國 立 交 通 大 學
應用數學系

碩 士 論 文

在一些集合系上的Fisher性質以及相交性質的探討

A Study of Fisher Type Property and Intersecting Properties over Some Set Systems

研 究 生：蔡 嘉 文

指導老師：黃 大 原　教 授

中 華 民 國 九 十 三 年 六 月

在一些集合系上的Fisher性質
以及相交性質的探討

A Study of Fisher Type Property and Intersecting Properties over Some Set Systems

研　究　生：蔡嘉文　　**Student：Chia-Wen Tsai**

指導老師：黃大原 教授　**Advisor：Dr. Tayuan Huang**

國　立　交　通　大　學

應用數學系

碩　士　論　文

A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
In partial Fulfillment of Requirement
For the Degree of Master
In
Applied Mathematics
June 2004
Hsinchu, Taiwan, Republic of China

中　華　民　國　九　十　三　年　六　月

在一些集合系上的Fisher性質
以及相交性質的探討

研 究 生：蔡嘉文　　　指導老師：黃大原　教授

國 立 交 通 大 學

應 用 數 學 系

摘　要

本論文首先針對在各種區組設計上的 Fisher 不等式進行研究，如 t−區組設計、可分解設計、以及 2*s*−區組設計，這些不等式利用關聯矩陣為出發點，以線性代數為工具予以證明。其次討論一些區組設計，如 2*s*−區組設計，3-(10, 4, 1)設計，以及 3-(2*k*, 4, 1)設計。我們也以區組設計的性質、強正則圖、以及正交陣列為例，來說明如何利用二次函數的技巧來推導出一些組合性質。

中 華 民 國 九 十 三 年 六 月

A Study of Fisher Type Property and Intersecting Properties over Some Set Systems

Student : Chia-Wen Tsai   Advisor : Dr. Tayuan Huang

Department of Applied Mathematics
National Chiao Tung University
Hsinchu 300, Taiwan, R.O.C.

Abstract

Fisher's type inequalities over various designs such as t-designs, resolvable designs, and $2s$-designs are studied in this thesis. The unified technique in terms of their incidence matrices, within the framework of linear algebra is emphasized. Moreover, the intersections among pairs of blocks of some designs such as $2s$-design, 3-(10, 4, 1) design and 3-($2k$, 4, 1) designs, are also studied, the technique of using quadratic functions is also emphasized.

A Study of Fisher Type Property and Intersecting Properties over Some Set Systems

Student: Chia-Wen Tsai Advisor: Tayuan Huang

Department of Applied Mathematics
National Chiao Tung University
Hsinchu 30050 Taiwan R.O.C.

June 17, 2004

Contents

1 Introduction

Let X be a finite set, and $\Phi \subseteq 2^X$, then (X, Φ) is called a *set system*. The notion of set systems usually provides a convenient model for combinatorial structures. Each set system can be expressed in terms of *its incidence matrix* M. More precisely, an incidence matrix M of a set system $\Pi = (X, \Phi)$ is row-indexed by elements of X and column-indexed by members of Φ respectively such that the entry of M along the x row and Fth column is 1 if and only if $x \in F$ and 0 otherwise. Many results and techniques in linear algebra can be used to derive some information for combinatorial structure under consideration through its incidence matrix.

In Chapter 2, we introduce some kinds of set systems. Strongly regular graph is a set system with $X = V(\Gamma)$, $\Phi = \{\Gamma_1(u) : \text{for all } u \in V(\Gamma)\}$ and some properties from adjacency of vertices. The family of t-designs forms another family of set system. Symmetric designs and quasi-symmetric designs are two classes of 2-designs whose incidence matrices can be associated with strongly regular graphs. Fisher's inequality over various designs, such as t-designs, resolvable designs and $2s$-designs, will be presented in Section 3.1 and proved by analyzing the incidence matrices. Besides, Fisher's inequality also can be stated in terms of the terminology of graphs. In Section 3.2, the intersections among pairs of blocks of some designs including $2s$-design, 3-(10, 4, 1) design and 3-($2k$, 4, 1) designs are discussed, and graphs which has friendship property are introduced. The technique using *quadratic functions* is introduced in Section 3.3, its applications will be illustrated in terms of a few examples including some properties of designs, strongly regular graphs and orthogonal arrays respectively.

Quasi-symmetric designs are those 2-designs with exactly two different intersection numbers among each pair of their blocks, and hence their block graphs are strongly regular. Since the examples of quasi-symmetric designs which are not symmetric designs, affine designs or linear spaces are rather rare. An approach in the study of quasi-symmetric designs with some disjoint blocks is to consider their substructures and induced substructures. In Section 4.1, we consider those quasi-symmetric designs having good blocks.

2 Preliminaries

2.1 Set Systems and Their Incidence Matrices

An incidence structure $\Pi = (X, \mathcal{B})$ consists of a finite set X, together with a collection $\mathcal{B}$ of subsets of X. The elements in X are called points, and the subsets in $\mathcal{B}$ are called blocks, we call a point x lies in a block B if $x \in B$. An incidence structure $\Pi = (X, \mathcal{B})$ is usually denoted by its incidence matrix M in the following sense that M is row-indexed (respective column-indexed) by points (respective by blocks) such that the entry $M(x, B)$ of M at the row indexed by the points x and the column indexed by the block B is 1 if $x \in B$ and 0 otherwise.

A residue structure associated with a point a is defined as follows. Let $N(a) = \{a\} \cup \{x \in X | x \text{ is collinear with } a\}$. If B is a block not containing a, the residue B^a of B with respect to a is defined to be $B - N(a)$. Let $\mathcal{B}^a = \{B^a | B \in \mathcal{B} \text{ with } a \in B\}$. Then the residue of $\Pi = (X, \mathcal{B})$ with respect to a is defined to be $\Pi^a = (X - N(a), \mathcal{B}^a)$.

Definition 1. *Let Γ be a k-regular graph of order v with the following properties that each pair of adjacent vertices has λ common neighbors and each pair of nonadjacent vertices has μ common neighbors. Then Γ is said to be a strongly regular graph with parameters (v, k, λ, μ).*

2.2 Combinatorial t-designs

Definition 2. *Let v, k and λ be positive integers such that $v > k \geq 2$. A (v, k, λ)-balanced incomplete block design (which we abbreviate to (v, k, λ)-BIBD), which is also called a 2-(v, k, λ) design, is a pair $(X, \mathcal{B})$ such that the following properties are satisfied:*

1. *X is a set of v elements called points,*
2. *$\mathcal{B}$ is a collection of subsets of X called blocks,*
3. *each block contains exactly k points, and*
4. *every pair of distinct points is contained in exactly λ blocks.*

Let $X = \{x_1, x_2, ..., x_v\}$ and $\mathcal{B} = \{B_1, B_2, ..., B_b\}$. The incidence matrix of $(X, \mathcal{B})$ is the (0, 1)-matrix $M = (m_{ij})$ of order $v \times b$ defined by the rule $m_{ij} = \begin{cases} 1 & \text{if } x_i \in B_j \\ 0 & \text{if } x_i \notin B_j \end{cases}$

A 2-(v, k, λ) design $\Pi = (X, \mathcal{B})$ with $b = v$ is called a *symmetric design*. It is known that the followings are equivalent.

1. $|B_i \cap B_j| = \lambda$ for any pair B_i, $B_j \in \mathcal{B}$,

2. Π is a symmetric design.

A *quasi-symmetric* design is a 2-design with the number of points in the intersection of two blocks takes only two values. Let $x < y$ be the two cardinalities of block intersection in the quasi-symmetric design. The *block graph* of a quasi-symmetric design has as vertices the blocks, two vertices adjacent if they intersect y points.

Definition 3. *A t-design with parameters (v, k, λ) (or a t-(v, k, λ) design) is a pair $\Pi = (X, \mathcal{B})$, where X is a set of 'points' of cardinality v, and $\mathcal{B}$ a collection of k-element subsets of X called 'blocks', with the property that any t points are contained in precisely λ blocks.*

3 Some Techniques in terms of Matrices and Discriminants

Some combinatorial informations can be derived from matrices and quadratic functions as well. For example, the fact that the rank of a matrix is always less than or equal to its numbers of rows and of columns leads to an inequality. Among many others, Fisher inequality is such an example. In this section, we will explore such a technique for Fisher inequalities for *BIBD* and their variations.

