\}$ and $B = E(G - v)\backslash\{(u_1, u_j)|j = 2, 3, \ldots, 2s - 1\}$. Then, it is not difficult to check that $G_2$ is an odd subgraph of $G$ and $G$ can be covered by three odd subgraphs $G_1$, $G_2$ and $G_3$ where $G_3$ is also an odd star and $E(G_3) = \{(v, u_l)|l = 1, 2, \ldots, 2s - 1\}$. Note that if $deg_G(u_1) = 1$, then $G$ is covered by $G_1$ and $G_2$.

**Case 3.** For each $u \in N(v)$, $deg_G(u)$ is odd and $|V(G - v)| > |N(v)|$.

For clearness, we consider two subcases depending on the parity of $deg_G(x)$ where $x \in N(u)\backslash N[v]$ for some $u$ such that $N(u)\backslash N[v]$ is not an empty set.

**Subcase 3.1.** $deg_G(x)$ is odd.

Since $G - v$ is 2-connected, by Lemma 2.1, $G - v$ has a spanning tree $T$ such that $deg_T(x) = 1$. By Proposition 1.3.3, since $|V(G - v)|$ is even, $G - v$ can be covered by an odd subgraph $G'$ and a forest $F$ such that $F$ is a subgraph of $T$, $deg_F(x) = 0$ and $deg_F(u)$ is odd($deg_{G-v}(u)$ is even). By a similar idea as in Case 1, we can replace $v$ with $v_1, v_2, \ldots, v_t$ where $N(v) = \{u_1(= u), u_2, \ldots, u_t\}$ and obtain a new forest $F^*$ by adding $(v_i, u_i)$, $i = 1, 2, \ldots, t$, to $F$. Now, $deg_{F^*}(u_1)$ is even and thus $F^*$ can be covered by two odd subgraphs $F_1^*$ and $F_2^*$ such that $e_1 = (v_1, u_1)$ is in $F_1^*$ and all the other edges incident to $u_1$(including $(u_1, y)$) are contained in $F_2^*$. By the same argument as we have in Case 1, if both $F_1^*$ and $F_2^*$ contain an odd number of edges from $\{(v_i, u_i)|i = 1, 2, \ldots, t\}$, then we have the three odd graphs we need, $G'$, $F_1$ and $F_2$, where $F_1$ and $F_2$ are obtained by replacing $(v_i, u_i)$'s in $F_1^*$ and $F_2^*$ respectively with $(v, u_i)$'s. On the other hand, if $F_1^*$ contains an even number of edges from $\{(v_i, u_i)|i = 1, 2, \ldots, t\}$, we let $F_1'$ and $F_2'$ be the graphs obtained by the same process as above and then let $F_1 = F_1' - e_1$ and $F_2 = F_2' + e_1 + (u_1, x)$. Since $e_1$ is in fact an independent edge in $F_1^*$, $F_1 = F_1' - e_1$ is an odd subgraph of $G$. $F_2$ is also an odd subgraph because of the fact that $deg_F(x) = 0$. Hence we have proved Subcase 3.1.

