國 立 交 通 大 學
應用數學系
碩 士 論 文

圖論上代數方法的探討

Algebraic Techniques in Graph Theory

研 究 生：李致維
指導老師：翁志文 教 授

中 華 民 國 九 十 三 年 六 月

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碩　士　論　文

A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
In partial Fulfillment of Requirement
For the Degree of Master
In
Applied Mathematics
June 2004
Hsinchu, Taiwan, Republic of China

中　華　民　國　九　十　三　年　六　月

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摘　　要

代數方法在圖論上被廣泛的使用。如圖上的自同態群的研究，利用特徵值及線性代數的方法來探討圖的性質、以及與圖有關的多項式。這篇論文的目的主要是收集了已知的圖論上使用的代數方法。

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Algebraic Techniques in Graph Theory

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Abstract

Algebraic methods provide many new and powerful ways in the study of graph theory. These include the study of the group of homomorphisms on graphs, the construction of graphs from a group, using the eigenvalue or other linear algebraic techniques in the study of graph theory and the study of polynomials associated with a graph. The purpose of this thesis is to collect the known results in graph theory with algebraic techniques involved.

誌謝

感謝我的指導教授翁志文老師，在這兩年的時間，悉心的教學以及論文的指導和修正，使得我能順利畢業。老師在做研究上細心驗證的精神，也讓我獲益良多。

非常感謝交大應數所的所有老師，老師們的教學熱誠以及對學生的關心，都讓我獲益良多。非常感謝黃光明老師、黃大原老師、傅橫霖老師、陳秋媛老師等，在課業上、生活上的多方指導及照顧。

感謝郭君逸學長、張飛黃學長在課業上以及生活上的照顧，感謝抮君、正傑、昭芳、喻培、宏嘉、榮丰、嘉文、建瑋、貴弘、啟賢、文祥等同學，讓我的研究所生活增色不少。

最後，感謝我的父母以及我的家人，一直給我支持，讓我能順利完成學業。

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Chapter 1

Introduction

Algebraic methods provide many new and powerful ways in the study of graph theory. These include the study of the group of homomorphisms on graphs, the construction of graphs from a group, using the eigenvalue or other linear algebraic techniques in the study of graph theory and the study of polynomials associated with a graph. The purpose of this thesis is to collect the known results in graph theory with algebraic techniques involved. The thesis is organized as follows.

In chapter 2, we use the concept of group acting on a set to study a graph. Here the group is usually the automorphism group of a given graph. We then introduce vertex transitive graphs and Cayley graphs. We study the edge connectivity, vertex connectivity, matchings, maximal cycles in a connected vertex transitive graph. We show a connected vertex transitive graph is a homomorphic image of some Cayley graph.

In chapter 3, we introduce the core of a graph. The core of a graph is the smallest homomorphism image of the graph. We show the core of a vertex transitive graph is vertex transitive. We give some sufficient conditions of a core.

In chapter 4, we introduce the adjacency matrix of a graph. We study the spectrum of an adjacency matrix. The classical Perron Frobenius Theorem of symmetric matrices with nonnegative entries is included in this chapter.

In chapter 5, we generalize the concept of sets interlacing to eigenvalues sequences interlacing and rational functions interlacing.

In chapter 6, we introduce the incidence matrix, the Laplacian, and more general, the weighted Laplacian of a graph. The Laplacian is an important matrix associated with a graph. We study the spectrum of the Laplacian.

We also show the number of spanning trees in a graph is determined by the spectrum of its Laplacian. We give an upper bound of the second least eigenvalue of the Laplacian in terms of some combinatorial structure of a graph.

In chapter 7, we introduce the rank function and matroid. We study their basic properties. We introduce the dual, the restriction and the contraction of a matroid.

All of the results in this thesis are classical. We learn most of them from [1]. We add more details in order to realize the content. For example, Example 2.2, Example 2.5, Definition 2.6, Lemma 2.16, Example 2.17, Lemma 2.25, Theorem 2.41, Lemma 2.42, Theorem 2.43, Lemma 2.44, Example 2.47, Example 2.50, Lemma 3.6, Example 3.10, Lemma 3.11, Example 3.12, Theorem 3.13, Lemma 3.14, Corollary 3.17, Example 3.21, Lemma 3.25, Example 3.28, Lemma 3.26, Lemma 3.27, Lemma 3.34, Lemma 4.8, Lemma 4.10, Lemma 4.11, Lemma 4.12, Lemma 4.13, Lemma 4.14, Lemma 4.15, Lemma 4.24, Definition 5.1, Example 5.2, Lemma 5.4, Theorem 5.7, Lemma 6.13, Lemma 6.14, Lemma 6.21, Corollary 6.48, Lemma 7.2. We rewrite some of the proofs for the readers easy to understand. For example, Theorem 2.13, Theorem 2.18, Lemma 4.8, Theorem 4.25, Theorem 5.7, Theorem 6.10. Some ideas come from [2], [3].

Chapter 2

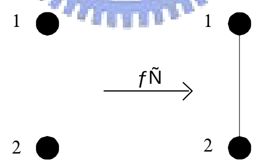
Transitive Graphs

Throughout this thesis, let $X = (X, R)$ be an undirected graph without loops or multiple edges. We abuse the notation X as both the graph and the vertex set of the graph. $R = \{xy \mid x, y \in X, x \neq y\}$ is the edge set.

2.1 Cayley Graphs

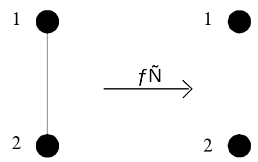
Definition 2.1. Let X, X' be graphs. A function $\varphi : X \to X'$ is a *homomorphism* from X into X' if $\varphi(x)\varphi(y) \in R'$ for all $x, y \in X$ with $xy \in R$.

Example 2.2. (1)



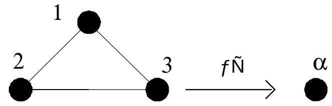
is a homomorphism.

(2)



is **not** a homomorphism.

(3)



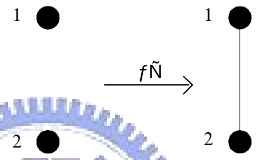
$f(1) = f(2) = f(3) = \alpha$, f is **not** a homomorphism.

Definition 2.3. (1) $\varphi : X \to X'$ is an *isomorphism* if φ is bijection and $xy \in R$ if and only if $\varphi(x)\varphi(y) \in R'$.

(2) If $\varphi : X \to X$ is an isomorphism, we say φ is an *automorphism* on X. We will use $Aut(X)$ to denote the set of automorphisms on X.

Note 2.4. $(Aut(X), \circ)$ *is a group, where* $\circ$ *is the composition operation.*

Example 2.5.



f is **not** a isomorphism.

The concept of group action on a set is widely used in algebraic graph theory. We give its definition below.

Definition 2.6. Let G be a group, and S be a set. We say G acts on S if there exists a function $\cdot : G \times S \to S$ such that

(1) $e \cdot s = s$;

(2) $(g \cdot h) \cdot s = g \cdot (h \cdot s)$

for all g, h$\in G$ and all $s \in S$, where e is the identity of G.

Note 2.7. (1) $g \cdot s = t$ *if and only if* $s = g^{-1} \cdot t$ *for all* $g \in G$ *and* s , $t \in S$.

(2) *Define a relation on* S *by* $s \sim t$ *if and only if* $t = g \cdot s$ *for some* $g \in G$. *Then* $\sim$ *is an equivalent relation, and* $\sim$ *defines a partition on* S.

Definition 2.8. Let G be a group and S be a set. We say G acts *transitively* on S if the partition defined from the equivalent relation $\sim$ has only one element(orbit).

Note 2.9. *A group G acts transitively on a set S if for any s, $t \in S$, there exists $g \in G$ such that $g \cdot s = t$.*

Definition 2.10. A graph X is *vertex transitive* if for any $x, y \in X$, there exists $\rho \in Aut(X)$ such that $\rho(x) = y$.

Note 2.11. *If X is vertex transitive. Then X is regular (i.e. each vertex in X has the same number of valency (neighbors)). We will use k to denote the valency of X.*

Definition 2.12. Fix $n \in \mathbb{N}$. Define

$$\begin{aligned} Q_n &:= \{(a_1, a_2, a_3, ..., a_n) \mid a_i = 0 \text{ or } 1\} \\ R &:= \{xy \mid x, y \in Q_n \text{ differ in exactly one coordinate}\}. \end{aligned}$$

The graph (Q_n, R) is called the *n-cube*.

Theorem 2.13. *The n-cube (Q_n, R) is vertex transitive.*

Proof. Pick any $x, y \in Q_n$. Define a map $\rho : Q_n \to Q_n$ by

$$\rho(z) = y - x + z (\text{mod } 2)$$

where the operations $+, -$ are the usual coordinatewise summation and subtraction. It is straightforward to check $\rho \in Aut(X)$ and $\rho(x) = y$. $\square$

Definition 2.14. Let G be a group and $\triangle \subseteq G$ be a subset such that

(1) $e \notin \triangle$,

(2) $g \in \triangle$ if and only if $g^{-1} \in \triangle$ for all $g \in G$.

Set $X = G$ and $R = \{xy \mid x, y \in G \text{ and } y = x \cdot g \text{ for some } g \in \triangle\}$. Then (X, R) is called the *Cayley graph* of G with respect to C. We will write $X(G, \triangle)$ for such a graph.

Note 2.15. (1) *If G is abelian then $X(G, \triangle)$ is a simple undirected graph.*

(2) *x, y are adjacent in $X(G, \triangle)$ if and only if $x^{-1}y \in \triangle$.*

Lemma 2.16. *Let $X(G, \triangle)$ be a Cayley graph. For each $g \in G$, define $\phi_g : G \to G$ by $\phi_g(h) = gh$. Then $\phi_g \in Aut(X)$.*

Proof. Since $X(G, \triangle)$ is a Cayley graph, the vertex set $X = G$ and the edge set $R = \{sh \mid h, s \in G \text{ and } h = sc, \text{ for some } c \in \triangle\}$. Pick $x,\ y \in G$. Observe

$$\begin{aligned} x \sim y &\Leftrightarrow x^{-1}y \in \triangle \\ &\Leftrightarrow x^{-1}g^{-1}gy \in \triangle \\ &\Leftrightarrow (gx)^{-1}gy \in \triangle \\ &\Leftrightarrow \phi_g(x) \sim \phi_g(y). \end{aligned}$$

Let $\phi_g(h) = \phi_g(k)$. Then $gh = gk$. Hence $g^{-1}gh = g^{-1}gk$. Then $h = k$. So ϕ_g is injective. Observe for any $x \in G$, there exists $g^{-1}x \in G$ such that $\phi_g(g^{-1}x) = g^{-1}gx = x$. Hence ϕ_g is surjective. So $\phi_g \in Aut(X)$. □

Example 2.17. Let $\mathbb{Z}_2 = \{0, 1\}$. Let $G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$(n copies) and $\triangle = \{a \in \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \mid$ exactly one coordinate of a is $1\}$. Then $Q_n = X(G, \triangle)$.

Generalizing the ideal of the proof of Theorem 2.13, we have the following Theorem.

Theorem 2.18. *The Cayley graph $X(G, \triangle)$ is vertex transitive.*

Proof. Pick any $x, y \in X = G$. Define a map $\phi_{yx^{-1}} : G \to G$ by $\phi_{yx^{-1}}(z) = yx^{-1}z$. Hence $\rho \in Aut(X)$ by Lemma 2.16. Clearly, $\rho(x) = y$. □

2.2 Edge Connectivity

Definition 2.19. Let $A \subseteq X$ be a vertex subset. The edge subset $\partial A := \{xy \in R \mid |\{x, y\} \cap A| = 1\}$ is called the *boundary* of A.