Moreover, a quadratic function can also be associated with some combinatorial problems, and its discriminant usually provides another inequality with combinatorial interests. We will explain this technique in Section 3.2 in terms of a few typical examples.

3.1 Fisher Inequalities over Various Designs

Some combinatorial informations can be derived from matrices and quadratic functions as well. For example the fact that the rank of a matrix is always less than or equal to its numbers of rows and of columns leads to an inequality. Among many others, Fisher inequality is such an example. In this section, we will explore such a technique for Fisher inequalities for *BIBD* and their variations.

Theorem 3.1.1 (Fisher's Inequality). *For a t-(v,k,λ) design $\Pi = (X, \mathcal{B})$, then*

1. *$b \geq v$ if $t = 2$;*
2. *$b \geq v + r - 1$ if $t = 2$, and it is resolvable with r parallel classes; the equality holds with $r = k + \lambda$; any two blocks from different classes meet at $\frac{k^2}{v}$ points.*
3. *$b \geq \binom{v}{s}$ if $t = 2s$, and $k \leq v - s$.*

Proof. Let $\Pi = (X, \mathcal{B})$ be a 2-(v,k,λ) design, where $X = \{x_1, \ldots, x_v\}$ and $B = \{B_1, \ldots, B_b\}$. Let M be the incidence matrix of this 2-design, and define s_j to be the jth column of M, note that $s_1, \ldots, s_b$ all are v-dimensional vectors in the real vector space $\mathbb{R}^v$. Let

$$\mathbf{S} = \{\sum_{j=1}^{b} \alpha_j s_j : \alpha_1, \ldots, \alpha_b \in \mathbb{R}\}$$

then $\mathbf{S}$ is the subspace of $\mathbb{R}^v$ spanned by $S = \{s_j : 1 \leq j \leq b\}$, it consists of all linear combinations of the vectors $s_1, \ldots, s_b$, it suffices to show that $e_i \in S$ for $1 \leq i \leq v$. First, we observe that $\sum_{j=1}^{b} s_j = (r, \ldots, r)^T$ and then

$$\sum_{j=1}^{b} \frac{1}{r} s_j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Next, fix a value i, $1 \leq i \leq v$. Then we have

$$\sum_{\{j:x_i \in B_j\}} s_j = (r-\lambda)e_i + \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix}$$

Combine them to obtain

$$e_i = \sum_{\{j:x_i \in B_j\}} \frac{1}{r-\lambda} s_j - \sum_{j=1}^{b} \frac{\lambda}{r(r-\lambda)} s_j,$$

a linear combination of $s_1, \ldots, s_b$, as required.

To prove 2, let $m_i = \frac{(i-1)v}{k} + 1$ and $n_i = \frac{iv}{k}$ for $1 \leq i \leq r$. Suppose that the blocks are labelled with the r parallel classes $\Pi_i = \{B_j : m_i \leq j \leq n_i\}$, $1 \leq i \leq r$, so we have that $\sum_{j=m_i}^{n_i} s_j = (1, \ldots, 1)^T$, for $1 \leq i \leq r$, and $s_{m_i} = \sum_{j=m_1}^{n_1} s_j - \sum_{j=m_i+1}^{n_i} s_j$ for $2 \leq i \leq r$. In other words, the $r-1$ vectors in the set $S' = \{s_{m_2}, \ldots, s_{m_r}\}$ can be expressed as linear combinations of the $b-r+1$ vectors in $S - S'$. Now, since the b vectors in S span $\mathbb{R}^v$, it follows that the $b-r+1$ vectors in $S-S'$ span $\mathbb{R}^v$. Since $\mathbb{R}^v$ has dimension v and is spanned by a set of $b-r+1$ vectors, it must be the case that $b \geq v + r - 1$.

To prove 3, we use a modified incidence matrix M_s, whose columns are indexed by blocks and rows by s-sets of points, with (S, B) entry 1 if $S \subset B$, 0 otherwise, then M_s is a $\binom{v}{s} \times b$ matrix, and it suffices to show that the columns of M_s span $\mathbb{R}^{\binom{v}{s}}$. Accordingly, let c_B be the column of M_s with label B, and let e_s be the vector with 1 in the position labelled S and 0 in all other positions (the unit basis vector corresponding to S). For $x_j = \sum_{|S' \cap S| = j} e_{S'}$, $0 \leq j \leq k$, and $y_i = \sum_{|B \cap S| = i} r_B$, $0 \leq i \leq s$. Because

$$y_i = \sum_{|B \cap S| = i} r_B = \sum_{j=0}^{i} \sum_{|S' \cap S| = j} \sum_{\substack{B \supseteq S' \\ |B \cap S| = i}} e_{S'} = \sum_{j=0}^{i} \binom{s-j}{i-j} v_{2s-j, s+i-j} (\sum_{|S \cap S'| = j} e_{S'})$$

where $v_{p,q}$ denotes the number of blocks intersecting a given p-set P in a given q-subset Q. Then we have a system of $s+1$ linear equations for the x_j in terms of the y_i. The coefficient matrix is triangular, and its diagonal entries $v_{2s-i,s}$ are non-zero (since $s \leq k \leq v - s$). So the equations have a unique solution. In particular, $x_s = e_S$ is a linear combination of the y_i, and so is in the row space of M_s. $\square$

Fisher Inequality can also be stated in terms of the terminology of graphs.

Theorem 3.1.2. *Let $G_1, \ldots, G_b$ be (not necessarily distinct) complete subgraphs of K_v, $v \geq 2$, each of order at most $n-1$ such that every edge of K_v belongs to the same number $\lambda \geq 1$ of G_is. Then $b \geq v$.*

Proof. Let $V(K_v) = \{a_1, \ldots, a_n\}$, and let M be the $v \times b$ incidence matrix of the cover $K_v = \cup_{i=1}^{b} G_i$, i.e., $M_{ij} = 1$ if $a_i \in G_j$; and $M_{ij} = 0$ otherwise. Then

$$MM^T = \begin{bmatrix} m_1 & \lambda & \cdots & \lambda \\ \lambda & m_2 & \lambda & \vdots \\ \vdots & \lambda & \ddots & \lambda \\ \lambda & \cdots & \lambda & m_v \end{bmatrix},$$

where m_i is the number of G_j, $1 \le j \le b$, containing a_i. Since each edge of K_v belongs to λ of G_is, each point of K_v belongs to at least λ of G_is. If there exists a point a_k of K_v belongs to exactly λ of G_is, then $\{a_1, \ldots, a_n\} - \{a_k\}$ belong to these λ of G_is. (Because each of the edges that connects a_k and $\{a_1, \ldots, a_n\} - \{a_k\}$ belongs to λ of G_is) thus, each of these λ of G_is has order v, a contradiction. So every point belongs to more than λ of G_is.

Since every point of K_v belongs to more than λ of G_is, let $m_i = \lambda + k_i$ with $k_i \ge 1$ for $1 \le i \le v$. Then

$$det(MM^T) = \begin{vmatrix} \lambda+k_1 & \lambda & \cdots & \lambda \\ \lambda & \lambda+k_2 & \lambda & \vdots \\ \vdots & \lambda & \ddots & \lambda \\ \lambda & \cdots & \lambda & \lambda+k_v \end{vmatrix} = \begin{vmatrix} \lambda+k_1 & \lambda & \ldots & \lambda \\ -k_1 & k_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ -k_1 & 0 & \ldots & k_v \end{vmatrix}$$

$$= \begin{vmatrix} \lambda+k_1+\lambda\frac{k_1}{k_2}+\ldots+\lambda\frac{k_1}{k_n} & \lambda & \ldots & \lambda \\ 0 & k_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & k_v \end{vmatrix} = (1+\lambda\sum_{i=1}^{v}\frac{1}{k_i})\prod_{i=1}^{v} k_i$$

is non-zero. Thus, $rank(MM^T) = v \le rank(M) \le \min\{v, b\}$, and $b \ge v$ required. $\square$

3.2 The Intersections among Pairs of Blocks of Some Designs

Theorem 3.2.1. *Suppose that $\Pi = (X, \mathcal{B})$ is a 2-(v, k, λ) design with its incidence matrix M, then the following are equivalent*

1. $b = v$,
2. $r = k$, *and*
3. $M^T M = (k-\beta)I + \beta J$ *for some constant* β.