**Subcase 3.2** $deg_G(x)$ is even.

Since $G$ is 3-connected, $G - u$ is 2-connected. By Lemma 2.1 $G - u$ has a spanning tree $T$ such that $deg_T(x) = 0$ and $G - u$ can be covered by an odd graph $G'$ and a forest $F$ which is a subgraph of $T$. Now, instead of replacing $v$ with $v_i$'s(Subcase 3.1) we replace u with $u_1, u_2, \ldots, u_s$ where $s = |N(u)|$ and join $u_i$ to $w_i$ for $i = 1, 2, \ldots, s$, where $N(u) = \{x = w_1, w_2, \ldots, w_s = v\}$ to obtain a forest $F^*$. By observation, since $x$ is adjacent to $u$, $deg_{G-u}(x)$ is odd. Moreover, $deg_T(x) = 0$ implies that $deg_F(x) = 0$ and $(u_1, x)$ is an independent edge in $F^*$. Therefore, we can cover $F^*$ with two odd forests $F_1^*$ and $F_2^*$ such that $e_1 = (u_1, w_1)$ in $F_1^*$ and $e_2 = (u_s, w_s)$ in $F_2^*$. Now, by the fact that $deg_G(u)$ is odd, $s$ is an odd integer. Hence, one of $F_1^*$ and $F_2^*$ contains an even number of edges in $\{(u_i, w_i)|i = 1, 2, \ldots, s\}$ and the other contains an odd number of such edges. First, if $F_2^*$ contains an even number of such edges, then let $F_2$ be obtained from $F_2^* + e_1$ by replacing $(u_1, w_1)$ and $(u_i, w_i)$'s in $F_2^*$ with $(u, w_1)$ and $(u, w_i)$'s respectively and let $F_1$ be obtained from $F_1^*$ by replacing $(u_j, w_j)$'s in $F_1^*$ with $(u, w_j)$'s. On the other hand, if $F_2^*$ contains an odd number of edges from $\{(u_i, w_i)|i = 1, 2, \ldots, s\}$, then $F_1^* + e_2$ and $F_2^*$ both contain an odd number of such edges. Thus, $F_1$ and $F_2$ can be obtained with a similar way. Since $F_1$ and $F_2$ are odd subgraphs of $G$, this concludes the proof of this subcase and the theorem. □

## 3 The Main Results in Edge-connectivity

At this moment, we are not able to solve the problem when the connectivity of $G$ is 1 or 2. But we can prove that $G$ can be covered by at most three odd subgraphs if $\kappa'(G) = 1$ or 2. First, we need a notion about edge pair. Observe that, by Lemma 2.3, a forest can be covered by at most two disjoint forests. In fact, we can cover $F$ with exactly two edge-disjoint odd forests, $(F_1, F_2)$, provided that $F$ is not an odd forest itself. Moreover, it is not difficult to see that there are more than one way to cover $F$ by two edge-disjoint odd forests. Therefore, it is interesting to know whether there exist two edges $e_1$ and $e_2$ such that there is a covering with edge-disjoint forests $(F_1, F_2)$ where $e_1$ and $e_2$ are belonged to the same odd forest and also there exists a covering with edge-disjoint forests $(F_1^*, F_2^*)$ of $F$ in which $e_1 \in E(F_1^*)$ and $e_2 \in E(F_2^*)$.

Obviously, if $F$ is $P_3$, then it is not possible. But, if $F$ is an even star with at least four edges, then we do have such a pair of edges. In fact, any two edges form a pair of such edges. For convenicence, we call them **a good pair** in $F$.

Let $P(e_1, e_2)$ be a path which connects two edges $e_1$ and $e_2$ in a graph (not including $e_1$ or $e_2$). It is well-known that in a tree there exists exactly one path $P(e_1, e_2)$ for each pair of non-adjacent edges $e_1$ and $e_2$. (If $e_1$ and $e_2$ are incident, then define $P(e_1, e_2)$ as the graph with vertex set $e_1 \bigcap e_2$ and edge set $\emptyset$.) Now, we have a result to characterize a good pair.

**Lemma 3.1.** Two edges $e_1$ and $e_2$ form a good pair in a tree $T$ if and only if there exists a vertex $x$ on $P(e_1, e_2)$ such that $deg(x) = 2t$, $t > 1$.