Note 2.20. (1) $\partial\emptyset = \emptyset$.

(2) *If X is connected then $|\partial(A)| \geq 1$ for any nonempty $A \subsetneq X$.*

(3) $|\partial A| + |\partial B| \geq |\partial(A \cup B)| + |\partial(A \cap B)|$ *for $A,\ B \subseteq X$.*

Definition 2.21. $\kappa_1(X) := \min\limits_{\substack{A \neq \emptyset \\ A \neq X}} |\partial A|$ is called the *edge connectivity* of X.

Note 2.22. (1) $\kappa_1(X) \leq \min\limits_{x \in X} deg(x)$.

(2) *$\kappa_1(X) = 0$ if and only if X is disconnected.*

Definition 2.23. $A \subseteq X$ is an *edge atom* if $|\partial(A)| = \kappa_1(X)$ and for any $B \subseteq X$, $|\partial(B)| = \kappa_1(X)$ implies $|B| \geq |A|$.

Note 2.24. *Suppose $A \subseteq X$ is an edge atom and ϕ is an automorphism on X. Then $\phi(A)$ is an edge atom.*

Lemma 2.25. *Suppose A is an edge atom. Then $|A| \leq \dfrac{|X|}{2}$.*

Proof. Since $\kappa_1(X) = |\partial(A)| = |\partial(X - A)|$, $|A| \leq |X - A|$. Thus $|A| \leq \dfrac{|X|}{2}$. □

Corollary 2.26. *Suppose A, B are edge atoms of X. Then $A = B$ or $A \cap B = \emptyset$.*

Proof. Suppose $A \cap B \neq \emptyset$. Then $A \cup B \neq X$ since $|A|, |B| \leq \dfrac{|X|}{2}$. Hence $|\partial(A \cup B)| \geq \kappa_1(X)$. By Note 2.20(3), $|\partial(A \cup B)| + |\partial(A \cap B)| \leq |\partial A| + |\partial B| = 2\kappa_1(X)$. Then $|\partial(A \cap B)| \leq \kappa_1(X)$. This proves $|A \cap B| = |A| = |B|$. Hence $A = B$. □

Theorem 2.27. *Suppose X is a connected vertex transitive graph. Then $\kappa_1(X) = k$, where k is the valency of X. Furthermore, $|\partial(A)| > k$ for all atoms A with $1 < |A| < |X|$.*

Proof. $\kappa_1(X) \leq k$ is clear. Let A be an atom. If $|A| = 1$, then $\kappa_1(X) = |\partial(A)| = k$. Suppose $|A| \geq 2$. Observe $\rho(A)$ is an atom for any $\rho \in Aut(X)$ by Lemma 2.24. Hence $\rho(A) = A$ or $\rho(A) \cap A = \emptyset$. By Corollary 2.26 we claim A is regular as an induced subgraph. Pick 2 vertices $x, y \in A$. Choose a function $\rho \in Aut(X)$ such that $\rho(x) = y$. Hence $\rho(A) = A$ by Corollary 2.26. Then all the neighbors z in A of x are one to one corresponding to neighbors $\rho(z)$ in A of y. Let ℓ denote the valency of A. Notice that $\ell < k$, since X is connected. Observe $|A| \geq \ell + 1$. Hence

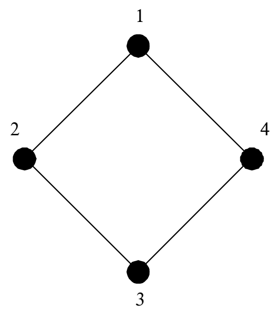
$$\begin{aligned} |\partial(A)| &= |A|(k - \ell) \\ &\geq |A|(k - (|A| - 1)) \\ &= |A|((k + 1) - |A|) \\ &\geq k. \end{aligned}$$

Observe above equality holds if and only if $|A| = 1$ or $|A| = X$. We obtain $\kappa_1(X) \geq k$. So $\kappa_1(X) = k$. □

2.3 Vertex Connectivity

Definition 2.28. A *vertex cutset* in a graph X is a set of vertices whose deletion increases the number of connected components of X.

Example 2.29. X:



$\kappa_1(X) = 2$. Let $A = \{1, 2\}$, $\partial A = \{14, 23\}$. Let $B = \{1, 3\}$, B is a vertex cutset.

Note 2.30. *X has a vertex cutset if X is not a complete graph.*

Definition 2.31. Let X be a connected graph with n vertices and let K_n be the complete graph with n vertices. If $X \neq K_n$, then the *vertex connectivity* of X is the minimum number of vertices in a vertex cutset, and will be denoted by $\kappa_0(X)$. We define $\kappa_0(K_n) = n - 1$.

Definition 2.32. Suppose A is a subset of vertices in X. Let $N(A)$ denote the vertices in $X \setminus A$ with a neighbor in A and $N[A] = A \cup N(A)$.

Note 2.33. (1) $A \cup N(A) \cup \overline{N[A]} = X$.

(2) $N(A) \supseteq N(\overline{N[A]})$.

(3) $\kappa_0(X) \leq \min\limits_{\substack{\overline{N[A]} \neq \emptyset \\ A \neq \emptyset}} |N(A)|$ *if X is connected.*

Definition 2.34. (1) A *fragment* of X is a subset A such that $\overline{N[A]} \neq \emptyset$ and $|N(A)| = \kappa_0(X)$.

(2) An *atom* of X is a fragment with minimum number of vertices.

Lemma 2.35. *Let X be a connected graph on n vertices with $\kappa_0 = \kappa_0(X)$. Suppose A and B are fragments of X and $A \cap B \neq \emptyset$. If $|A| \leq |\overline{N[B]}|$, then $A \cap B$ is a fragment.*

Proof. We present the proof as a number of steps.

(a) $|A \cup B| < n - \kappa_0$.

Observe

$$\begin{aligned} |A| + |B| &\leq |\overline{N[B]}| + |B| \\ &= n - |B| - |N(B)| + |B| \\ &= n - \kappa_0. \end{aligned}$$

Since $A \cap B$ is nonempty, the claim follows.

(b) $|N(A \cup B)| \leq \kappa_0$.

We observe

$$\begin{aligned} |N(A \cup B)| &\leq |N(A)| + |N(B)| - |N(A \cap B)| \\ &\leq \kappa_0 + \kappa_0 - \kappa_0 \\ &= \kappa_0. \end{aligned}$$

Hence the claim follows.

(c) $\overline{N[A \cup B]} \neq \emptyset$.

From (a), (b) observe

$$\begin{aligned} |\overline{N[A \cup B]}| &= n - |A \cup B| - |N(A \cup B)| \\ &> n - (n - \kappa_0) - \kappa_0 \\ &= 0. \end{aligned}$$

Hence the claim follows.

(d) $A \cup B$ is a fragment.

Clearly $A \cup B \neq \emptyset$. Since $\overline{N[A \cup B]} \neq \emptyset$, $|N(A \cup B)| \geq \kappa_0$ is clear from the definition of κ_0. Hence $|N(A \cup B)| = \kappa_0$ from (b).

(e) $A \cap B$ is a fragment.

By assumption, $A \cap B \neq \emptyset$. From (c) we observe

$$N[A \cap B] \subseteq N[A] \cap N[B] \neq X$$

Hence $\overline{N[A \cap B]} \neq \emptyset$. Observe

$$\begin{aligned} |N(A \cap B)| &\leq |N(A \cup B)| \\ &\leq |N(A)| + |N(B)| - |N(A \cup B)| \\ &= \kappa_0 + \kappa_0 - \kappa_0 \\ &= \kappa_0. \end{aligned}$$

Hence $|N(A \cap B)| = \kappa_0$. □

Corollary 2.36. *Let X be a connected graph. If A is an atom and B is a fragment of X. Then $A \subseteq B$, $A \subseteq N(B)$, or $A \subseteq \overline{N[B]}$.*

Proof. Note $|A| \leq |B|$ and $|A| \leq |\overline{N[B]}|$ since $\overline{N[B]}$ is a fragment. Observe $|A| \leq |B| \leq |\overline{N[\overline{N[B]}]}|$. Hence by previous Lemma $A \cap B$, $A \cap \overline{N[B]}$ are fragments if they are nonempty. Suppose $A \not\subseteq B$ and $A \not\subseteq \overline{N[B]}$. Then $A \cap B = \emptyset$ and $A \cap \overline{N[B]} = \emptyset$, otherwise we have a contradiction since $A \cap B$, $A \cap \overline{N[B]}$ are atoms with size less than $|A|$. Hence $A \subseteq N(B)$. □

Theorem 2.37. *Let X be a vertex transitive graph with valency $k \geq 2$. Then*

$$\kappa_0(X) \geq \frac{2}{3}(k+1).$$

Proof. If X is not connected, then all the connected components of X are the same. We can assume X is connected. Let A be an atom in X. If $\rho \in Aut(X)$, then $\rho(A)$ is an atom. Hence by Corollary 2.36, $\rho(A) \subseteq A$, $\rho(A) \subseteq N(A)$ or $\rho(A) \subseteq \overline{N(A)}$. Since X is vertex transitive, we can choose $\rho \in Aut(X)$ such that $\rho(A) \subseteq N(A)$. For another $\psi \in Aut(X)$ with $\psi(A) \in N(A)$, either $\psi(A) = \rho(A)$ or $\psi(A) \cap \rho(A) = \emptyset$. This proves $|N(A)| = t|A|$ for some positive integer t. We shall claim $t \geq 2$. Suppose $t = 1$. Then $|N(A)| = |A|$ and $N(A) = \rho(A)$. Hence $N(A)$ is an atom. Then

$$|N(N(A))| = |N(A)| = |A|. \tag{2.1}$$

Since $N(N(A)) \cap A \neq \emptyset$, we have $A \subseteq N(N(A))$ by previous Corollary. Hence by equation(2.1), $A = N(N(A))$. This shows $\overline{N[A]} \neq \emptyset$, a contradiction to A being an atom. Observe each vertex in A has valency k, and

$$\begin{aligned} k &\leq |A| - 1 + |N(A)| \\ &= (t+1)|A| - 1. \end{aligned}$$

Hence $|A| \geq \dfrac{k+1}{t+1}$. Then

$$\kappa_0 = |N(A)| = t|A| \geq t\frac{k+1}{t+1} \geq \frac{2}{3}(k+1).$$

□

2.4 Matchings

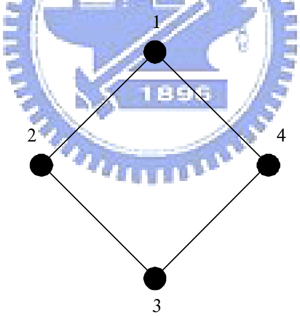
Definition 2.38. (1) A *matching* M in a graph X is a set of edges such that each pair of edges does not have a common vertex.

(2) A *maximum matching* is a matching with the maximum possible number of edges.

(3) A matching M that covers every vertex of X is called a *perfect matching.*

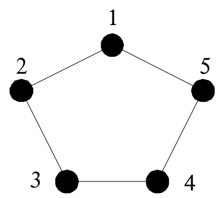
Note 2.39. *If X has a perfect matching then $|X|$ is even.*

Example 2.40. (1)



$M = \{12, 34\}$ is a maximum matching and also a perfect matching.

(2)



$M = \{12, 34\}$ is a maximum matching, but **not** a perfect matching.

Theorem 2.41. *Let X be a connected vertex transitive graph. Then $|M| \geq \lfloor \frac{|X|}{2} \rfloor$ for any maximum matchings M of X.*

Proof. It is suffices to prove for any distinct vertices u, $v \in X$, either u is in an edge of M, or v is in an edge of M. We prove by induction on the distance of $\delta(u, v)$. $\delta(u, v) = 1$ is clear, otherwise we can add $e = uv$ into M a contradiction to M being maximum.