Proof. For a 2-(v, k, λ) design $\Pi = (X, \mathcal{B})$, $bk = vr$ and hence the first two are equivalent.

The matrix M^TM is a $b \times b$ matrix. If $k \neq \beta$ then $det(M^TM) \neq 0$, so $rank(M^TM) = b \leq rank(M) \leq min\{v, b\} \leq v$, hence $b = v$ because of the Fisher inequality $b \geq v$. This shows that 3. $\Rightarrow$ 1.

Let $B = \{B_1, \ldots, B_b\}$ and define s_j to be the jth column of M, note $s_1, \ldots\ s_b$ are all v-dimensional vectors in the real vector space $\mathbb{R}^v$. Fix h, $1 \leq h \leq b$, then

$$\sum_{\{i:x_i\in B_h\}} \sum_{\{j:x_j\in B_j\}} s_j = \sum_{\{i:x_i\in B_h\}} ((r-\lambda)e_i + \begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix})$$
$$= (r-\lambda)s_h + k\begin{pmatrix} \lambda \\ \vdots \\ \lambda \end{pmatrix} = (r-\lambda)s_h + \sum_{j=1}^{b} \frac{\lambda k}{r} s_j,$$

and $\sum_{\{i:x_i\in B_h\}} \sum_{\{j:x_j\in B_j\}} s_j = \sum_{j=1}^{b} |B_h \cap B_j| s_j$. Since $b = v$, $B = \{B_1, \ldots, B_b\}$ is a basis of $\mathbb{R}^v$, the left coefficients of s_j and the right coefficients of s_j must equal, it follows that $|B_h \cap B_j| = \lambda$ whenever $h \neq j$, $M^TM = (k-\lambda)I + \lambda J$ as required, this shows that 1. $\Rightarrow$ 3.. $\square$

Theorem 3.2.2. *Let $\mathcal{B}$ be a collection of k-subsets of a v-set X, where $2s \leq k \leq v - s$. Then any two of the following conditions imply the third one:*

1. *$(X, \mathcal{B})$ is a $2s$-design;*
2. *the cardinality of the intersection of two distinct blocks in $\mathcal{B}$ takes just s distinct values;*
3. *$|B| = \binom{v}{s}$.*

Proof. $1, 2 \Rightarrow 3$: $(X, \mathcal{B})$ is a $2s$-design, and $2s \leq k \leq v - s$. Then $b \geq \binom{v}{s}$ and the cardinality of the intersection of two distinct blocks in $\mathcal{B}$ takes just s distinct values. Now, we consider its incidence matrix M_s as above. Then

$$M_s^T M_s = \begin{bmatrix} \binom{k}{s} & 1 & \cdots & 1 \\ 1 & \binom{k}{s} & 1 & \vdots \\ \vdots & 1 & \ddots & 1 \\ 1 & \cdots & 1 & \binom{k}{s} \end{bmatrix}_{b\times b}.$$

Because $\binom{k}{s} > 1$, $det(M_s^T M_s) > 0$. Thus, $rank(M_s^T M_s) = b$. Then $min\{b, \binom{v}{s}\} = rank(M_s) \geq rank(M_s^T M_s) = b$, i.e., $b \leq \binom{v}{s} = |B|$ $\square$

An example of a 3-(10, 4, 1) design is given below:

$$\begin{array}{cccccc}
\{1,2,4,5\} & \{1,2,3,7\} & \{1,3,5,8\} & \{2,3,5,6\} & \{2,3,4,8\} & \{2,4,6,9\} \\
\{3,4,6,7\} & \{3,4,5,9\} & \{3,5,7,0\} & \{4,5,7,8\} & \{4,5,6,0\} & \{1,4,6,8\} \\
\{5,6,8,9\} & \{1,5,6,7\} & \{2,5,7,9\} & \{6,7,9,0\} & \{2,6,7,8\} & \{3,6,8,0\} \\
\{1,7,8,0\} & \{3,7,8,9\} & \{1,4,7,9\} & \{1,2,8,9\} & \{4,8,9,0\} & \{2,5,8,0\} \\
\{2,3,9,0\} & \{1,5,9,0\} & \{1,3,6,9\} & \{1,3,4,0\} & \{1,2,6,0\} & \{2,4,7,0\}
\end{array}$$

If $Y = \{1,2,3,5,0\}$, then Y does not contain any block, and similarly, Y^C does not contain any block either. The following theorem shows that can be generalized to any 3-(10, 4, 1) design. We then further show that similar condition holds in general cases.

Theorem 3.2.3. *If Y is a set of 5 points of a 3-(10, 4, 1) design containing no block, then the complement of Y contains no block either.*

Proof. For a 3-(10, 4, 1) design, it has 30 blocks, and $n_1 = 12, n_2 = 4, n_3 = 1$, where n_i is the number of blocks containing a fixed set of i points.

Suppose there exists a set Y of 5 points containing no blocks, but $\overline{Y}$ containing a block, say B_1, without loss of generality, we may assume that $Y = \{1,2,3,4,5\}$, and the transpose of the first column of its incidence matrix is $(0,0,0,0,0,1,1,1,1,0)$, then $Y \cap B_1 = \varnothing$. Let $D = \{B_i : Y \cap B_i \neq \emptyset, i = 1,2,\ldots,30\}$, then $|D| \leq 29$ because $Y \cap B_1 = \varnothing$. On the other hand, the principal of inclusion and exclusion shows that

$$|D| = \binom{5}{1} \times 12 - \binom{5}{2} \times 4 + \binom{5}{1} \times 1 = 30,$$

contradicts the fact that $|D| \leq 29$, as required. □

Hanani (1960) proved that a necessary and sufficient condition for the existence of a 3-(v, 4, 1) design is that $v \equiv 2$ or 4 (mod 6).

Lemma 3.2.4. *If there exists a 3-(v, 4, 1) design, then $v \equiv 2$ or 4 (mod 6).*

Proof. Each 3-(v, 4, 1) design is also a 2-(v, 4, $\frac{v-2}{2}$) design with $\frac{\binom{v}{3}}{4}$ blocks. The integrality of $\frac{v-2}{2}$ shows that v is even. Moreover, as a 2-design, we have $\frac{\binom{4}{3}}{4} \times 4 = vr$, and then $r = \frac{(v-1)(v-2)}{6}$. Since $gcd(v-1, v-2) = 1$, if $2|(v-1)$ and $3|(v-2)$, then $v = 2m+1$ and $v = 3l+2$ for some m and l, and hence, $v = 6k+5$; similarly if $2|(v-2)$, $v = 6k+4$; similarly if $6|(v-1)$ (or $6|(v-2)$), then $v = 6k+1$ (or $v = 6k+2$ respectively). However, v is odd contradicting the fact v is even. It follows that $v \equiv 2$ or 4(mod 6). □

Similar arguments 3-(v, 4, 1) work well for designs with even number of vertices.

Theorem 3.2.5. *If Y is a set of k points of a 3-(2k, 4, 1) design containing no block, then the complement of Y contains no block either.*

Proof. For a 3-$(2k$, 4, 1) design, it has $b = \frac{2k(2k-1)(2k-2)}{24}$ blocks, and $n_1 = \frac{(2k-1)(2k-2)}{6}$, $n_2 = k-1$, $n_3 = 1$, where n_i is the number of blocks containing a fixed set of i points.

Suppose there exists a set Y of k points containing no blocks, but $\overline{Y}$ containing a block, say B_1, without loss of generality, we may assume that $Y = \{1, 2, \ldots, k\}$, and the transpose of the first column of its incidence matrix is {0, 0, 0, 0, . . . , 0, 1, 1, 1, 1, 0, . . . , 0}, then $Y \cap B_1 = \varnothing$. Let $D = \{B_i : Y \cap B_i \neq \varnothing, i = 1, 2, \ldots, b\}$, then $|D| \leq b-1$ because $Y \cap B_1 = \varnothing$. On the other hand, the principal of inclusion and exclusion shows that $|D| = k \times \frac{(2k-1)(2k-2)}{6} - \binom{k}{2} \times (k-1) + \binom{k}{3} \times 1 = \frac{k(k-1)(2k-1)}{6} = b$, contradicts the fact that $|D| \leq b-1$, as required. □

This following result, known as the Friendship Theorem, is due to Erdös, Rényi and Sós(1966), as referenced by van Lint [9, pp.45].