**Proof.** For the sufficiency, let $x$ be the vertex on $P(e_1, e_2)$ such that $deg(x) = 2t$, $t > 1$. W.L.O.G. we let $e_1$ and $e_2$ lie in the same odd forest $F$, $e_3$ and $e_4$ be the edges incident to $x$ in $P(e_1, e_2) + \{e_1, e_2\}$. Let $e_3$ and $e_4$ lie in different forests $F_1$ and $F_2$ respectively and $e_5$ be an edge in $F_2$ which is also incident to $x$. By Lemma 2.3, we can get two new disjoint odd forests $F_1^*$ and $F_2^*$ by switching $e_3$ and $e_5$. Now, $e_1$ and $e_2$ lie in $F_1^*$ and $F_2^*$ respectively. Therefore, $e_1$ and $e_2$ form a good pair. For the necessity, if we have assigned a forest to contain $e_1$, then all edges that joined to the vertices on $P(e_1, e_2)$ should be fixed; of course, these edges include $e_2$. Without having a vertex $x$ with $deg(x) = 2t$, $t > 1$, there is no way to put $e_1$ and $e_2$ in edge-disjoint odd forests provided $e_1$ and $e_2$ are in the same odd forest and vice versa. This concludes the proof. □

**Lemma 3.2.** Let $e_1$ and $e_2$ be two edges in $T$ which are not a good pair in $T$ and let $e$ be an edge which is incident to a vertex on $P(e_1, e_2)$ and a vertex not in $V(T)$. If $e_1$ and $e_2$ lie in the same(respectively different) forest(s) when covering $T$ by two disjoint odd forests, then $e_1$ and $e_2$ form a good pair in $T + e$ or they could lie in the different(respectively same) forest(s) when covering $T + e$ by two odd disjoint forests.

**Proof.** By Lemma 3.1, if $e_1$ and $e_2$ are not a good pair in $T$, then for each vertex $v$ on $P(e_1, e_2)$, $deg(v)$ is either odd or 2. Since $v$ is on $P(e_1, e_2)$, $deg(v) \geq 2$. If $deg(v)$ is odd, then $deg_{T+e}(v) = 2t$, $t > 1$. By Lemma 3.1, $e_1$ and $e_2$ form a good pair in $T + e$. On the other hand, since adding $e$ to $T$ will change the status of $e_1$ and $e_2$(in the same odd forest or not), the proof follows. □

**Lemma 3.3.** Let $e_1$ and $e_2$ be two edges in $T$ which are not a good pair in $T$ and $e \notin \{e_1, e_2\}$ such that $e$ is incident to a vertex on $P(e_1, e_2)$. If $e_1$ and $e_2$ lie in the same(respectively different) forest(s) when covering $T$ by two disjoint odd forests, then $e_1$ and $e_2$ form a good pair in $T - e$ or they could lie in the different(respectively same) forest(s) when covering $T - e$ by two odd disjoint forests.

**Proof.** If $e \in P(e_1, e_2)$, then $e_1$ and $e_2$ lie in two different connected components of $T - e$. It's easy to see that $e_1$ and $e_2$ form a good pair in $T - e$. So we can suppose $e \notin P(e_1, e_2)$. By Lemma 3.1, if $e_1$ and $e_2$ are not a good pair in $T$, then for each vertex $v$ on $P(e_1, e_2)$, $deg(v)$ is either odd or 2. Let $e = (x, v)$, $x$ be a vertex on $P(e_1, e_2)$ and $v$ be any vertex in $T$. Since $deg_T(x) \geq 2$ hence $x \in P(e_1, e_2)$. Now, consider the following three cases.

**Case 1.** $deg_T(x) = 2$.

$e$ must be on $P(e_1, e_2)$. Therefore, $e_1$ and $e_2$ are a good pair in $T - e$.

**Case 2.** $deg_T(x) = 3$.

Suppose $e \notin P(e_1, e_2)$ and $deg_{T-\{e\}}(x) = 2$. Hence, exactly one edge incident to $x$ in $P(e_1, e_2)$ will be changed when $e$ is removed from $T$ and also exactly one edge of $\{e_1, e_2\}$ will be changed.

**Case 3.** $deg_T(x)$ is odd but not 3.

Suppose $e \notin P(e_1, e_2)$, then $deg_{T-\{e\}}(x)$ is even and not 2. By Lemma 3.1, $e_1$ and $e_2$ are a good pair in $T - e$. This concludes the proof. □

A graph $U$ is **unicyclic** if $U$ is connected and $U$ contains exactly one cycle. Clearly, $|E(G)| = |V(G)|$ if $G$ is unicyclic.