Suppose $\delta(u, v) \geq 2$. Choose $x \in X$ such that $\delta(x, v) = 1$ and $\delta(u, x) + \delta(x, v) = \delta(u, v)$. Suppose u, v do not appear in any edges of M. Since $\delta(u, x) < \delta(u, v)$ and by induction, x is in an edge of M. Pick $\rho \in Aut(X)$ such that $\rho(u) = x$. Then $M' := \rho(M)$ is a maximum matching and x is not in an edge in M'. Hence u is in an edge of M' by induction. We set $M \triangle M' := (M - M') \cup (M' - M)$(view as a subgraph of X). Observe each vertex in $M \triangle M'$ has degree 1 or 2, and $deg(u) = deg(x) = 1$ in $M \triangle M'$. Let P be a path in $M \triangle M'$ with u as its endpoint. Observe each second edge from u in P is in M. Hence $|P \cap M| = |P \cap M'|$ or $|P \cap M| + 1 = |P \cap M'|$. The latter is impossible, otherwise $M \triangle P = (M \setminus P) \cup (P \setminus M)$ is a matching of size $|M| + 1$ a contradiction. Thus $M' \triangle P$ is a maximum matching and u is not in an edge of $M' \triangle P$. Then x is in an edge of $M' \triangle P$ by induction. Hence x is in an edge of P, since x is not in an edge of M'. Thus x is the other endpoint of P. Since x is not in an edge of M', and x, v are adjacent, we obtain that v is in an edge of M'. Hence $deg(v) = 1$ in $M \triangle M'$. As above arguments, we can find a path P' in $M \triangle M'$ from v to x which x is in the last edge of P'. Since $deg(u) = deg(v) = deg(x) = 1$ and other vertices of P and P' have degree 2, we have $P = P'$ and $u = v$, a contradiction. □

Lemma 2.42. *Let e be an edge of X that is not contained in any maximum matchings of X. Then for any $\phi \in Aut(X)$, $\phi(e)$ is not contained in any maximum matchings of X.*

Proof. Suppose $\phi(e)$ is contained in a maximum matching M. Since $\phi^{-1} \in Aut(X)$, we know that $\phi^{-1}(M)$ is also a maximum matching. But e is contained in $\phi^{-1}(M)$ a contradiction. □

Theorem 2.43. *Let X be a connected vertex transitive graph. Then each edge of X is in a maximum matching.*

Proof. Let e be an edge that is not in any maximum matchings of X. For $e = xy$, $\rho(e) := \rho(x)\rho(y)$ is an edge in X for any $\rho \in Aut(X)$. Let $Y :=$

$\{\rho(e) \mid \rho \in Aut(X)\}$(view as a subgraph). Since X is vertex transitive, Y is a spanning subgraph of X, and Y is transitive. We prove this theorem by induction on $|X| + |R|$.

Suppose $Y = X$, we pick a maximum matching M and an edge $e' \in M$. Then we choose $\rho \in Aut(X)$ such that $\rho(e) = e'$. Hence $e \in \rho^{-1}(M)$.But $\rho^{-1}(M)$ is a maximum matching a contradiction. Suppose $Y \neq X$ and $Y = Y_1 \cup Y_2 \cup Y_3 \cup \cdots \cup Y_t$ (union of connected components). Observe Y_i is isomorphic to Y_j for any i, j. Suppose $e \in Y_1$, by induction, there exists a maximum matching M_1 of Y_1 containing e. We observe for $\rho_j \in Aut(X)$ with $\rho_j(Y_1) = Y_j$, $\rho_j(M_1)$ is a maximum matching of Y_j. If M_1 is perfect then $M_1 \cup \rho_2(M_1) \cup \cdots \cup \rho_t(M_t)$ is perfect in Y (and then in X) a contradiction.

Suppose M_1 misses exactly one vertex. Then so does $\rho_j(M_1)$ for $j = 2, \cdots, t$. We define a new graph Z with t vertices $\{Y_1, Y_2, \cdots, Y_t\}$ and Y_i, Y_j are adjacent if and only if there exists $y_i \in Y_i$, $y_j \in Y_j$ such that y_i, y_j are adjacent in X. Note that Z is connected vertex transitive. We can find a maximum matching of Z. Let Y_iY_j be an edge in Z. We choose $y_i \in Y_i$, $y_j \in Y_j$ such that y_i, $y_j$are adjacent in X. Notice if there is one Y_k not in the matching, we pick any vertex y_k in Y_k. We collect the maximum matchings in Y_i that misses y_i for each $i = 1, \cdots, t$, together those y_iy_j appears in the matching of Z. This will form a maximum matching of Y(then of X). This contradicts the fact that each edge of Y is not in any maximum matching of X. □

2.5 Cycles

We show the maximal length of a cycle in a vertex transitive graph is at least $\sqrt{3n}$, where $n = |X| \geq 3$.

Lemma 2.44. *Let G be a finite group and let G act on a finite set S. Fix $x \in S$. Let $G_x := \{f \mid f \in G, f(x) = x\}$.*

(1) *G_x is a subgroup of G.*

(2) *Fix $y \in S$, and $h \in G$ such that $h(x) = y$. Then $\{f \mid f \in G, f(x) = y\} = hG_x$.*

(3) *Suppose G acts transitively on S. Then $|S| = \dfrac{|G|}{|G_x|}$.*

(4) *Let $G \subseteq Aut(X)$ be a group, and $C = \{g \in G \mid x \sim g(x)\}$. Suppose G acts transitively on X. Then X is isomorphic to G/G_x, where G/G_x is the graph with vertices being the left cosets of G_x and two left cosets gG_x, hG_x have an edge if and only if $g^{-1}h \in C$.*

Proof. (1) For $f, g \in G_x$,

$$fg^{-1}(x) = fg^{-1}(g(x)) = f(x) = x.$$

Hence $fg^{-1} \in G_x$. This proves G_x is a subgroup of G.

(2) (a) $\{f \mid f \in G, f(x) = y\} \subseteq hG_x$.
Pick $f_1 \in \{f \mid f \in G, f(x) = y\}$. Observe $h^{-1}(y) = x$. Hence

$$h^{-1}f_1(x) = h^{-1}(y) = x.$$

Then $h^{-1}f_1 \in G_x$. Hence $f_1 \in hG_x$.

(b) $hG_x \subseteq \{f \mid f \in G, f(x) = y\}$.
Pick $f_2 \in G_x$. Then $hf_2(x) = h(x) = y$. Hence $h_2 f \in \{f \mid f \in G, f(x) = y\}$.

From (a), (b) $\{f \mid f \in G, f(x) = y\} = hG_x$.

(3) From (2), there is a $1-1$ correspondence between the set S and the left cosets of G_x.

(4) Fix $x \in X$. Define $\phi : X \to G/G_x$ by $\phi(y) = hG_x$, where $y \in X$ and $h \in G$ satisfying $h(x) = y$. ϕ is well-defined since G acts transitively on X, and by (2) and the fact from group theory that for all $h' \in hG_x$, $h'G_x = hG_x$. It is also clear from (2) that ϕ is one to one and onto. Last, for any $y, z \in X$(say $\phi(y) = hG_x$ and $\phi(z) = gG_x$),

$$\begin{aligned} y \sim z \text{ (in } X) &\Leftrightarrow h(x) = y \sim z = g(x) \text{ (in } X) \\ &\Leftrightarrow g^{-1}h(x) \sim x \text{ (in } X) \\ &\Leftrightarrow g^{-1}h \in C \\ &\Leftrightarrow hG_x \sim gG_x \text{ (in } G/G_x). \end{aligned}$$

□

Lemma 2.45. *Let X be a vertex transitive graph and S be a subset of X where $c := \min_{g \in Aut(X)} |S \cap g(S)|$. Then $|S| \geq \sqrt{c|X|}$.*

Proof. Set $G = Aut(X)$. Observe

$$c|G| \leq |\{(g,\ x) \mid g \in G,\ x \in S \cap g(S)\}|. \tag{2.2}$$

Note that for each $x \in S$ there are $|S||G_x|$ $g \in G$ such that $g^{-1}(x) \in S$ by Lemma 2.44(2). Hence

$$|\{(g,\ x) \mid g \in G,\ x \in S \cap g(S)\}| = |S|^2|G_x|. \tag{2.3}$$

From equations(2.2), (2.3), $|S|^2 \geq \dfrac{c|G|}{|G_x|}$. Since X is vertex transitive, $\dfrac{|G|}{|G_x|} = |X|$ by Lemma 2.44(3). Hence $|S|^2 \geq c|X|$ and the Lemma follows. □

Lemma 2.46. *Let X be a graph with $\kappa_0(X) \geq 3$. Then any two cycles of maximum length intersect at least three vertices.*

Proof. Let C_1, C_2 be two cycles of maximum length. Suppose C_1, C_2 intersect less than three vertices. We divide the proof into 3 cases.
Case 1: C_1, C_2 intersect in two vertices s, t: Since $X - \{s, t\}$ is connected, we can find a path P from a vertex $x \in C_1 - C_2$ to a vertex $y \in C_2 - C_1$ such that x, y are the only two vertices that P intersects C_1 and C_2. Without loss of generality, assume the length of the path $s \overset{C_1}{-} x \overset{P}{-} y \overset{C_2}{-} t$ is longer than the path $s \overset{C_1}{-} t$. Then

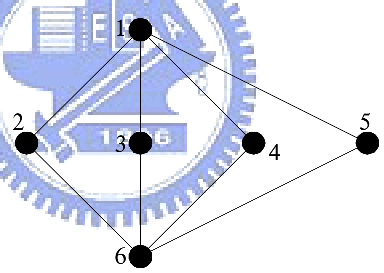
$$s \overset{C_1}{-} x \overset{P}{-} y \overset{C_2}{-} t \overset{C_1}{-} s$$

is a cycle of length larger than C_1, a contradiction.

Case 2: C_1, C_2 intersect in a unique vertex s : Since $X - \{s\}$ is connected, we can find $x \in C_1 - C_2$ and $y \in C_2 - C_1$ such that the distance $\delta(x, y)$ is minimum among all such pairs. Find a shortest path P from x to y. Clearly, P intersects C_1 and C_2 in x, y only. Now go from s to x by a longer path in C_1, then from x to y by P, then from y to s by a longer path in C_2. This is a cycle of length longer than the length of C_1 a contradiction.

Case 3: Suppose C_1, C_2 have no common vertices. We need to find two disjoint paths from C_1 to C_2. If we can do so, we can use these two paths as "bridges" to construct a cycle of larger length in a similar way to previous two cases and obtain a contradiction. Pick $s \in C_1$ and $t \in C_2$ such that the distance $\delta(s, t)$ is the distance from C_1 to C_2. Let P be the shortest path from s to t. Clearly, $P \cap C_1 = \{s\}$, $P \cap C_2 = \{t\}$. The difficulty is to find another path P' from another vertex s' in C_1 to another vertex t' in C_2, and that P, P' have no common vertices. To prove the existence of P', we quote a theorem that states that in a k-connected graph, every $k+1$ vertices $x_0, x_1, \cdots, x_k$ can form a *fan*. That means there are k paths from x_0, to each x_i with x_0 being the only common vertex. Now we apply this theorem to find such P'. Pick any $s' \in C_1 - \{s\}$. There are two disjoint paths P_1, P_2 from s' to some vertices t_1 and t_2(respectively) in $C_2 - \{t\}$. Replacing s', t_1, t_2 if possible, we can assume $P_1 \cap C_1 = \{s_1\}$, $P_2 \cap C_1 = \{s_2\}$, $P_1 \cap C_2 = \{t_1\}$, $P_2 \cap C_2 = \{t_2\}$, $P_1 \cap P_2 - (C_1 \cup C_2) = \emptyset$, where s_1, $s_2 \neq s$, t_1, $t_2 \neq t$ and $t_1 \neq t_2$. If P_1 does not intersect P, then $P = P_1$ and we are done. Hence we assume $P_1 \cap P \neq \emptyset$. Similarly, we assume $P_2 \cap P \neq \emptyset$. We construct two disjoint paths Q_1, Q_2 by using P, P_1, P_2. Q_1 is the path starting from s following the path P to the first vertex that P intersects P_1 or P_2(say P_1), and then following the path P_1 to the end. With this Q_1, we set $Q_2 = P_2$. It is clear from the construction that $Q_1 \cap Q_2 = \emptyset$. □

Example 2.47. The following graph X has vertex connectivity $\kappa_0(X) = 2$.