Theorem 3.2.6. *Let Γ be a graph in which any two vertices have a unique common neighbor, then either Γ is a windmill, or Γ is regular (and hence strongly regular).*

A similar result works for a family of subtrees of a tree.

Theorem 3.2.7. *Let $T_1, \ldots, T_k$ be subtrees of a tree T such that the trees T_i and T_j have a vertex in common for all $1 \leq i < j \leq k$. Then T has a vertex that is in all the T_i.*

1st proof. We first claim that $T_i \cap T_j$ is a subtree of T for any $i \neq j$. Suppose, to the contradictory, that there exist $p, q \in T_i \cap T_j$ such that p and q are not connected in $T_i \cap T_j$. There exist $t_i \in T_i$ and $t_j \in T_j$ such that p, t_i, q, t_j, p are on a cycle, a contradiction.

Without loss of generality, suppose $|T_1 \cap T_2|$ is the maximal intersection number For trees T_1 and T_2, we show that $T_i \cap (T_1 \cap T_2)$ is nonempty for each i. Let $T_1 \cap T_2$ be a tree T'. Suppose to the contradictory, $T_i \cap T'$ is empty. Let $x \in (T_1 \cap T_i - T_2)$, and $y \in (T_2 \cap T_i) - T_1$. Thus, there exists $t' \in T'$ such that x, t' and y are on a cycle, a contradiction.

We further show that $(T_i \cap T') \cap (T_j \cap T')$ is nonempty for distinct i and j. Suppose to the contradictory that it is empty. Let $x \in (T_i \cap T_j) - T'$, then

1. x, t_i, t'_i, t_1, t'_j are on a cycle where $t_i \in T_i$, $t'_i \in T_i \cap T'$, and $t_1 \in T_i$ whenever $x \notin T_1 \cup T_2$, or

2. x, t_i, t'_i, t_1, t'_j are on a cycle where $t_i \in T_i$, $t'_i \in T_i \cap T'$, $t_1 \in T_1$ (resp. T_2) and $t'_j \in T_j \cap T'$ whenever $x \in T_1$ (resp. T_2).

Therefore $T_i \cap T'$ is a subtree of T', $i = 3, 4, \ldots, k$, and $(T_i \cap T') \cap (T_j \cap T') \neq \varnothing$, the condition is the same as original assumption, by this way, we finally obtain that $T_1 \cap T_2 \cap, \ldots, \cap T_k$ is a subtree of T. □

2nd proof. If there exists a subtree T_k with $|V(T_k)| = 1$, then it is trivial. Assume there is no subtree with only one vertex. We prove it by introduction on $|V(T)| = n$. Since the case $n = 2$ is trivial, we suppose that T has a vertex that in all the T_i, i.e., $\cap_{i=1}^{k} T_i \neq \varnothing$, for $n \leq m$. When $n = m + 1$, we choose a leaf x of T. If $x \in \cap_{i=1}^{k} T_i$, then $\cap_{i=1}^{k} T_i \neq \varnothing$. If $x \notin \cap_{i=1}^{k} T_i$. then deleting x from the tree T, and we obtain a tree called T' such that $|V(T')| = m$. Let $T'_i = T_i - \{x\}$. $T'_1, T'_2, \ldots, T'_k$ are subtrees of T' so that T'_i and T'_j have a vertex in common for all $1 \leq i < j \leq k$. then $\cap_{i=1}^{k} T'_i \neq \varnothing$. So $\cap_{i=1}^{k} T_i \neq \varnothing$. By induction, T has a vertex in all the T_i. □

3.3 Combinatorial Information derived from Quadratic functions

Quadratic functions sometimes can be associated with some combinatorial problems, and their discriminants usually provide some inequalities with combinatorial interests. This technique will be illustrated in section in terms of a few typical examples.

Theorem 3.3.1 ([9],pp.). *Each block of a 2-(v, k, λ)design meets nontrivially at least*

$$\frac{k(r-1)^2}{(k-1)(\lambda-1)+(r-1)}$$

other blocks. The sufficient and necessary conditions for the equal sign holds is that any two blocks intersect in precisely $\dfrac{(k-1)(\lambda-1)+(r-1)}{r-1}$ *points, and hence it is a symmetric design.*

Proof. Let B be a block of $\mathcal{B}$. Let $a = |\{B_i : B_i \neq B \text{ and } |B_i \cap B| \neq 0\}|$, and n_i be the number of blocks meet B in precisely i points. Therefore $\sum_{i=1}^{k} n_i = a$, $\sum_{i=1}^{k} i n_i = kr - 1$, and hence $\sum_{i=1}^{k} i(i-1)n_i = k(k-1)(\lambda - 2)$ and

$$\sum (i - x)^2 n_i = ax^2 - 2k(r-1)x + k((k-1)(\lambda-1) + (r-1)) \geq 0.$$

So its discriminant

$$D = (-2k(r-1))^2 - 4ak((k-1)(\lambda-1) + (r-1))$$

is negative, and hence $a > \frac{k(r-1)^2}{(k-1)(\lambda-1)+(r-1)}$ as required. Further more, the equal sign holds if and only if the quadratic equation

$$\sum_{i=1}^{k}(i-x)^2 n_i = ax^2 - 2k(r-1)x + k((k-1)(\lambda-1)+(r-1)) = 0$$

has a unique solution $x = \frac{2k(r-1)}{2a} = \frac{(k-1)(\lambda-1)+(r-1)}{r-1}$, i.e., $n_i > 0$ if and only if $x = i$. $\square$

An *n-arc* of a symmetric 2-(v,k,λ)design $\Pi = (X, \mathcal{B})$ is a set of n points such that no three of which are contained in a block. A block B of Π is called a *tangent* to an n-arc S if $|S \cap B| = 1$. An n-arc A is called an *oval of type I* if each point of A lies on a unique tangent, and an *oval of type II* if it has no tangents.

Theorem 3.3.2 ([9],pp.). *Let A be a type I oval in a symmetric 2-(v,k,λ) design with $k-\lambda$ even. Then any point of the design lies on either one or all tangents to A.*

Proof. Observe that k, λ and n are all odd, and so any point lies on at least one tangent to A. We apply a different version of the 'variance trick'. Let n_i be the number of points which lie on i tangents. Then $\sum n_i = v$, $\sum in_i = nk$, and $\sum i(i-1)n_i = n(n-1)\lambda$. Therefore $\sum(i-1)(i-n)n_i = 0$, whence every point lies on one or all the tangents. $\square$

Theorem 3.3.3 ([15],pp.232). *Let Γ be a strongly regular graph with parameters (v,k,λ,μ), connected complement and Spec$(\Gamma) = (k^1, \theta^{m_\theta}, \tau^{m_\tau})$. If $k < m_\theta$, then*

$$(m_\theta - 1)(k\lambda - \lambda^2 - (k-m_\theta)\tau^2) - (\lambda + (k-m_\theta)\tau)^2 > 0.$$

Proof. Consider the quadratic polynomial $p(x)$

$$\begin{aligned} p(x) &= (m_\theta - 1)(ka - \lambda^2 - k - m_\theta x^2) - (\lambda + (k-m_\theta)x^2) \\ &= (m_\theta - 1)k\lambda - m_\theta\lambda^2 + 2\lambda(m_\theta - k)x + (k-1)(m_\theta - k)x^2, \end{aligned}$$

we find that its discriminant is $-4a(m_\theta - k)(m_\theta - 1)k(k-1-\lambda)$. Since $k < m$ and $1 < m$, we see that this is negative unless $\lambda = 0$. If $\lambda = 0$, then $p(x) = (k-1)(m_\theta - k)x^2$, and consequently $p(\tau) \neq 0$, unless $\tau = 0$.

If $\lambda = 0$ and $\tau = 0$, then Γ is the complete bipartite graph $K_{k,k}$ with eigenvalues k, 0, and $-k$. However, if $\tau = 0$, then $\theta = -k$ and $m = 1$, which contradicts the condition that $k < m_\theta$. $\square$

Orthogonal arrays provide a convenient method of obtaining a sequence of pseudo-random sample points. Suppose that A is an orthogonal array $OA(k,n)$ on the set X with $|X| = n$. A *Two-point sampling* is accomplished as follows: let r be a random row in A, use the k values $A(r,1),\ldots,A(r,k)$ along the rth row of A as the k sample points, note that these k samples points are not necessarily all distinct. If the rows of A are indexed by $X \times X$, then a random row of A is specified by choosing two points independently at random from X.