**Lemma 3.4.** Let $U$ be a unicyclic graph which is not an odd graph itself and also $U$ can not be covered by two disjoint odd subgraphs. Let $e$ be an edge which joins a vertex on the cycle of $U$ and a vertex not in $V(U)$. Then $U + e$ is an odd graph itself or $U + e$ can be covered by two disjoint odd subgraphs.

**Proof.** W.L.O.G. we suppose $e = (x_1, v)$ is the new edge in $U + e$ and $x_1$ is a vertex in $V(U)$. Hence, there exists an edge $e_1 = (x_1, x_2)$ on the cycle of $U$. Clearly, $T = U$ - $e_1$ is a tree and $T$ is not an odd graph(or $U$ can be covered by $T$ and $e_1$). By Lemma 2.3, $T$ can be covered by two disjoint odd forests. We shall color the edges of two disjoint odd forests by black and white respectively. Then the proof follows if we can find a way to color the edges of $U + e$ by using black and white and both colors induce an odd forest.

**Case 1.** $deg_T(x_1)$ and $deg_T(x_2)$ are both odd.

All edges joined to $x_1$(or $x_2$) must be colored with the same color. If the colors of the edges that joined to $x_1$ and $x_2$ are the same, then it's done by coloring $e_1$ with the other color. Otherwise, we suppose that the edges which join to $x_1$ are colored with white and $e$ and $e_1$ are also colored with white.

**Case 2.** $deg_T(x_1)$ is even and $deg_T(x_2)$ is odd.

If the edges joined to $x_2$ can be colored with black, then we can color $e$ and $e_1$ with white.

**Case 3.** $deg_T(x_1)$ is odd and $deg_T(x_2)$ is even.

At first, we color all edges joined to $x_2$ with black and the others by the method used in Lemma 2.3. If the edges joined to $x_1$ are colored with white, then we color $e_1$ with black. Otherwise, we color $e$ and $e_1$ with black.

**Case 4.** Both $deg_T(x_1)$ and $deg_T(x_2)$ are even.

At first, we color all edges joined to $x_2$ with black and the others by the method used in Lemma 2.3. Then, it is done by coloring $e$ and $e_1$ with black. □

**Lemma 3.5.** Let $G$ be a connected graph such that $|V(G)|$ is odd. Then $cov^*(G) \leq 3$ provided $G$ has a vertex of degree 1.

**Proof.** Let $deg_G(x) = 1$ and $G' = G[V(G)-x]$. Then, $|V(G')|$ is even and $G'$ is connected. By Proposition 1.3.3, $G'$ can be covered by an odd subgraph and a forest $F$. Let $e$ be the edge incident to $x$ in $G$. Clearly, $F+e$ is a forest. By Lemma 2.3, $F + e$ can be covered by at most two disjoint odd forests. This concludes the proof. □

**Theorem 3.6.** If $G$ is a connected graph which contains a bridge (in the other words, $\kappa'(G) = 1$) and $|V(G)|$ is odd, then $cov^*(G) \leq 3$.

**Proof.** By Lemma 3.5, we can assume that there exists no vertex which is of degree one in $G$. Let $e$ be a bridge of $G$. Then $G - e$ contains two connected components $G_1$ and