Let $C_1 = \{1, 2, 6, 4\}$ and $C_2 = \{1, 3, 6, 5\}$. Observe C_1, C_2 are cycles of maximum length. But $| C_1 \cap C_2| = 2$.

Theorem 2.48. *Let X be a connected vertex transitive graph with $n \geq 3$ vertices. Then X contains a cycle of length at least $\sqrt{3n}$.*

Proof. We observe the valency of X is k and $k \geq 2$ since $|n| \geq 3$. If $k = 2$ then we find X is a cycle and the theorem follows since $n \geq \sqrt{3n}$. Suppose $k \geq 3$. Then by Theorem 2.37, $\kappa_0(X) \geq \frac{2}{3}(k+1) \geq \frac{8}{3}$, so $\kappa_0(X) \geq 3$. From previous lemma we obtain $|C \cap g(C)| \geq 3$ for any cycle C of maximum length and $g \in Aut(X)$. By Lemma 2.45, $|C| \geq \sqrt{3|n|}$. □

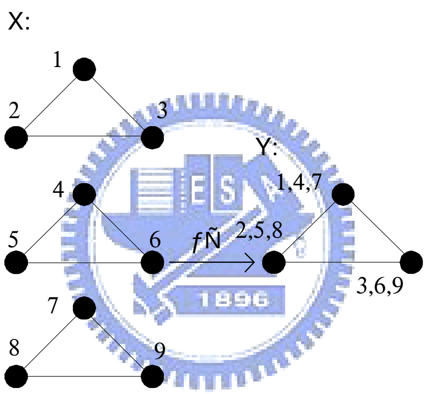
2.6 Retract

In Theorem 2.18, we showed a Cayley graph is vertex transitive. In this section, we show every vertex transitive graph is a retract of a Cayley graph.

Definition 2.49. A subgraph Y of X is a *retract* if there exists a homomorphism ρ from X to Y such that $\rho(y) = y$ for all $y \in Y$. Then ρ is called a *retraction* from X into Y.

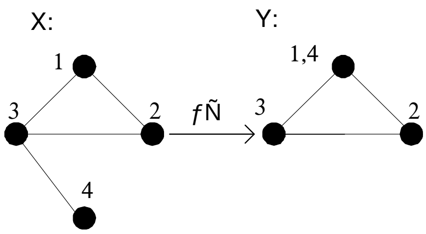
Example 2.50. (1) X is a retract of X. Let $I : X \to X$, I is a retraction.

(2)



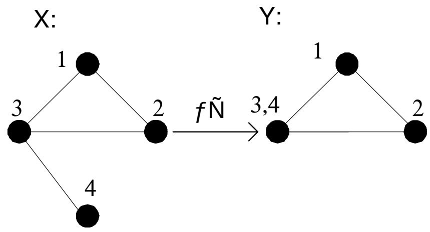
Y is a retract of X.

(3)



f is a retraction.

(4)



f is **not** a retraction.

Theorem 2.51. *Any connected vertex transitive graph is isomorphic to a retract of a Cayley graph.*

Proof. Fix $x \in X$. Let $C = \{g \in Aut(X) \mid x \sim g(x)\}$, and let G be the subgroup of $Aut(X)$ generated by C. Note that G acts on X transitively, since the orbit containing x of the action of G is a regular graph with the same valency as X and this will make the orbit is X. Let $X' = X(G, C)$ be the Cayley graph. Let $H = G_x$ be the stablizer of x under the action of G. Let $Z = \{g_1H, g_2H, \cdots, g_tH\}$ be the left cosets of H, where g_i are fix representatives of these cosets. View Z as the induced subgraph $\{g_1 \cdots g_t\}$of X'. We claim the map $\psi : Z \to X$ defined by $\psi(g_i) = g_i(x)$ is an isomorphism. ψ is a bijection since ψ is the standard one to one correspondence between the left cosets of H and the vertices in X. Observe

$$\begin{aligned} g_i \sim g_j \text{ in } Z &\Leftrightarrow g_i^{-1}g_j \in C \\ &\Leftrightarrow x \sim g_i^{-1}g_j(x) \text{ (in } X) \\ &\Leftrightarrow g_i(x) \sim g_j(x) \\ &\Leftrightarrow \psi(g_i) \sim \psi(g_j). \end{aligned}$$

This prove the claim. We will identify Z and X, and to prove the theorem, it remains to show that Z is a retract of X'. Define $\phi : X' \to Z$ by $\phi(w) = g_i$, where $w \in g_iH$. Clearly $\phi(g_i) = g_i$. Observe for $w_1 = g_ih_1$, $w_2 = g_jh_2 \in X'$,

$$\begin{aligned} w_1 \sim w_2 \text{ (in } X') &\Leftrightarrow w_1^{-1}w_2 \in C \\ &\Leftrightarrow h_1^{-1}g_i^{-1}g_jh_2 \in C \\ &\Leftrightarrow x \sim h_1^{-1}g_i^{-1}g_jh_2(x) \text{ (in } X) \\ &\Leftrightarrow x = h_1(x) \sim g_i^{-1}g_jh_2h_2(x) = g_i^{-1}g_j(x) \text{ (in } X) \\ &\Leftrightarrow g_i^{-1}g_j \in C \\ &\Leftrightarrow \phi(w_1) = g_i \sim g_j = \phi(w_2) \text{ (in } Z). \end{aligned}$$

This completes the proof of the theorem. □

Chapter 3

Homomorphisms

3.1 Cores

Before giving our first definition in this chapter, we consider the following remark first.

Remark 3.1. (1) $\phi : N_2 \to K_2$ is a bijective homomorphism, but ϕ is not an isomorphism.

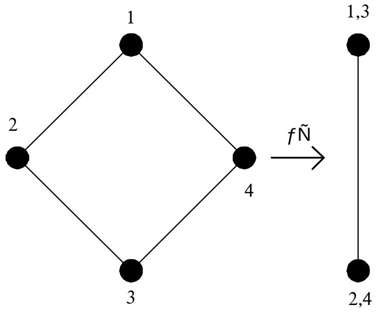
(2) Suppose $|X| < \infty$. Then any bijective homomorphism $\phi : X \to X$ is a isomorphism.

(3) Suppose $|X| < \infty$. Suppose $\phi : X \to X'$, $\psi : X' \to X$ are bijective homomorphisms. Then there is an isomorphism $\varphi : X \to X'$.

Definition 3.2. A graph X is a *core* if for any homomorphism $\rho : X \to X$, $\rho \in Aut(X)$.

Example 3.3. (1) K_n is a core since K_n has no loop.

(2)



For $f(1) = f(3) = 1$ and $f(2) = f(4) = 2$, The cycle C_4 of four vertices is **not** a core.

Definition 3.4. $\chi(X)$ is the smallest positive integer n such that there is a homomorphism $\rho : X \to K_n$. $\chi(X)$ is called the *chromatic number* of X.

Definition 3.5. A subgraph Y of X is a core of X if

(1) Y is a core.

(2) There is a homomorphism from X to Y.

Lemma 3.6. *A core of X is a retract of X.*

Proof. Let Y be a core of X. Then there is a homomorphism $f : X \to Y$. The restriction of f into the domain Y is a homomorphism of Y into itself. Since Y is a core, this restriction is an automorphism, so it has an inverse $(f \restriction Y)^{-1}$. Then $(f \restriction Y)^{-1} \circ f$ is the desired retraction map. □

From Lemma 3.6, we immediately have the following Lemma.

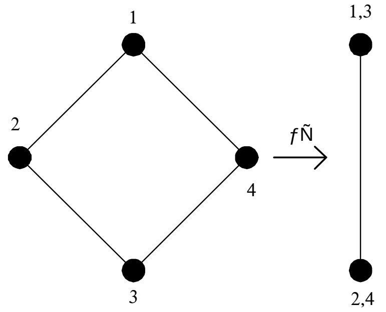
Lemma 3.7. *A core of X is an induced subgraph of X.*

Proof. Obviously by previous Lemma. □

Definition 3.8. A graph X is *critical* if $\chi(Y) < \chi(X)$ for any proper subgraph Y of X.

Note 3.9. *For a subgraph Y of X, $\chi(Y) \leq \chi(X)$.*

Example 3.10. For the following graphs $K_2 \subseteq C_4$ and homomorphism $f : C_4 \to K_2$.

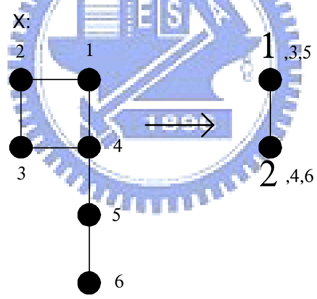


$\chi(C_4) = \chi(K_2) = 2$. Hence C_4 is **not** critical.

Lemma 3.11. *If X is critical then X is a core.*

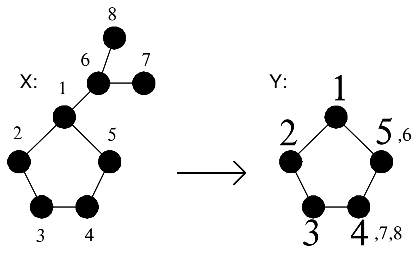
Proof. Suppose not. Let $\rho : X \to Y$, $Y \subsetneq X$ be a homomorphism. Set $\chi(Y) = n$ and let $\psi : Y \to K_n$ be a homomorphism. Then $\psi \circ \rho : X \to K_n$ is a homomorphism. Hence $\chi(X) \leq n = \chi(Y)$. Thus X is not critical, a contradiction. □

Example 3.12. (1)



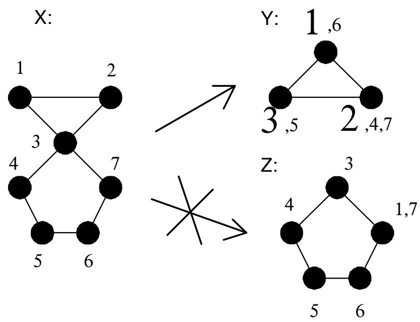
K_2 is a core of X. Similarly, any edge is a core of X.

(2)



Y is a core of X.

(3)



Hence Y is a core of X but Z is **not** a core of X.