An elementary combinatorial analysis of the two-point sampling technique which allows us to calculate a bound on the error probability is presented below. Suppose that I is a yes-instance, and define the set S of *witnesses* (note that we do not know the set S explicitly).

$$S = \{x \in U : f(I,x) = 1\},$$

We have $|S| = m = (1-\epsilon)n$.

Theorem 3.3.4 ([35],pp.96-99). $err(S) \leq 1 - \dfrac{km}{n} + m(k-1)$

Let a_i denote the number of rows of A in which there are exactly I occurrences of elements from S. Call a row of the matrix a *bad row* if none of the elements in the row is a witness. Then the error probability is simply the probability that the randomly selected row is a bad row. Hence, the error probability, when we run the algorithm A using k ample points chosen from a random row of A, is

$$err(S) = \frac{a_0}{n^2} \tag{1}$$

As mentioned above, we do not know the set S explicitly. However, an upper bound on the error probability can be obtained by computing $err = max\{err(A) : A \in \binom{X}{m}\}$. Since an $OA(k,n)$ has n^2 rows,

$$\sum_{i=0}^{n} a_i = n^2. \tag{2}$$

Counting the number of occurrences of witnesses in A in two ways, there are exactly a_i rows in which there are i occurrences of witnesses, in any column of A. It follows that each point occurs exactly n times.

$$\sum_{i=0}^{n} ia_i = knm. \tag{3}$$

Similarly, counting the number of occurrences of pairs of witnesses occurring in the same row in two ways, it yields:

$$\sum_{i=0}^{n} i(i-1)a_i = k(k-1)m^2. \tag{4}$$

For a real number x, we have

$$0 \leq \sum_{i=1}^{n}(i-x)^2 = \sum_{i=1}^{n}(i^2 - 2xi + z^2)a_i = \sum_{i=1}^{n} i^2 a_i - 2x\sum_{i=1}^{n} i a_i + x^2 \sum_{i=1}^{n} a_i$$
$$= k(k-1)m^2 + knm - 2xknm + x^2 \sum_{i=1}^{n} a_i.$$

It follows from equations (2), (3) and (4), that

$$\sum_{i=1}^{n} a_i \geq \frac{2knmx - knm - k(k-1)m^2}{x^2} \tag{5}$$

The right hand side of (5) is maximized when $x = \frac{n+(k-1)m}{n}$. Hence, we get

$$\sum_{i=1}^{n} a_i \geq \frac{kmn^2}{n+(k-1)m}. \tag{6}$$

Now, from (2) and (6), we have

$$a_0 \leq n^2 - \frac{knm^2}{n+(k-1)m}.$$

Then, we get the following bound on the error probability from (1):

$$err(S) \leq 1 - \frac{km}{n+(k-1)m}.$$

Note that this bound on the error probability approaches 0 only linearly quickly as a function of k. A small example which meets the bound is given by the following $OA(3,4)$:

0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
2	3	0	1	3	2	1	0	0	1	2	3	1	0	3	2

4 Quasi-symmetric designs and Strongly Regular designs

We are interested in 2-(v, k, λ) designs with constraints over intersections between their blocks. Those of symmetric designs, quasi-symmetric designs, semi-symmetric design or even quasi-semi-symmetric designs. Based on the paper [29], the class of quasi-symmetric designs with specific blocks will be surveyed in Section 4-1. Based on the papers [14] by T. S. Fu and Y. Huang, the class of quasi-semi-symmetric designs will be surveyed in Section 4-2.

4.1 Quasi-symmetric designs with Good Blocks

The family of quasi-symmetric designs form a broad class of 2-designs containing all affine and symmetric designs and linear spaces. Quasi-symmetric designs not in these subclasses are sparse. One motive for investigating quasi-symmetric designs with prescribed geometric conditions, such as the existence of subdesigns, is that perhaps a new construction might become apparent. The structure of quasi-symmetric designs which have special blocks on which are induced quasi-symmetric designs will be surveyed in this section, we show that-with the exception of linear, affine, and projective spaceswthere are only two possible parameter sets for such designs. Only one example is known of a design of one parameter set type and none of the other. In this section, quasi-symmetric designs with $(x, y) = (0,\ y)$ or $(1,\ y)$ are studied.

Definition 4. *Let x and y be non-negative integers with $x \leq y$. 2-design is quasi-symmetric with intersection numbers x and y if $|B \cap C| \in \{x, y\}$ for any two distinct blocks B and C and both intersection numbers are realized. The design is proper if $x \neq y$; otherwise it is improper.*

It is well-known that a 2-(v, k, λ) design with $\lambda > 1$ is symmetric if and only if any two distinct blocks meet in λ points. A design is *resolvable* if its blocks can be partitioned into subsets (parallel classes) each of which partitions the whole point set. The partition of blocks is called a *parallelism*. Blocks in the same parallel class are parallel. Two distinct parallel blocks are disjoint. If, further the number of points common to any two nonparallel blocks is constant, the design is said to be *affine*. Clearly, affine designs are examples of quasi-symmetric designs with 0 as an intersection number. Linear spaces, i.e., 2-$(v, k, 1)$ designs, are also examples.

Theorem 4.1.1 ([29]). *For any quasi-symmetric design with intersection numbers x and y:*

1. *$y - x$ divides both $k - x$ and $r - \lambda$ if the design is proper;*

2. $(b-1)xy + k(k-1)(\lambda-1) = k(r-1)(x+y-1)$; *and*

3. $(k-1)(\lambda-1) = (r-1)(y-1)$ *whenever* $x = 0$.

Similar to affine planes and to symmetric designs, the notion of good blocks for quasi-symmetric designs is introduced.

Lemma 4.1.2 ([29]). *Let* $\Pi = (X, \mathcal{B})$ *be a quasi-symmetric 2-*(v, k, λ) *design with intersection number* x *and* y*, where* $x < y$*. Let* B *and* C *be blocks meeting in* y *points.*

1. *The intersection of any two distinct blocks containing* $B \cap C$ *is* $B \cap C$*.*

2. *Any point not in* $B \cap C$ *is on at most one block containing* $B \cap C$*.*

3. *There are at most* $\frac{v-y}{k-y}$ *blocks containing* $B \cap C$*. Equality holds if and only if any point not in* $B \cap C$ *is on a unique block containing* $B \cap C$*.*

Proof. 1 follows since $y \geq x$ and $|B \cap C| = y$. Since two distinct blocks cannot have $y+1$ or more common points, then 2. follows. Finally, 3. is an easy deduction from 1. and 2. □

Definition 5. *Let* Π *be a quasi-symmetric 2-*(v, k, λ) *design, a block* B *is good if there is a (necessarily unique) block containing both* p *and* $B \cap C$ *whenever a block* C *meeting* B *in* y *points and any point* p *not in* $B \cap C$*.*

Note that if $\lambda = 1$ or, equivalently, $y = 1$ then all blocks in Π are good. The quotient $m = \frac{v-k}{k-y}$ will be proved to be a useful parameter for a quasi-symmetric 2-(v, k, λ) design. The next lemma is a simple consequence of the definition of a good block and lemma 4.1.2.

Lemma 4.1.3 ([29]). *Let* Π *be as in Lemma 4.1.2 and let* B *be any block. Then* B *is a good block if and only if for any block* C *meeting* B *in* y *points there are exactly* $m+1$ *blocks containing* $B \cap C$*.*

As an easy consequence, $m = \frac{v-k}{k-y}$ must be a positive integer for any quasi-symmetric design with a good block. Those quasi-symmetric designs having 0 as an intersection number and having a good block are classified parametrically as shown in the following theorem.

We show that a quasi-symmetric design with intersection number 1 and $y > 1$ and a good block belongs to one of three types: Examples of quasi-symmetric designs which are not symmetric designs or affine designs or linear spaces are rather rare, so construction methods for quasi-symmetric designs are of interest. The classification problem for quasi-symmetric designs, even in case $x = 0$ appears to be a difficult open problem.