$G_2$. Since $|V(G)|$ is odd, $|V(G_1)| + |V(G_2)|$ is odd. W.L.O.G. we suppose $|V(G_1)|$ is odd and $|V(G_2)|$ is even. So, $|V(G_1 + e)|$ is even and $|V(G_2 + e)|$ is odd and $G_2 + e$ contains a vertex of degree 1. By Proposition 1.3.3, Lemma 2.3 and Lemma 3.5, $G_1 + e$ and $G_2 + e$ can be covered by at most 3 disjoint odd subgraphs, denoted by $\{H_1^*, H_2^*, H_3^*\}$ and $\{H_1^{**}, H_2^{**}, H_3^{**}\}$ respectively. Furthermore, $e \in H_1^*$ and $e \in H_1^{**}$.(Note that $H_2^*, H_3^*, H_2^{**}$ and $H_3^{**}$ may be null graphs.) Then $H_1 = H_1^* \bigcup H_1^{**}$, $H_3 = H_3^* \bigcup H_3^{**}$ and $H_3 = H_3^* \bigcup H_3^{**}$ are three disjoint odd graphs(or null graphs) which covers $G$. This concludes the proof. □

A forest $F$ is **minimum** if a connected graph $G$ with even order is covered by a disjoint odd spanning subgraph $G_1$ and a forest $F$ such that $|E(F)|$ is the minimum. Since a connected graph $G$ with even order can be covered by an odd subgraph $G_1$ and a forest $F$ such that $G_1$ and $F$ are edge-disjoint, minimum forest $F$ does exist.

**Proposition 3.7.** If $F$ is a minimum forest of $G$, then $F + e$ is a forest for any edge $e$ in $G - F$.

**Proof.** Suppose not. So $F + e$ contains a cycle $C$ and $e$ is an edge on the cycle. Because $G_1 = G - F$ is spanning, $F^* = (F + e) - (C - e)$ is a forest and $G_1^* = (G_1 - e) + (C - e)$ is an odd spanning subgraph. Clearly, $F^*$ and $G_1^*$ are disjoint and $|E(F^*)| < |E(F)|$ hence $|E(C)| \geq 3$. We have a contradiction. □

**Lemma 3.8.** Let $G$ be a connected graph such that $|V(G)|$ is odd. If $G$ contains a vertex of degree 2, then $cov(G) \leq 3$.

**Proof.** First, if $G$ contains a bridge, then by Theorem 3.6, $cov(G) \leq 3$.

So we suppose $G$ contains no bridges and $deg(v) = 2$. Hence, $G' = G[V(G) - v]$ is connected and $|V(G')|$ is even. By Proposition 1.3.3, $G$ can be covered by an odd subgraph $G_1$ and a minimum forest $F_m$. Let $v$ be incident to $u_1$ and $u_2$ in $G$. By a similar idea as in the proof of Theorem 2.6, we can replace $v$ with $v_1, v_2$ to obtain a new forest $F_m^*$ by adding $e_i = \{v_i, u_i\}$, $i = 1, 2$, to $F_m$. If $F_m^*$ can be covered by disjoint odd forests $F_1$ and $F_2$ such that $e_1 \in F_1$ and $e_2 \in F_2$, then $G - G_1$ can be covered by 2 disjoint odd subgraphs. On the other hand, if $e_1$ and $e_2$ is a good pair in $G - G_1$, then we are done. Otherwise, if $e_1$ and $e_2$ is not a good pair in $G - G_1$, by Proposition 3.7, $F_m^* + e_3$ is a forest where $e_3$ is an edge incident to $u_1$ in $G_1$. The proof follows by covering $G - G_1 + e_3$ with two odd subgraphs. □

Note that in Lemma 3.8 each edge incident to the vertex of degree 2 belongs to exactly one odd subgraph.

Now, we are ready for the main result.

**Theorem 3.9.** Let $G$ be a connected graph such that $|V(G)|$ is odd. Then $cov(G) \leq 3$ provided $\kappa'(G) = 2$.

**Proof.** Let $\{e_1, e_2\}$ be the edge-cut of $G$. Therefore, $G - e_1 - e_2$ contains two connected components $X$ and $Y$.

For simplicity, we split the proof into three cases.

**Case 1.** $G$ contains a vertex of degree 2.

By Lemma 3.8, the proof follows.