Theorem 3.13. *Any two cores of X are isomorphic.*

Proof. Let Y, $Y' \subseteq X$ be two cores of X and $\varphi : X \to Y$, $\psi : X \to Y'$ are the corresponding homomorphisms. Then $\psi \circ \varphi \restriction Y' : Y' \to Y'$ is a homomorphism. Since Y' is a core, we know $\psi \circ \varphi \restriction Y' : Y' \to Y'$ indeed is an automorphism. Then $\varphi \restriction Y' : Y' \to Y$ is one to one. On the other hand, since $\varphi \circ \psi \restriction Y : Y \to Y$ is a homomorphism and Y is a core, we have $\varphi \circ \psi \restriction Y : Y \to Y$ is an automorphism. This shows $\varphi \restriction Y' : Y' \to Y$ is onto. Hence $\varphi \restriction Y' : Y' \to Y$ is a bijection. It is an isomorphism. □

Lemma 3.14. *Every graph has a core.*

Proof. Let X be a graph. Set $S = \{Y \subseteq X \mid \text{there exists a homomorphism } f : X \to Y\}$. Pick $Y \in S$ with least vertices. We claim $Y \in S$ is a core. Let $\rho : X \to Y$ be a homomorphism. Suppose Y is not a core. Let $\psi : Y \to Y$ be a homomorphism which is not onto. Then $\psi \circ \rho : X \to Y$ is a homomorphism with image $\psi \circ \rho(X) \subsetneq Y$, a contradiction to the choice of Y. □

From Theorem 3.13 and Lemma 3.14, we have a conclusion: Every graph X has a unique core (up to isomorphism). We denoted it by $X^\bullet$.

Theorem 3.15. *Suppose X is vertex transitive. Then $X^\bullet$ is vertex transitive.*

Proof. Pick any x, $y \in X^\bullet$, choose $f \in Aut(X)$ such that $f(x) = y$. Pick a retraction $g : X \to X^\bullet$. Then

$$g \circ (f \restriction X^\bullet) : X^\bullet \to X^\bullet$$

is a homomorphism. Observe $X^\bullet$ is a core and $g \circ (f \restriction X^\bullet) \in Aut(X^\bullet)$. Note $g \circ (f \restriction X^\bullet)(x) = g(f(x)) = g(y) = y$. Hence $X^\bullet$ is vertex transitive. □

Theorem 3.16. *If X is a vertex transitive graph, then $|X^\bullet|$ divides $|X|$.*

Proof. Let $f : X \to X^\bullet$ be a homomorphism. We want to prove $|f^{-1}(y)|$ is independent of $y \in X^\bullet$. We claim for any $g \in Aut(X)$, for any $y \in X^\bullet$, $|f^{-1}(y) \cap g(X^\bullet)| = 1$. Since $f \circ (g \restriction X^\bullet) : X^\bullet \to X^\bullet$ is a homomorphism, $f \circ (g \restriction X^\bullet) \in Aut(X^\bullet)$. Observe

$$1 = |f \circ (g \restriction X^\bullet)^{-1}(y)| = |(g \restriction X^\bullet)^{-1}(f^{-1}(y))|.$$

Thus $|f^{-1}(y) \cap g(X^\bullet)| = 1$, since $g \restriction X^\bullet$ is one to one. This claim says for each $y \in X^\bullet$, $g \in Aut(X)$, there exists a unique pair (z, x) such that $z \in X^\bullet$, $x \in f^{-1}(y)$ and $g(z) = x$. On the other hand by Lemma 2.44(2), for each pair (z, x)such that $z \in X^\bullet$, $x \in f^{-1}(y)$ there are $|G_x|$ elements $g \in Aut(X)$ such that $g(z) = x$, where G_x is the stablizer of X under the action of $Aut(X)$. (i.e. $G_x = \{f \mid f(x) = x, f \in Aut(X)\}$). Note $|G_x|$ is independent of x. Hence $|Aut(X)| = |X^\bullet||f^{-1}(y)||G_x|$. Thus $|f^{-1}(y)| = \dfrac{|Aut(X)|}{|X^\bullet||G_x|}$ is independent of y. □

Corollary 3.17. *If X is a vertex transitive graph such that $|X|$ is a prime number and X has at least one edge, then X is a core.*

Proof. From Theorem 3.16, we know $|X^\bullet|$ divides $|X|$. So $|X^\bullet| = 1$ or $|X|$. Observe $|X^\bullet| \neq 1$, since X has at least one edge. Hence $|X^\bullet| = |X|$. We have $X = X^\bullet$ by Lemma 3.7. □

Corollary 3.18. *If X is a vertex transitive graph with $\chi(X) = 3$ and $3 \nmid |X|$, then X has no triangle.*

Proof. There exists a homomorphism $f : X \to K_3$ because $\chi(X) = 3$. Suppose X has a triangle. Then there is no $Y \subseteq X$ such that $|Y| \leq 2$ and there exists a homomorphism $g : X \to Y$. Hence K_3 is a core of X. Hence 3 divides $|X|$ and by Theorem 3.16 a contradiction. □

3.2 Folding

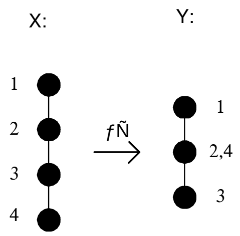
Definition 3.19. Let X be a graph and $Y \subseteq X$ is a induced subgraph. A retraction $f : X \to Y$ is *simple folding* if

(1) $|X| = |Y| + 1$,

(2) If $u, v \in X$ with $f(u) = f(v)$ then $u = v$ or $\partial(u, v) = 2$ (in X).

Note 3.20. *We always assume $Y \subseteq X$, and f is a retraction.*

Example 3.21.



f is a simple folding.

Definition 3.22. Suppose Y is an induced subgraph of X. Then a retraction $f : X \to Y$ is a *folding*, if either $X = Y$ or there exist induced subgraphs $Y_1, Y_2, \cdots, Y_n = Y$ of X and simple foldings $f_1 : X \to Y_1$, $f_2 : Y_1 \to Y_2$, $\cdots$, $f_n : Y_{n-1} \to Y_n$ such that $f = f_n \circ \cdots \circ f_2 \circ f_1$ for X is connected.

Lemma 3.23. *Suppose Y is an induced subgraph of X and $f : X \to Y$ is a retraction. Then f is a folding.*

Proof. Induction on $|X| - |Y|$. If $X = Y$ then f is a folding by the definition. Suppose $Y \subsetneq X$. Pick $y \in Y$ and $x \in X \setminus Y$ such that $x \sim y$. Define Y_1 by identifying x and $f(x)$ in X, hence $|Y_1| = |X| - 1$. Define $f_1 : X \to Y_1$ by

$$f_1(u) = \begin{cases} u, & \text{if } u \neq x, \\ f(x), & \text{if } u = x. \end{cases} \tag{3.1}$$

Then $f_1 : X \to Y_1$ is a simple folding. Define $f_2 : Y_1 \to Y$ by $f_2(u) = f(u)$. Then $f = f_2 \circ f_1$. Observe $f_2 : Y_1 \to Y$ is a retraction and $|Y_1 - Y| = |X - Y| - 1$. By induction, f_2 is a folding, hence $f = f_2 \circ f_1$ is a folding. □

Definition 3.24. A homomorphism $f : X \to Y$ is a *local injection*, if for any $y \in Y$, and for any $u, v \in f^{-1}(y)$, $u = v$ or $\partial(u, v) \geq 3$.

Lemma 3.25. *Let X be a connected graph and Y be a induced subgraph of X. Suppose $f : X \to Y$ is a homomorphism. Then for any $y_1, y_2 \in Y$ with $f(y_1) = y_1$ and $f(y_2) = y_2$, we have $\partial_Y(y_1, y_2) = \partial_X(y_1, y_2)$.*

Proof. $\partial_Y(y_1, y_2) \geq \partial_X(y_1, y_2)$ since $Y \subseteq X$, and $\partial_Y(y_1, y_2) \leq \partial_X(y_1, y_2)$ since f is a homomorphism. Hence $\partial_Y(y_1, y_2) = \partial_X(y_1, y_2)$. □

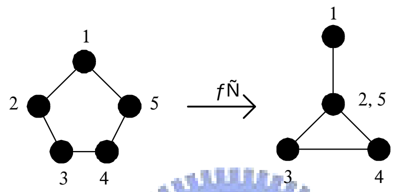
Lemma 3.26. *Suppose X be a graph and Y is a proper induced subgraph of X. If $f : X \to Y$ is a folding, then $f : X \to Y$ is not a local injection.*

Proof. Suppose $f = f_t \circ \cdots \circ f_2 \circ f_1$ where f_i are simple folding with $f_i : Y_{i-1} \to Y_i$. Pick $y \in Y$ and u, $v \in Y_{t-1}$ such that $\partial_{Y_{t-1}}(u,v) = 2$ and $f_t(u) = f_t(v) = y$. Then $f(u) = f(v)$ and $\partial_X(u,v) = \partial_{Y_{t-1}}(u,v) = 2$. $\square$

Lemma 3.27. *Let n be odd, Y be a graph. If $\phi : C_n \to Y$ be a homomorphism with C_n be a cycle of length n. Then Y contains an odd cycle.*

Proof. Suppose Y does not contain odd cycles. Then Y is bipartite. Observe $\phi(C_n)$ is a closed walk of odd length in Y, a contradiction. $\square$

Example 3.28.



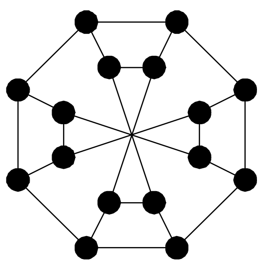
A example with odd cycle.

Definition 3.29. For x, y, $z \in X$ if $x \sim y \sim z$ and $x \neq z$, then $\{x, y, z\}$ is called a *2-arc* of X.

Theorem 3.30. *If X is a connected graph and every 2-arc of X is in a shortest odd cycle, then X is a core.*

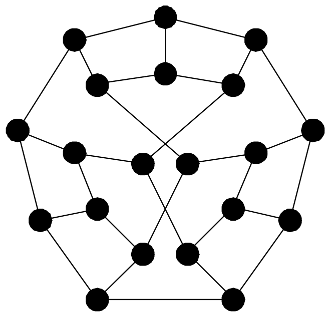
Proof. Suppose $f : X \to X^\bullet$ is a retraction and $X^\bullet \neq X$. Then f is a folding. Hence f is not a local injection. Hence there exist u, $v \in X$ with $\partial(u,v) = 2$ and $f(u) = f(v)$. Observe u, v are contained in a shortest odd cycle of C. And $f(u)$, $f(v)$ are contained in the odd cycle $f(C)$ which has the same length as C. This implies $f(u) \neq f(v)$, a contradiction. $\square$

Example 3.31. (1)



The length of shortest odd cycle is seven. By Theorem 3.30 the graph is a core.

(2)



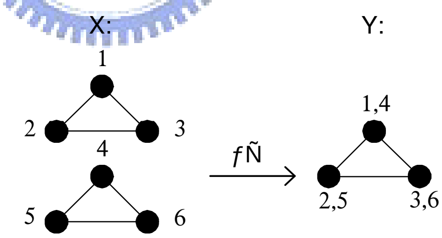
The length of shortest odd cycle is seven. By Theorem 3.30 the graph is a core.

Definition 3.32. Let X, Y be graphs. A homomorphism $f : X \to Y$ is *local bijective*(respectively *isomorphic*) if for any $y \in Y$, there exists $x \in X$ such that

(1) $f(x) = y$,

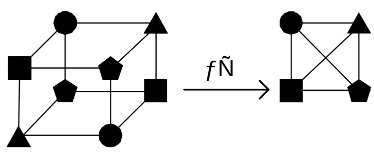
(2) $f \restriction N[x] : N[x] \to N[y]$ is bijective(respectively isomorphic) where $N[x] = N[\{x\}]$.

Example 3.33. (1)



Observe f is local bijective and local isomorphic.

(2)



Observe f is local bijective, but is **not** local isomorphic.