One approach in the study of such designs is to impose some additional parametric or structural condition. Another approach is to consider substructures and induced substructures of such quasi-symmetric designs with a possible view to obtain new construction methods. They consider quasi-symmetric designs with $x = 0$ and having a certain type of block, referred to as a good block. According to them, in any quasi-symmetric design with intersection numbers x, y $(0 \le x \le y)$, a block B is said to be *good*, if for any block C with $|B \cap C| = y$ and any point $p \notin B \cap C$, there is a (unique) block containing p and $B \cap C$. It is clear that if $\lambda = 1$ (or, equivalently, $y = 1$) in a quasi-symmetric design, then all blocks are good. Their notion of good blocks in quasi-symmetric designs extends the earlier notion for affine designs and symmetric designs.

One of the main results of [29] is the following theorem.

Theorem 4.1.4 ([29]). *Let $\Pi = (X, \mathcal{B})$ be a quasi-symmetric 2-(v, k, λ) design with intersection numbers $x = 0$ and y. Suppose that Π has a good block. Then Π is a symmetric design or an affine design or a linear space or else has one of the following two parameter sets:*

1. $v = y^4(y^3 - y^2 + 2y - 1)$, $k = y^2(y^2 - y + 1)$, $\lambda = y^3 + y + 1$;
2. $v = y(y^4 - y^3 + y^2 - y + 1)$, $k = y(y^2 - y + 1)$, $\lambda = y^2 + 1$.

Those QSD in case 1. and 2. Π induces a 2-$(k, y, 1)$ design on the points of any good block B, the blocks of the induced design being distinct non-empty intersections of B with the other blocks of Π. It is remarked in [29] that the smallest possible value of y in case 1. is $y = 3$, the value $y = 2$ being excluded by the non-existence of a projective plane of order 10. It is also noted that the unique 2-(22, 6, 5) Witt design having $x = 0$, $y = 2$ satisfies case 2., but no examples with $y \ge 3$ are known.

Theorem 4.1.5 ([29]). *Let Π be a proper quasi-symmetric 2-(v, k, λ) design with $\lambda > 1$ and having an intersection number 0. Let B be a good block and let Π_0 be the quasi-symmetric 2-$(k, y, \frac{\lambda-1}{m})$ design induced on B, where $m = \frac{v-k}{k-y}$. If there is a good block which is disjoint from B, then Π_0 is resolvable.*

Theorem 4.1.6 ([29]). *Let Π be a quasi-symmetric 2-(v, k, λ) design with $\lambda > 1$ and having an intersection number 0. Then the following statements 1. and 2. are equivalent:*

1. *All blocks of Π are good.*

2. *Either* Π *is the design of points and hyperplanes of projective or affine space, or all lines (see definition below) of* Π *have size* y *and* Π *has parameter of type 1. or 2. in Theorem 4.1.4.*

Let Π_p denote the incidence structure whose points and blocks are the lines and blocks of Π on a point p, respectively. In case 2.(ii), Π_p is a projective plane of order $y^3 + y$ or y^2 for any point p. If $y = 2$, Π is the unique Witt 3-(22,6,1)design.

Proper quasi-symmetric designs with $x = 1$ and which have a good block is reviewed in this section. The following are two well-known examples of such designs. Example:

1. Let $PG(4, q)$ be the four-dimensional projective geometry over $GF(q)$. Let $\Pi = PG_2(4, q)$, respectively. Then Π is a quasi-symmetric 2-(v, k, λ) design with $v = \frac{q^5-1}{q-1}$, $k = \lambda = \frac{q^3-1}{q-1}$, $x = 1$ and $y = q + 1$. All blocks of Π are good.

2. The unique 2-(23, 7, 21) Witt design Π, which is also a 4-(23, 7, 1) design, is a quasi-symmetric design with $x = 1$ and $y = 3$. All blocks in this design are good since $y = 3$ and there is exactly one block on any four points.

Theorem 4.1.7 ([28]). *Let* $\Pi = (X, \mathcal{B})$ *be a proper quasi-symmetric 2-*(v, k, λ) *design with intersection numbers 1 and* y *and with a good block. Then* Π *is one of the following:*

1. *a quasi-symmetric design with parameters* $v = \frac{q^5-1}{q-1}$, $k = \lambda = \frac{q^3-1}{q-1}$, $b = (q^2 + 1)v$, $r = (q^2 + 1)k$, $x = 1$ *and* $y = q + 1$ $(q \geq 2)$ *as those of* $PG(4, q)$*.*

2. *The unique 2-(23, 7, 21) Witt design with* $v = 23$, $k = 7$, $\lambda = 21$, $b = 253$, $r = 77$, $x = 1$, $y = 3$.

3. *A quasi-symmetric design with parameters* $v = 1 + ((\alpha - 1)\lambda + 1)(y - 1)$ *and* $k = 1 + \alpha(y - 1)$, *for some integer* $\alpha > y \geq 5$, *and in which the design* Π_1 *induced on a good block is a 2-*$(k, y, 1)$ *design.*

No example of designs satisfying case 3. of the theorem has been found.

Theorem 4.1.8 ([28]). *Let* $\Pi = (X, \mathcal{B})$ *be a quasi-symmetric design, with parameters as those of* $PG_2(4, q)$ *and with* $(x, y) = (1, q + 1)$*. All blocks of* Π *are good if and only if* Π *is isomorphic to* $PG_2(4, q)$*.*

4.2 A Class of Strongly Regular Designs *SRD*

4.2.1 Quasi-semi symmetric designs

Motivated by the study of the geometric structures associated with the half dual polar graph $D_{n,n}(q)$ and the alternating forms graphs $Alt(n,q)$, some specific conditions over incidence structures were considered by Fu and Huang [14]:

(QSS1) every two distinct points are in 0 or λ common blocks,

(QSS2) every two distinct blocks intersect in 0 or μ points,

(QSS3) if $\lambda = 1$, then there are constants k and r such that every block contains k points and every point is on r blocks,

(QSS4) if (x, B) is a nonincident pair of point x and block B, then there are exactly α blocks of x intersecting B.

Let λ, μ, and α be positive integers. A finite incidence structure $\Pi = (X, \mathcal{B})$ is called a *quasi-semi-symmetric* design (abbreviated 'QSSD') for λ, μ with *nexus* if conditions (QSS1)-(QSS4) are satisfied. Clearly, $\lambda = 1$ if and only if $\mu = 1$, and hence Π is a semilinear space or a partial linear space (see Brouwer et al., 1989, for the definition). Condition (QSS3) is necessary to ensure the k-uniformity and r-regularity of Π (i.e., every blocks of Π contains k points, and every point of Π is in r blocks). An example that satisfies (QSS1) and (QSS2) with $\lambda = \mu = 1$ but does not satisfy (QSS3) is given in Huang an Pan (1988). Partial geometries, first studied by Bose, are examples of QSSDs with $\lambda = \mu = 1$, and partial λ-geometries, introduced by Cameron and Drake (1980) are QSSDs with $\lambda = \mu$.

QSSDs with multiple intersections, i.e., $\lambda \geq \mu \geq 2$, were treated []. Basic properties, associated combinatorial structures, some examples constructed from vector spaces, and some existence conditions for QSSDs with $\mu = \lambda - 1 \geq 2$ are described. Two extremal conditions that provide an upper bound and a lower bound, respectively, for α. The following two equivalent conditions, called the (*)-*conditions*, were studied for $(s, r; \mu)$-nets in Huang and Laurent (1993) and for partial λ-geometries in Cameron and Drake (1980). Indeed, $\alpha = \frac{\lambda^2(\mu-1)}{\mu}$ under these extremal conditions. For a nonflag (x, B), $|x^{\perp} \cap B|$ is a constant β, where $\beta\lambda = \alpha\mu$, and we let $\Pi_{x,B}$ be the incidence structure defined over $x^{\perp} \cap B$. The structure of $\Pi_{x,B}$, together with the (*)-condition, gives a sharp lower bound for β (and hence for α).

Symmetric designs, semisymmetric designs, and partial λ-geometries are among such structures. In this paper, in addition to some general properties, we study the existence conditions for QSSDs with $\mu = \lambda - 1 \geq 2$ and the properties of

QSSDs satisfying the following extremal condition: if B_1 and B_2 are two blocks with a nonempty intersection, then there are another $\lambda - 2$ blocks $B_3, \ldots, B_\lambda$ such that $\cap_{1\leq i\leq\lambda} B_i = B_1 \cap B_2$. We show that $\alpha \geq \frac{\lambda^2(\mu-1)+\lambda}{\mu}$ under such a condition, and QSSDs with equality are classified whenever $\mu = \lambda$ or $\mu = \lambda - 1$ following a classification of affine polar spaces by Cohen and Shult (*Geometraic Dedicata* 35 (1990), 43-76).