**Case 2.** $G$ contains $e_1 = (x_1, y_1)$ and $e_2 = (x_1, y_2)$, where $x_1 \in X$ and $y_1, y_2 \in Y$.

**Subcase 2.1.** $|V(X)|$ is odd and $|V(Y)|$ is even.

By Lemma 3.8, $Y + e_1 + e_2$ can be covered by three odd subgraphs $Y_1$, $Y_2$ and $Y_3$. W.L.O.G. we let $e_1 \in Y_1$ and $e_2 \in Y_2$. Since $|V(X + e_1)|$ is even, by Proposition 1.3.3, $X + e_1$ can be covered by a spanning odd subgraph $X_3$ and a minimum forest $F_m$. By Lemma 3.2 and Proposition 3.7, either $F_m + e_2$ or $F_m + e_2 + e_4$ can be covered by two odd disjoint forests $X_1$ and $X_2$ such that $e_1 \in X_1$ and $e_2 \in X_2$, $e_4$ is an edge incident to $x_1$ in

$X_3$. Then $G$ can be covered by three odd subgraphs $G_1$, $G_2$ and $G_3$ where $G_i = X_i \bigcup Y_i$.

**Subcase 2.2.** $|V(X)|$ is even and $|V(Y)|$ is odd.

By a similar idea in Subcase 2.1. By Proposition 1.3.3, $Y + e_1 + e_2$ can be covered by three odd subgraphs $Y_1$, $Y_2$ and $Y_3$ and $e_1 \in Y_1$ and $e_2 \in Y_2$. By Lemma 3.2 and Proposition 3.7, either $F_m + e_2$ or $F_m + e_2 + e_4$ can be covered by two odd disjoint forests $X_1$ and $X_2$ such that $e_1 \in X_1$, $e_2 \in X_2$ and $e_4$ is an edge incident to $x_1$ in $X_3$. Then $G$ can be covered by three odd subgraphs $G_1$, $G_2$ and $G_3$ where $G_i = X_i \bigcup Y_i$.

**Case 3** $G$ contains $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Consider the two graphs $X + (x_1, z_1) + (x_2, z_1)$ and $Y + (y_1, z_2) + (y_2, z_2)$ where $z_i$ is new vertex not in $X$ and $Y$. By Proposition 1.3.3 and Lemma 3.8, the two graph both can be covered by at most three odd subgraphs. This implies $Y + e_1 + e_2$ can be covered by three odd subgraphs $Y_1$, $Y_2$ and $Y_3$ such that $e_1 \in Y_1$ and $e_2 \in Y_2$. Also, $X + e_1 + e_2$ can be covered by three odd subgraphs $X_1$, $X_2$ and $X_3$ such that $e_1 \in X_1$ and $e_2 \in X_2$. Then $G$ can be covered by three odd subgraphs $G_1$, $G_2$ and $G_3$ where $G_i = X_i \bigcup Y_i$. □

# 4 Conclusion

In this thesis, we manage to prove that a 3-connected graph, or a k-edge-connected graph for $k = 1, 2$, can be covered by three of its odd subgraphs. But, to solve the entire problem posed by Pyber needs more effort to finish the whole proof. We do hope that this can be done in the near future. Recently, we have received an infomation from Professor Pyber that this problem was proved by a Hungarian Tama's Mätrai several years ago. But, due to the length and complicated proof technique, his proof was not accepted by an elite journal and therefore he decided not to publish the work. We wish that our proof is clear and short enough to be checked with resonable effort.

## References

[1] P. Erdős , A. W. Goodman and L. Pósa, *The representation of graphs by set intersections*, Canad. J. Math.18(1966), 106-112.

[2] L. Pyber, *Covering the edges of a graph by ...*, Lecture notes.

[3] P. D. Seymour, *Nowhere-zero 6-flows*, J. Combin. Th.(B)30(1981), 130-135.

[4] D. B. West, *Introduction to Graph Theory*, 2nd ed. Prentice Hall(2001), Inc.

[5] D. H. Younger, *Integer flows*, J. Graph Th. 7(1983), 349-357.