Lemma 3.34. *If X is a connected graph and $f : X \to Y$ is local isomorphic. Then $f : X \to Y$ is isomorphic.*

Proof. We only need to prove f is one to one. Suppose not. Pick x, $y \in X$ such that $f(x) = f(y)$ and $\partial(x, y)$ is minimum. Note $\partial(x, y) \geq 2$. Let x, z, $\cdots$, y be the shortest path from x to y. Then $f(x)$, $f(z)$, $\cdots$, $f(y) = f(x)$ is a cycle in Y. Hence $f(y) \sim f(z)$ in Y. Thus $y \sim z$ in X. Since $f(y)$, $f(x) \in N(f(z))$ and $f(y) = f(x)$, we must have $y = x$ by the assumption of local isomorphism, a contradiction to $\partial(x, y) \geq 2$. □

Corollary 3.35. *If Y is a tree, and $f : X \to Y$ is local bijective. Then X is disjoint copies of Y.*

Proof. Since for each $y \in Y$, $N(y)$ contains no edges, f in fact is a local isomorphism. Then the corollary follows from Lemma 3.34. □

Chapter 4

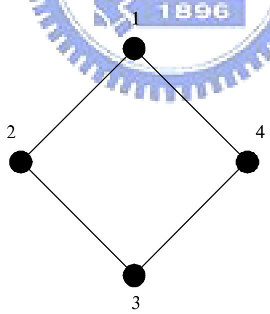
The Adjacency Matrix

4.1 Definition

Definition 4.1. The *adjacency matrix* $A = A(X)$ of a graph X is the matrix with rows and columns indexed by X such that

$$A_{xy} = \begin{cases} 1, & \text{if } x \sim y, \\ 0, & \text{if } x \nsim y, \ (x, y \in X.) \end{cases}$$

Example 4.2. X:



For the graph X, the adjacency matrix $A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$.

Definition 4.3. A *walk* of length r in X is a sequence of vertices x_0, x_1, x_2 ,$\cdots$, x_r such that $x_i \sim x_{i+1}$ for $i = 0, 1, \cdots r - 1$.

Lemma 4.4. *Let $A = A(X)$ be the adjacency matrix of X. For $x, y \in X$ the number of walks of length r from x to y is $(A^r)_{xy}$.*

Proof. $(A^r)_{xy} = \sum_{x_1,x_2,\cdots,x_{r-1}\in X} A_{xx_1}A_{x_1x_2}\cdots A_{x_{r-1}y} = |\{(x_1, x_2, \cdots, x_{r-1}) \mid x \sim x_1 \sim x_2 \sim \cdots \sim x_{r-1} \sim y\}|$. □

4.2 Spectrum

Definition 4.5. Let $A = A(X)$ be the adjacency matrix of X. Then θ is an *eigenvalue* of A if there exists a nonzero column vector $U \in \mathbb{C}^x$ such that $AU = \theta U$. Then U is called an *eigenvector* of A associated with θ.

Note 4.6. *An $n \times n$ symmetric matrix over $\mathbb{R}$ has n orthogonal eigenvectors over $\mathbb{R}$. The multiset of the eigenvalues of $A(X)$ is called the spectrum of X.*

Throughout this chapter, we assume the base field is $\mathbb{R}$.

Example 4.7. Let $X = K_n$ and we have the adjacency matrix $A = A(X)$. Observe $A + I = J$ (J is all 1's matrix). Hence we have rank$(J) = 1$ and J has $n-1$ orthogonal eigenvectors $U_1, U_2, \cdots, U_{n-1}$ associated with 0. Set $U_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. So $JU_n = nU_n$. Then

$$AU_i = (J - I)U_i = \begin{cases} -U_i, & \text{for } i \le n-1, \\ (n-1)U_n, & \text{for } i = n. \end{cases}$$

Hence A has eigenvalues $-1, -1, \cdots, -1(n-1 \text{ times}), n-1$.

Lemma 4.8. *Let X be a regular graph with valency k. Then*

(1) *The valency k is an eigenvalue of $A = A(X)$.*

(2) *For any eigenvalues θ of A, $|\theta| \le k$.*

(3) *The multiplicity of k is the number of connected components in X.*

Proof. (1) Observe $A\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=\begin{pmatrix}k\\k\\\vdots\\k\end{pmatrix}=k\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$. So k is an eigenvalue of A.

(2) Suppose $AU=\theta U$ for some $U=\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}\neq 0$ where $n=|X|$. Pick j such that $|u_j|=\max_i |u_i|$. Hence

$$|\theta u_j|=|(\theta U)_j|=|(AU)_j|=|\sum_i A_{ji}u_i|\leq\sum_i A_{ji}|u_i|\leq k|u_j|.$$

Hence $|\theta|\leq k$.

(3) If $\theta=k$, then all of the above inequalities are equalities. This means $u_i=u_j$ if $i\sim j$. If we replace the role of u_j by u_k for some $k\sim j$ and keep doing this, we obtain that u_i are all the same when i was in the same connected components of X.

□

Throughout the end of this section, we fix a graph X and its adjacency matrix $A(X)$.

Definition 4.9. (1) The set of eigenvalues of $A(X)$ is denoted by $ev(X)$.

(2) For $\theta\in ev(X)$, let $\mathbb{V}(\theta)$ denote the set of eigenvectors of $A(X)$ corresponding to θ. ($\mathbb{V}(\theta)$ is a subspace of $\mathbb{R}$).

(3) For $\theta\in ev(X)$, define a matrix $E_\theta:\mathbb{R}^X\rightarrow\mathbb{R}^X$ such that E_θ is the projection of $\mathbb{R}^X$ into $\mathbb{V}(\theta)$. E_θ is called the *primitive idempotent* of θ.

Lemma 4.10. $E_\theta^2=E_\theta$.

Proof. For any $U\in\mathbb{R}^X$,

$$E_\theta^2U=E_\theta(E_\theta U)=E_\theta U$$

since $E_\theta U\in\mathbb{V}(\theta)$. □

Lemma 4.11. *For θ, $\eta \in ev(X)$ with $\theta \neq \eta$, then $E_\theta E_\eta = 0$.*

Proof. For $U \in \mathbb{R}^X$, then $E_\eta U \in \mathbb{V}(\eta)$. Since $\mathbb{V}(\eta)$ is orthogonal to $\mathbb{V}(\theta)$, $E_\theta(E_\eta U) = 0$. □

Lemma 4.12. $I = \sum_{\theta \in ev(X)} E_\theta$.

Proof. Pick $U \in \mathbb{R}^X$. Then $U = \sum_{\tau \in ev(X)} U_\tau$, for some $U_\tau \in \mathbb{V}(\tau)$. Hence by Lemma 4.10 and Lemma 4.11

$$\begin{aligned} \sum_{\theta \in ev(X)} E_\theta U &= \sum_{\theta \in ev(X)} E_\theta \sum_{\tau \in ev(X)} U_\tau \\ &= \sum_{\theta \in ev(X)} E_\theta \sum_{\tau \in ev(X)} E_\tau U_\tau = \sum_{\theta,\tau} E_\theta E_\tau U_\tau \\ &= \sum_{\theta \in ev(X)} E_\theta U_\theta = \sum_{\theta \in ev(X)} U_\theta = U. \end{aligned}$$

Hence $I = \sum_{\theta \in ev(X)} E_\theta$. □

Lemma 4.13. $A = \sum_{\theta \in ev(X)} \theta E_\theta$.

Proof. Pick $U \in \mathbb{R}^X$. Suppose $U = \sum_{\theta \in ev(X)} U_\theta$ where $U_\theta \in \mathbb{V}(\theta)$. Then

$$\begin{aligned} AU &= \sum_{\theta \in ev(X)} AU_\theta = \sum_{\theta \in ev(X)} \theta U_\theta = \sum_{\theta \in ev(X)} \theta E_\theta U_\theta \\ &= \sum_{\theta \in ev(X)} \theta E_\theta \sum_{\tau \in ev(X)} U_\tau = (\sum_{\theta \in ev(X)} \theta E_\theta) U. \end{aligned}$$

Hence $A = \sum_{\theta \in ev(X)} \theta E_\theta$ □

Lemma 4.14. *For any polynomial f, $f(A) = \sum_\theta f(\theta) E_\theta$.*

Proof. For $U \in \mathbb{V}(\theta)$, $AU = \theta U$,

$$A^2U = A(AU) = A(\theta U) = \theta(AU) = \theta^2 U.$$

So $A^n U = \theta^n U$. Hence $f(A)U = f(\theta)U$.

For $U \in \mathbb{R}^X$ we let $U = \sum_{\theta \in ev(X)} U_\theta$. Hence

$$\begin{aligned} f(A)U &= \sum_\theta f(A)U_\theta = \sum_\theta f(\theta)U_\theta \\ &= \sum_\theta f(\theta)E_\theta U_\theta = \sum_\theta f(\theta)E_\theta U. \end{aligned}$$

□

Lemma 4.15. *For $\theta \in ev(A)$, set $P_\theta(x) = \prod_{\substack{\eta \in ev(A) \\ \eta \neq \theta}} (x - \eta)$. Then $E_\theta = \frac{1}{P_\theta(\theta)} P_\theta(A)$. In particular, E_θ is a polynomial of A with degree $|ev(A)| - 1$.*

Proof. Observe by Lemma 4.14

$$\begin{aligned} P_\theta(A) &= \sum_{\tau \in ev(A)} P_\theta(\tau)E_\tau = \sum_{\tau \in ev(A)} (\prod_{\eta \neq \theta} \tau - \eta)E_\tau \\ &= (\prod_{\eta \neq \theta}(\theta - \eta))E_\theta = P_\theta(\theta)E_\theta. \end{aligned}$$

□

Lemma 4.16. *Suppose $f(x), g(x) \in \mathbb{R}[x]$ and $g(\theta) \neq 0$ for all $\theta \in ev(A)$. Then*

$$\frac{f(A)}{g(A)} = \sum_{\theta \in ev(A)} \frac{f(\theta)}{g(\theta)} E_\theta.$$

Proof. Observe the eigenvalues of $g(A)$ are $g(\theta)$, where $\theta \in ev(A)$. (In fact, A and $g(A)$ have the same set of eigenvectors). Hence $g(A)$ is invertible by the assumption $g(0) \neq 0$. Observe by Lemma 4.14, Lemma 4.11

$$\begin{aligned} g(A) \sum_{\theta \in ev(A)} \frac{f(\theta)}{g(\theta)} E_\theta &= \sum_{\theta \in ev(A)} g(\theta)E_\theta \sum_{\theta \in ev(A)} \frac{f(\theta)}{g(\theta)} E_\theta \\ &= \sum_{\theta \in ev(A)} f(\theta)E_\theta^2 = \sum_{\theta \in ev(A)} f(\theta)E_\theta \\ &= f(A). \end{aligned}$$

Hence

$$\frac{f(A)}{g(A)} = \sum_{\theta \in ev(A)} \frac{f(\theta)}{g(\theta)} E_\theta.$$

□

Lemma 4.17. $\{E_\theta \mid \theta \in ev(A)\}$ *are linear independent.*

Proof. Suppose $\sum_{\theta \in ev(A)} c_\theta E_\theta = 0$. Then for any nonzero $U_\tau \in \mathbb{V}(\tau)$,

$$(\sum_{\theta \in ev(A)} c_\theta E_\theta) U_\tau = 0.$$

Hence

$$(\sum_{\theta \in ev(A)} c_\theta E_\theta) U_\tau = c_\tau E_\tau U_\tau = c_\tau U_\tau = 0.$$

Hence $c_\tau = 0$ for all $\tau \in ev(A)$. □

From Lemma 4.10~Lemma 4.17, we can conclude

$$\langle A \rangle = \langle \{E_\theta \mid \theta \in ev(A)\} \rangle = \text{Span}\{E_\theta \mid \theta \in ev(A)\}$$

where $\langle A \rangle$ is the algebra generated by A. Hence $\dim_{\mathbb{R}} \langle A \rangle = |ev(A)|$.