4.2.2 Strongly Regular Designs

The notion of strongly regular designs was first introduced by D.G. Higman as a class of 1-design arising in the investigation of coherent configurations of small types. Indeed, *SRD*'s are $1\frac{1}{2}$-designs in the sense of Neumaier [31] and form a self-dual class. An *SRD* has a point graph and a block graph of which are strongly regular.

An incidence structure $\Pi = (X, \mathcal{B})$ with a point-block incidence matrix M is called a *strongly regular design* if there exists nonnegative integers a_i, b_i, N_i, P_i and S_i, $i = 1, 2$, such that

1. $JM = S_1 J$, $MJ = S_2 J$,
2. $MM^T = S_2 I + a_2 A_1 + b_2(J - I - A_1)$, $M^T M = S_1 I + a_1 A_2 + b_1(J - I - A_2)$,
3. $A_1 M = N_1 M + P_1(J - M)$, $MA_2 = N_2 M + P_2(J - M)$,

where A_1 and A_2 are square matrices of orders v and b, respectively. Note that A_1 and $\overline{A_1} = J - I - A_1$, respectively, A_2 and $\overline{A_2} = J - I - A_2$, form the adjacency matrices of a pair of complementary strongly regular graphs.

These conditions can be interpreted in terms of the relationship between points and blocks as follows:

1. each block consists of k points, and point lies in r blocks;
2. any two points lie in either a_2 or b_2 common blocks, and any two blocks meet in either a_1 or b_1 points;
3. for a pair of a point x and a block B, x is collinear with either P_1 points or N_1 points of B_1 depending on whether x is in B or not, the number of blocks containing x and meeting B is either P_2 or N_2 depending on whether $x \in B$ or not.

We will focus on those strongly regular designs with $b_1 = b_2 = 0$, i.e., we are concerned with those strongly regular designs with incidence matrix M satisfying the following conditions: $JM = kJ$, $MJ = rJ$, $MM^T = rI + aA_1$, $M^T M = kI + bA_2$, $A_1 M = aM + b(J - M)$ and $MA_2 = N_2 M + P_2(J - M)$ (note that k for S_1, r for

S_2, A for A_1, B for A_2). This class of strongly regular designs is called a *quasi-semisymmetric design* with *nexus*, and it was used as a geometric framework for a characterization of *alternating bilinear forms graphs* of order 4.

An $SRD(v, k, a, b)$ is called *resolvable* if the blocks can be partitioned into classes such that each class form a partition of X and any two blocks have common points if they are in different classes.

Lemma 4.2.1. *For a resolvable $SRD(v, k, a, b)$, then*

1. *for a pair (x, B) of nonincident point x and block B, there is a unique block on x and parallel to B, and hence x is collinear with $\frac{b(r-1)}{a}$ points on B,*
2. *two distinct points x and y are collinear with $k-2+\frac{(r-a)(br-b-a)}{a^2}$ points whenever they are collinear, or otherwise they are collinear with $\frac{br(r-1)}{a^2}$ points; and hence its point graph is a strongly regular graph on v points with $k = r(k-1)/a, f =, andg = r(r-1)b/a2.$*

4.2.3 Some properties of $SRD(v, k, \lambda, \mu)$

Lemma 4.2.2 ([14]). *Let $\Pi = (X, \mathcal{B})$ be a $SRD(v, k, \lambda, \mu)$, $\mu \geq 2$, with nexus α. The following two conditions are equivalent:*

1. *if B_1, B_2 are two distinct blocks, with $B_1 \cap B_2 \neq \varnothing$, then there exist $B_3, \ldots, B_\lambda \in \mathcal{B}$ such that $\cap_{1\leq i\leq\lambda} B_i = B_1 \cap B_2$, which consists of μ points.*
2. *if B_1, B_2, B_3 are three distinct blocks with $|B_1 \cap B_2 \cap B_3| \geq 2$, then $|B_1 \cap B_2 \cap B_3| = \mu$.*

Corollary 4.2.3 ([14]). *Let (x, B) be a nonflag of a SRD satisfying the (*)-condition, and let A_1 and A_2 be two distinct blocks of x intersecting B. Then $|A_1 \cap A_2 \cap B| \leq 1$.*

Lemma 4.2.4 ([14]). *Let $\Pi = (X, B)$ be a $SRD(v, k, \lambda, \mu)$, $g \geq 2$, satisfying the (*)-condition with nexus α and let (x, B) be a nonflag. Then*

1. *$\beta \geq \lambda(\mu-1)+1$, and hence $\alpha \geq \frac{\lambda^2(\mu-1)+\lambda}{\mu}$,*
2. *equality holds if and only if the structure $\Pi_{x,B}$ is a 2-$(\lambda(\mu-1)+1, \mu, 1)$ design.*

Substituting $\mu = \lambda$ $(= q+1)$ and $\mu = \lambda - 1$ $(= q)$ in the previous lemma, we have $\alpha \geq q^2+q+1$ and $\alpha \geq q^2+q$, respectively. Examples (iii) and (iv) in the previous section show that both bounds are sharp. Moreover, the 2-designs mentioned above in the QSSDs of Examples (iii) and (iv) are projective planes and affine planes of order q, respectively.

An upper bound for β (and hence for α) is obtained by the following extremal condition, called the (Δ)-condition.

Any three distinct pairwise collinear points are in at least one common block.

Lemma 4.2.5 ([14]). *Let $\Pi = (X, \mathcal{B})$ be a $SRD(v, k, \lambda, \mu)$, $g \geq 2$, satisfying the (Δ)-condition with nexus α. Then $\beta \leq \lambda(\mu - 1) + 1$, and hence $\alpha \leq \frac{\lambda^2(\mu-1)+\lambda}{\mu}$.*

Corollary 4.2.6 ([14]). *Let $\Pi = (X, \mathcal{B})$ be a $SRD(v, k, \lambda, \mu)$, satisfying the (*)-condition with nexus $\alpha = \frac{\lambda^2(\mu-1)}{\mu}$. Then the (Δ)-condition holds.*

For a nonflag (x, B), the incidence structure $\Pi_{x,B}$ is determined under the (*)- and (Δ)-condition.

Corollary 4.2.7 ([14]). *Let $\Pi = (X, \mathcal{B})$ be a $SRD(v, k, \lambda, \mu)$, $\mu \geq 2$ with nexus α. The following are equivalent:*

1. *Π satisfies the (*)- and (Δ)-conditions,*
2. *$\Pi_{x,B}$ is a 2-$(\lambda(\mu - 1) + 1, \mu, 1)$ design.*

Cameron and Drake (1980) showed that a $SRD(v, k, \lambda, \mu)$ satisfying the (*)-condition with nexus $\alpha = \lambda^2 - \lambda + 1$ is obtained from a polar space of type $D_4(q)$ with one family of maximal totally isotropic subspaces as the block set. As a result, its point graph is isomorphic to $D_{4,4}(q)$. In this section, we shall prove a similar result for a $SRD(v, k, \lambda, \lambda - 1)$ with nexus $\alpha = \lambda^2 - \lambda$.

Let $\Pi = (X, \mathcal{B})$ be a $SRD(v, k, \lambda, \mu)$ satisfying the with nexus $\alpha = \frac{\lambda^2(\mu-1)+\lambda}{\mu}$ (i.e., $\beta = \lambda(\mu - 1) + 1$). Associate Π with an incidence structure $\Pi' = (X, \mathcal{L})$ with a collection $\mathcal{P}$ of substructures, where $\mathcal{L} = \{A \cap B : A, B \in \mathcal{B}$ are distinct with $A \cap B \neq \varnothing\}$ and let $\mathcal{P} = \{x^\perp \cap B : x \in X, B \in \mathcal{B}, x \notin B\}$. Members of $\mathcal{L}$ and $\mathcal{P}$ are called *lines* and *planes*, respectively. Clearly, the point graphs of Π and Π' are identical. For any two collinear points x and y, let $A_1, \ldots, A_\lambda$ be the blocks containing x and y and denoted by xy the line $A_1 \cap A_2 = \cap_{1 \leq i \leq \lambda} A_i$ (by the (*)-condition). Since α reaches the lower bound, the (Δ)-condition also holds, by Corollary 4.2.6. Thus $x^\perp \cap y^\perp = \cup_{1 \leq i \leq \lambda} A_i$, and $\{A_i - xy : 1 \leq i \leq \lambda\}$ forms a partition of $x^\perp \cap y^\perp - xy$. Hence the incidence structure $\Pi' = (X, \mathcal{L})$ is a gamma space, and each block of Π induces a maximal singular subspace in Π' (refer to Brouwer et al., 1989, for the definitions of gamma spaces and singular subspaces). Note also that each plane in $\mathcal{P}$ is a singular subspace too. A triple of points is called a *triangle* if they are pairwise collinear but not contained in a common line. The main theorem in this section is as follows:

Theorem 4.2.8 ([14]). *Let* $\Pi = (X, \mathcal{B})$ *be a* $SRD(v, k, \lambda, \mu)$ *satisfying the (*)-condition with* $\alpha = \frac{\lambda^2(\mu-1)+\lambda}{\mu}$ *(i.e.,* $\beta = \lambda(\mu-1)+1$*). Then*

1. *if* $\mu = \lambda$ *(*$= q+1 \geq 3$*), then* $\Pi' = (X, \mathcal{L})$ *is the polar space of type* $D_4(q)$ *and the point graph of* Π *is isomorphic to* $D_{4,4}(q)$.

2. *if* $\mu = \lambda - 1$ *(*$= q \geq 4$*), then either* $\Pi' = (X, \mathcal{L})$ *is the affine polar space of type* $D_4(q) - \infty^{\perp}$ *and the point graph of* Π *is isomorphic to* $Alt(4, q)$, *or* $k = 5^5, 11^5$.

Construction for families of resolvable SRD with parameters $(v, k,\ a, b) = (q^3,\ q^2,\ 2, q)$, $(q^3,\ q^2,\ \frac{q}{2}, q)$ for $q = 2^n$ a power of 2. As a consequence, a few families of other designs as well as strongly regular graphs are derived. Among others, this gives a strongly regular graph $SRG(196, 135, 94, 90)$ which is new to the list of such graphs up to 280 vertices complied by A.E. Brouwer in the CRC Handbook of Combinatorial Designs.

References

[1] A. Baartmans and M. S. Shrikhande, "Designs with no three mutually disjoint blocks", *Discrete Math*, Vol. 40, 129-139, 1982.

[2] B. Bachi, "On quasi-symmetric designs", *Designs, Codes and Cryptography*, 2, 69-79, 1992.

[3] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, 2nd ed., Vols. 1, 2, Cambridge University Press. 162 MAVRON ET AL., 1999.

[4] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs* , Springer, Berlin, 1989.

[5] F. Buekenhout, "Une caracterisation des espaces affins basee sur la notion de droite", *Math. Z.*, 111, 367-371, 1969.

[6] P. J. Cameron, "Quasi-symmetric designs possessing a spread", In (A. Barlotti et al., ed.), *Combinatorics'88*, Mediterranean Press, Rome, Italy, 231-236, 1992.

[7] P. J. Cameron, "Strongly regular graphs", In: L. W. Beineke and R. J. Wilson, Eds., *Selected Topics in Graph Theory*, Academic Press, New York, 337-360, 1978.

[8] P. J. Cameron and D. K. Drake, "Partial λ-geometries of small nexus", In: J. Srivatsava, Ed., *Combinatorial Math., and Optimal Designs and Their Appl., Fort Collins 1978, Ann. Discrete Math. 6*, North-Holland, Amsterdam, 19-29.

[9] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, London Math. Society Student Texts 22, Cambridge University Press, Cambridge, 1991.

[10] A. M. Cohen and E. E. Shult, "Affine polar spaces", *Geom. Dedicata*, 35, 43-76, 1990.

[11] B. N. Cooperstein and E. E. Shult, "Geometric hyperplanes of embeddable Lie incidence geometries", In: J. W. P Hirschfeld, D. R. Hughes and J. A. Thas, Eds., *Advances in Finite Geometries and Designs*, Oxiford Science-Publications, Oxford, 81-91, 1991.

[12] D. A. Drake, "Partial λ-geometries and generalized Hadamard matrices over groups", *Canad. J. Math*, 31, 617-627, 1979.

[13] T. S. Fu and T. Huang, "A characterization of two classes of strongly regular graphs", *J. Geom*, 52, 91-100, 1995.

[14] T. S. Fu and T. Huang, "Quasi semisymmetric designs with extremal conditions", *Journal of Statistical Planning and Inference*, 51, 261-271, 1996.

[15] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer GTM 207, 2001.

[16] D. G. Higman, "Strongly regular designs and coherent configurations of type $\begin{bmatrix} 3 & 2 \\ & 3 \end{bmatrix}$", *European J. Combin*, 9, 411-422, 1988.

[17] T. Huang and M.Laurent, "(s, r; g)-nets and alternating forms graphs", *Discrete Math*, 114, 237-252, 1993.

[18] T. Huang and Y. Pan, "A diagram geometry for the association schemes of the alternating bilinear forms", *Proc. NSC, Part A, ROC 12*, 44-49, 1988.

[19] D. R. Hughes and F. C. Piper, *Design Theory*, Cambridge University Press, 1985.

[20] Y. J. Ionin, M. S. Shrikhande, "Strongly regular graphs and designs with three intersection numbers", *Designs, Codes and Cryptography*, 21, 113-125, 2000.

[21] W. M. Kantor, "Characterizations of finite projective and affine spaces", *Canadian J. Math.*, Vol. 21, 64-75, 1969.

[22] W. M. Kantor, "Automorphisms and isomorphisms of symmetric and affine designs", *J. Algebraic Comb*, Vol. 3, 307-338, 1994.

[23] M. E. Kimberley, "On the construction of certain Hadamard designs", *Math. Z.*, Vol. 119, 41-59, 1971.

[24] M. Y. Lin, "A study of quasi semisymmetric designs", Master thesis, National Chiao-Tung Univ., 1992.

[25] N. B. Limaye, S. S. Sane and M. S. Shrikhande, "The structure of triangle-free quasi-symmetricdesigns", *Discrete Math.*, Vol. 64, 199-207, 1987.

[26] V. C. Mavron, "On the structure of affine designs", *Math. Z.*, Vol. 125, 298-316, 1972.

[27] V. C. Mavron, "On a class of symmetric designs", *Geom. Dedicata*, Vol. 1, 468-478, 1973.

[28] V. C. Mavron, T. P. McDonough, M. S. Shrikhande, "Quasi-symmetric designs with good blocks and intersection number one", *Designs, Codes and Cryptography*, 28, 147-162, 2003.

[29] T. P. McDonough and V. C. Mavron, "Quasi-symmetry designs with good blocks", *Journal of Combinatorial Designs*, Vol.3, No.6, 433-441, 1995.

[30] V. C. Mavron and M. S. Shrikhande, "Designs with intersection numbers 0 and 2", *Arch. Math.*, Vol. 52, 407-412, 1989.

[31] A. Neumaier, "Regular cliques in graphs and special $1\frac{1}{2}$-designs", In: P. J. Cameron, J. W. P. Hirschfeld and D. R. Hughes, Eds., *Finite Geometries and Designs*, London Math. Soc. Lecture Note Series, Vol. 49, Cambridge Univ. Press, Cambridge, 244-259, 1981.

[32] S. S. Shrikhande, "Affine resolvable balanced incomplete block designs: a survey", *Aequationes Math.*, Vol. 14, 251-269, 1976.

[33] M. S. Shrikhande and S. S. Sane, *Quasi-symmetric Designs*, London Mathematical Society Lecture Note Series, Vol. 164, Cambridge University Press, 1991.

[34] S. S. Sane and M. S. Shrikhande, "Some characterizations of quasi-symmetric designs with a spread", *Designs,Codes and Cryptography*, Vol. 3, 155-166, 1993.

[35] D. R. Stinson, *Combinatorial Designs with Selected Applications Lecture Notes*, 1996.

[36] A. L. Wells, "Universal projective embeddings of the Grassmannian, half-spinor, and dual orthogonal geometries", *Quart. J. Math.*, Oxford, 34, 375-386, 1983.

[37] F. Zara, "Graphs lies aux espaces polaires", *European J. Combin.*, 5, 255-290, 1984.