Theorem 4.18. *Let X be the graph with diameter d. Then $|ev(A)| \geq d+1$.*

Proof. Suppose $|ev(A)| \leq d$. Then I, A, A^2, $\cdots$, A^{d-1} span E_θ for all $\theta \in ev(A)$. Hence they span A^d. That is $A^d = c_0 I + c_1 A + \cdots + c_{d-1} A^{d-1}$ for $c_i \in \mathbb{R}$. Pick $x, y \in X$ with $\partial(x, y) = d$. Then

$$0 \neq (A^d)_{xy} = (c_0 I + c_1 A + \cdots + c_{d-1} A^{d-1})_{xy} = 0$$

a contradiction. Hence $|ev(A)| \geq d+1$. □

Corollary 4.19. *The path P_n of length $n-1$ has n distinct eigenvalues.*

Proof. Let $A = A(P_n)$. Then $|ev(A)| \leq n$ since A is an $n \times n$ matrix. $|ev(A)| \geq n$ from Theorem 4.18. Hence $|ev(A)| = n$. □

4.3 Perron Frobenius Theorem

Lemma 4.20. *Let C be an $n \times n$ symmetric matrix. Assume all eigenvalues of C are nonnegative. Then $C = D^tD$ for some $n \times n$ matrix D.*

Proof. Observe

$$\begin{aligned} C &= P^t \begin{pmatrix} \theta_1 & 0 & 0 & \cdots & 0 \\ 0 & \theta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \theta_n \end{pmatrix} P \\ &= P^t \begin{pmatrix} \sqrt{\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\theta_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \sqrt{\theta_n} \end{pmatrix} \begin{pmatrix} \sqrt{\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\theta_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \sqrt{\theta_n} \end{pmatrix} P \\ &= D^tD \end{aligned}$$

where $D = \begin{pmatrix} \sqrt{\theta_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\theta_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \sqrt{\theta_n} \end{pmatrix} P$, and θ_i are eigenvalues with nonnegative values. □

Definition 4.21. Let C be a symmetric matrix with rows and columns indexed by X. C is *bipartite*(resp. *reducible*) if there exists $Y_1, Y_2 \subseteq X$ such that

(1) $Y_1 \cup Y_2 = X$;

(2) $Y_1 \cap Y_2 = \emptyset$;

(3) Y_1, $Y_2 \neq \emptyset$;

(4) $C_{xy} = 0$ if x, $y \in Y_1$ or x, $y \in Y_2$(resp. $C_{xy} = 0$ if either $x \in Y_1$, $y \in Y_2$, or $x \in Y_2$, $y \in Y_1$).

Lemma 4.22. *Let C be a bipartite symmetric matrix and let θ be an eigenvalue of C. Then $-\theta$ is also an eigenvalue of C and the multiplicity of θ is equal to the multiplicity of $-\theta$ in C.*

Proof. Suppose

$$\left(\begin{array}{c|c} O & B \\ \hline B^t & O \end{array}\right)\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \theta \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.$$

Then $BU_2 = \theta U_1$ and $B^t U_1 = \theta U_2$. Observe

$$\left(\begin{array}{c|c} O & B \\ \hline B^t & O \end{array}\right)\begin{pmatrix} U_1 \\ -U_2 \end{pmatrix} = \begin{pmatrix} -BU_2 \\ B^t U_1 \end{pmatrix} = \begin{pmatrix} -\theta U_1 \\ \theta U_2 \end{pmatrix} = -\theta \begin{pmatrix} U_1 \\ -U_2 \end{pmatrix}.$$

□

Note 4.23. *If C is bipartite, then C^2 is reducible.*

Lemma 4.24. *Let C be an irreducible $n \times n$ symmetric matrix with positive entries. If C^2 is reducible, then C is bipartite.*

Proof. Let X be the graph associated with C. Let Y, Z be a partition of the vertex set of X such that $C^2_{ij} = 0$ if $i \in Y$ and $j \in Z$. This means that two ends of each walk of length 2 must in the same set Y or in the same set Z. Observe there is an edge connecting Y and Z, since X is connected(this is from the irreducible of C). It is not too difficult from above comments that there is no edges and loops in Y and in Z. Hence X is bipartite and then C is bipartite. □

Theorem 4.25. (*Perron Frobenius Theorem*)
Let C be an $n \times n$ symmetric irreducible matrix with nonnegative entries. Let θ_1 be the largest eigenvalue of C and θ_r is the smallest eigenvalue of C. Suppose that V is an eigenvector of C corresponding to θ_1. Then

(1) *All entries of V have the same sign (no zero entries).*

(2) *θ_1 has multiplicity 1.*

(3) *$\theta_r \geq -\theta_1$.*

(4) *$\theta_r = \theta_1$ if and only if C is bipartite.*

Proof. (1) Observe $\theta_1 I - C$ has nonnegative eigenvalues. Hence $\theta_1 I - C = P^t P$ for some matrix P by Lemma 4.20. Observe

$$||PV||^2 = (PV)^t(PV) = V^t P^t PV = V^t(\theta_1 I - C)V = V^t \theta_1 IV - V^t \theta_1 V = 0.$$

Hence $PV = 0$. i.e. $\sum_{x=1}^{n} v_x P_x = 0$, where P_x is the xth column of P. Set $S = \{x \mid v_x > 0\}$, we assume $S \neq \emptyset$(otherwise use $-V$ instead of V). Set $W = \sum_{x \in S} P_x v_x$ and observe $W = -\sum_{y \notin S} P_y v_y$. For $x \in S$,

$$\langle P_x, W\rangle = \langle P_x, -\sum_{y\notin S} P_y v_y\rangle = -\sum_{y\notin S} v_y \langle P_x, P_y\rangle = -\sum_{y\notin S} v_y(\theta_1 I - C)_{xy} \leq 0.$$

Observe

$$0 \leq \langle W, W\rangle = \langle \sum_{x\in S} v_x P_x, W\rangle = \sum_{x\in S} v_x \langle P_x, W\rangle \leq 0.$$

Hence $W = 0$. For $y \notin S$,

$$0 = \langle P_y, W\rangle = \langle P_y, \sum_{x\in S} v_x P_x\rangle = \sum_{x\in S} v_x \langle P_y, P_x\rangle = \sum_{x\in S} v_x(\theta_1 I - C)_{yx}.$$

Since $(\theta_1 I - C)_{yx} \leq 0$ and $v_x > 0$, we have $C_{yx} = 0$ for $y \notin S$, $x \in S$. Hence C is reducible, a contradiction.

(2) Suppose θ_1 has multiplicity at least 2. Let $\mathbb{V}(\theta_1)$ be the eigenspace of C corresponding to θ_1. Then $\dim(\mathbb{V}(\theta_1)) \geq 2$. Since $\dim(\text{span}_{\mathbb{R}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\perp}) = n-1$, $\mathbb{V}(\theta_1) \cap \text{span}_{\mathbb{R}}\{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^{\perp}\} \neq \emptyset$. Hence there exists nonzero vector of the form $\begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix}$ in $\mathbb{V}^2(\theta_1)$, a contradiction to (1).

(3) We consider two cases.

Case 1: C^2 is reducible by Lemma 4.24. Hence C is bipartite. Thus the eigenvalues of C are symmetry to the origin. Hence $\theta_r \geq -\theta_1$.

Case 2: C^2 is irreducible. Observe $C^2V = \theta_1^2 V$. Hence V is an eigenvector of C^2 corresponding to θ_1^2. Let U be an eigenvector of C. Suppose $\theta_r < -\theta_1$. Then θ_r^2 is the maximal eigenvalue of C^2 with corresponding eigenvector U. By (1), the entries of U have the same sign. But U is orthogonal to V, a contradiction. Hence $\theta_r \geq -\theta_1$.

(4) ($\Rightarrow$)Suppose $\theta_r = \theta_1$. Then $\theta_r^2 = \theta_1^2$ are eigenvalues of C^2 with multiplicity at least 2. By (2), C^2 is reducible. Hence C is bipartite by Lemma 4.24.
($\Leftarrow$)Obvious from Lemma 4.22.

□

Chapter 5

Interlacing

5.1 Interlacing of sets

Definition 5.1. Let $S = \{\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m\}$ and $T = \{\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n\}$ are multisets of $\mathbb{R}$, where $n \geq m$. We say S *interlaces* T if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$ for all $i = 1, 2, \cdots, m$.

Example 5.2. (1) Let $S = \{5, 3, 1\}$, $T = \{5, 5, 4, 3, 2, 1\}$. Hence S interlaces T.

(2) Let $S = \{2.5, 1\}$, $T = \{3, 3, 2, 1\}$. Hence S interlaces T.

Note 5.3. *If $S \subseteq T$, then S interlaces T.*

Lemma 5.4. *Suppose S, T, U are multisubsets of $\mathbb{R}$.*

(1) *Suppose S interlaces T. Then S interlaces $S \cup T$.*

(2) *S interlaces T if and only if $S \cup U$ interlaces $T \cup U$.*

(3) *Let $f(x)$, $g(x)$ be real polynomials. Suppose*

$$\frac{f(x)}{g(x)} = \sum_{s \in S} \frac{1}{x - s}$$

for some finite set $S \subseteq \mathbb{R}$. Then the zero's of $f(x)$ interlace the zero's of $g(x)$.

Proof. (1) We claim that "interlacing" is a transitive relation. Let S interlaces T, and T interlaces U. We show then S interlaces U. Let $S = \{\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m\}$, $T = \{\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n\}$, $U = \{\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p\}$ where $p \geq n \geq m$. By the definition, we have

$$\begin{aligned} \theta_i &\geq \eta_i \geq \theta_{n-m+i} \ (1 \leq i \leq m), \\ \gamma_i &\geq \theta_i \geq \gamma_{p-n+i} \ (1 \leq i \leq n). \end{aligned}$$

Hence $\gamma_{n-m+i} \geq \theta_{n-m+i} \geq \gamma_{p-m+i} \geq \gamma_{p-n+i}, (1 \leq i \leq m)$. Hence we obtain $\gamma_i \geq \theta_i \geq \eta_i \geq \theta_{n-m+i} \geq \gamma_{p-m+i}, (1 \leq i \leq m)$. Hence S interlaces U. This proves "interlacing" is a transitive relation. Observe S interlaces T, and T interlaces $T \cup U$. Hence the result follows.

(2) To prove this, we can assume that $U = \{u\}$ has only one element. By a small perturbation on u, we can assume $u \in S \cup T$. Now (2) follows.

(3) By deleting the common linear factors in $f(x)$, $g(x)$, and using (2), we can assume $f(x)$ and $g(x)$ have no common linear factors. From the right hand side, we know $g(x) = \prod_{s \in S}(x - s)$ has degree $n = |S|$ and $f(x)$ has degree at most $n - 1$. Hence $g(x)$ has n zero's, and $f(x)$ has at most $n - 1$ zero's. Since

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \sum_{s \in S} \frac{-1}{(x-s)^2} < 0,$$

the graph of $y = \frac{f(x)}{g(x)}$ decreases. Hence $f(x)$ has exactly $n - 1$ zeros and they appear between two consecutive zeros of $g(x)$.

□

Definition 5.5. The interlacing is *tight* if for each $i = 1, 2, \cdots, m$, one of the equality holds.

Example 5.6. (1) $\{4, 3, 2, 1\}$ interlace $\{4, 3, 3, 3, 2, 1\}$ tightly.

(2) $\{4, 3, 2, 1\}$ interlace $\{4, 4, 2, 2, 1\}$. This interlacing is **not** tight.

5.2 Interlacing of eigenvalues

Theorem 5.7. *Let A be an $n \times n$ real symmetric matrix. Suppose P is an $n \times m$ matrix satisfying $P^tP = I_{m\times m}$ and $B = P^tAP$ when $n \geq m$. Then*

(1) *The eigenvalues of B interlace the eigenvalues of A.*

(2) *If the interlacing is tight, then $AP = PB$.*

Proof. (1) Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ be eigenvalues of A with corresponding orthogonal eigenvectors $U_1, U_2, \cdots, U_n$. Let $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ be eigenvalues of B with corresponding orthogonal eigenvectors $V_1, V_2, \cdots, V_m$. Set $\mathcal{U}_i = \text{span}\{U_1, U_2, \cdots, U_i\}$ and $\mathcal{V}_j = \text{span}\{V_1, V_2, \cdots, V_j\}$. Observe

$$\dim(P^t\mathcal{U}_{i-1}) \leq \dim(\mathcal{U}_{i-1}) \leq i-1.$$

Hence

$$\begin{aligned}\dim((P^t\mathcal{U}_{i-1})^\perp \cap \mathcal{V}_i) &= \dim(P^t\mathcal{U}_{i-1})^\perp + \dim(\mathcal{V}_i) - \dim((P^t\mathcal{U}_{i-1})^\perp + \mathcal{V}_i) \\ &\geq (m-i+1) + i - m = 1.\end{aligned}$$

Pick a nonzero vector $Y \in (P^t\mathcal{U}_{i-1})^\perp \cap \mathcal{V}_i$. Observe $Y^tBY \geq \eta_i Y^tY$ since $Y \in \mathcal{V}_i$. Observe

$$\begin{aligned}Y \in (P^t\mathcal{U}_{i-1})^\perp &\Leftrightarrow \langle Y, P^tU\rangle = 0 \text{ for all } U \in \mathcal{U}_{i-1} \\ &\Leftrightarrow \langle PY, U\rangle = 0 \text{ for all } U \in \mathcal{U}_{i-1} \\ &\Leftrightarrow PY \in \mathcal{U}_{i-1}^\perp = \text{span}\{U_i, U_{i+1}, \cdots, U_n\}\end{aligned}$$

Hence $(PY)^tA(PY) \leq \theta_i(PY)^t(PY)$. Observe $PY \neq 0$ and

$$\theta_i \geq \frac{(PY)^tA(PY)}{(PY)^t(PY)} = \frac{Y^tP^tAPY}{Y^tP^tPY} = \frac{Y^tBY}{Y^tY} \geq \eta_i.$$

If we use $-A$, $-B$ to replace A, B, we obtain $-\theta_{n-i} \geq -\eta_{m-i}$ for $i = 0, 1, 2, \cdots, m-1$. This is $\eta_i \geq \theta_{n-m+i}$ for $i = 1, 2, \cdots, m$.

(2) In the proof of (1), the equality holds if and only if PY is an eigenvector of A corresponding to θ_i for an eigenvector Y of B corresponding to η_i. Suppose $\theta_i = \eta_i$ $(1 \leq i \leq k)$ and $\eta_i = \theta_{n-m+i}$ $(k+1 \leq i \leq m)$ for some k $(1 \leq k \leq m)$. Let $Y_1, Y_2, \cdots, Y_m$ be the eigenvectors of B corresponding to $\eta_1, \eta_2, \cdots, \eta_m$ such that $PY_1, PY_2, \cdots, PY_m$ be the eigenvectors of

A corresponding to $\theta_1, \theta_2, \cdots, \theta_k, \theta_{n-m+k+1}, \theta_{n-m+k+2}, \cdots, \theta_n$. So for $1 \le i \le m$

$$\begin{aligned} PBY_i &= \eta_i PY_i \\ APY_i &= \theta_j PY_i = \eta_i PY_i, \end{aligned}$$

where

$$j = \begin{cases} i, & \text{if } i \le k, \\ n-m+i, & \text{else.} \end{cases}$$

Hence $PB = AP$.

□

Definition 5.8. Let A be an $n \times n$ matrix. Then B is a *principle submatrix* of A if B is obtained by deleting some rows and columns with the same indices from A.

Example 5.9. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Then $(1), (5), (9)$, $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$, $\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix}$ and A are all the principle submatrices of A. Observe $\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix}$ is **not** a principle submatrix of A since it is obtained by deleting row 3 and column 2 from A.

Corollary 5.10. *Let A be an $n \times n$ real symmetric matrix. Suppose B is an $m \times m$ principle submatrix of A. Then the eigenvalues of B interlace the eigenvalues of A.*

Proof. By reordering the indices, we can assume that B appears in the upper left corner of A. Then $B = P^t AP$ for $n \times m$ matrix $P = \begin{pmatrix} I \\ O \end{pmatrix}$. Hence the result follows from Theorem 5.7. □

Corollary 5.11. *Let X be a graph and fix a vertex $x \in X$. Suppose θ is an eigenvalue of X with multiplicity $m > 1$. Then θ is an eigenvalue of the graph induced on $X - x$ with multiplicity at least $m - 1$ and at most $m + 1$.*

Proof. Let A be the adjacency matrix of X and B be the adjacency matrix of $X-x$. Observe A is a real symmetric matrix and B is a principal submatrix of A. By Corollary 5.10, we know the eigenvalues of B interlace the eigenvalues of A. Since θ is an eigenvalue of A with multiplicity m, the result follows. □

5.3 Equitable partition of a graph

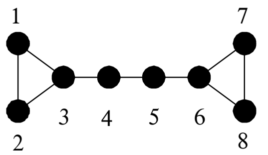
Definition 5.12. Let X be a graph, and let $\pi = \{C_1, C_2, \cdots, C_r\}$ be a partition of X. Then π is *equitable* if for all $i, j \in \{1, 2, \cdots, r\}$, there exists a number b_{ij} such that for all $x \in C_i$ we have $|N(x) \cap C_j| = b_{ij}$.

Definition 5.13. Suppose π is equitable. Then X/π is a weighted digraph where $X/\pi = \{C_1, C_2, \cdots, C_r\}$ and define $C_i \overset{b_{ij}}{\to} C_j$ if $b_{ij} \neq 0$. Let $A = A(X/\pi)$ be the *adjacency matrix*. That is, A is an $r \times r$ matrix such that $A_{ij} = b_{ij}$.

Definition 5.14. Let $\pi = \{C_1, C_2, \cdots, C_r\}$ be a partition of X. The *characteristic matrix* of π is an $|X| \times |\pi|$ matrix P such that

$$P_{xi} = \begin{cases} 1, & \text{if } x \in C_i, \\ 0, & \text{otherwise.} \end{cases}$$

Example 5.15.



For this graph, let $C_1 = \{1, 2, 4, 5, 7, 8\}$ and $C_2 = \{3, 6\}$. Then $b_{11} = 1$,

$b_{12} = 1$, $b_{21} = 3$, $b_{22} = 0$, $A(X/\pi) = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}$ and

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \end{array}.$$

Note 5.16. (1) *The columns of P is linear independent.*

(2) *Observe*

$$(P^tP)_{ij} = \sum_{x \in X} P^t_{ix} P_{xj} = \sum_{x \in X} P_{xi} P_{xj} = \begin{cases} 0, & i \neq j, \\ |C_i|, & i = j. \end{cases}$$

Hence P^tP is an invertible diagonal matrix.

Lemma 5.17. *Let π be an equitable partition of X with characteristic matrix P. Then $A(X)P = PA(X/\pi)$. (Equivalently, $(P^tP)^{-1}P^tA(X)P = A(X/\pi)$).*

Proof. Suppose $\pi = \{C_1, C_2, \cdots, C_r\}$ and $x \in C_i$. Observe

$$(A(X)P)_{xj} = \sum_{y \in X} A(X)_{xy} P_{yj} = b_{ij},$$

and

$$(PA(X/\pi))_{xj} = \sum_{k=1}^{r} P_{xk} A(X/\pi)_{kj} = \sum_{k=1}^{r} P_{xk} b_{kj} = b_{ij}.$$

Hence $A(X)P = PA(X/\pi)$. □

Corollary 5.18. *Let π be an equitable partition of X. Then the minimal polynomial of $A(X/\pi)$ divides the minimal polynomial of $A(X)$.*

Proof. Let $A = A(X)$, $B = A(X/\pi)$. From Lemma 5.17, we know $AP = PB$. Observe $A^2P = APB = PBB = PB^2$. In general, $A^nP = PB^n$ holds for all $n \in \mathbb{N}$. Hence $f(A)P = Pf(B)$ for any polynomials $f(x)$. Suppose $g(x)$ is the minimal polynomial of A. Then $g(A) = 0$, and $g(A)P = Pg(B) = 0$. Since the columns of P are linear independent, we have $g(B) = 0$. Then $g(x)$ is a multiple of the minimal polynomial of $B = A(X/\pi)$. □

Theorem 5.19. *The characteristic polynomial of B divides the characteristic polynomial of A where $A = A(X)$, $B = A(X/\pi)$.*

Proof. Let P be the characteristic matrix of the partition π of X. Set $T = \left(P \,|\, Q \right)$ for some $n \times (n-r)$ matrix Q such that T is invertible. Then

$$\begin{aligned} AT &= A\left(P \,|\, Q \right) = \left(AP \,|\, AQ \right) = \left(PB \,|\, AQ \right) \\ &= \left(P \,|\, Q \right) \left(\begin{array}{c|c} B & C \\ \hline \mathrm{O} & D \end{array} \right) = T \left(\begin{array}{c|c} B & C \\ \hline \mathrm{O} & D \end{array} \right) \end{aligned}$$

for some matrices C, D of size $r \times (n-r)$, $(n-r) \times (n-r)$ respectively. Then $T^{-1}AT = \left(\begin{array}{c|c} B & C \\ \hline \mathrm{O} & D \end{array} \right)$. Hence

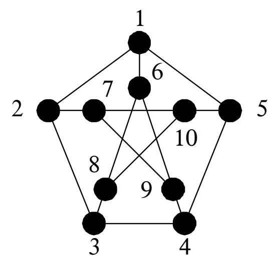
$$\begin{aligned} \det(xI - A) &= \det(T^{-1})\det(xI - A)\det(T) \\ &= \det(T^{-1}(xI - A)T) = \det(xI - T^{-1}AT) \\ &= \det(xI - \left(\begin{array}{c|c} B & C \\ \hline \mathrm{O} & D \end{array} \right) \\ &= \det(\left(\begin{array}{c|c} xI - B & C \\ \hline \mathrm{O} & xI - D \end{array} \right)) = \det(xI - B)\det(xI - D). \end{aligned}$$

Hence $\det(xI - B)$ divides $\det(xI - A)$. □

Note 5.20. (1) *The set of eigenvalues of $A(X/\pi)$ is a subset of the set of eigenvalue of A.*

(2) *θ is an eigenvalue of $A(X/\pi)$ with multiplicity t. Then θ is an eigenvalue of A with multiplicity at least t.*

Example 5.21. Petersen Graph X:

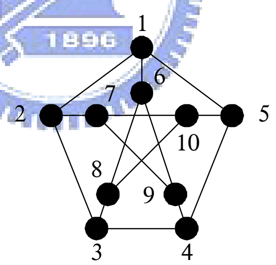


Let $\pi = \{\{1\}, \{2,5,6\}, \{3,4,7,8,9,10\}\}$. Then $A(X/\pi) = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$. We obtain $\det(xI - A(X/\pi)) = (x-1)(x-3)(x+2)$. Hence 1, 3, -2 are eigenvalues of Pertersen Graph.

Theorem 5.22. *Let G act on X with orbits $C_1, C_2, \cdots, C_r$. Then $\pi = \{C_1, C_2, \cdots, C_r\}$ is an equitable partition of X.*

Proof. Pick $x, y \in C_i$. Choose $g \in G$ such that $y = g(x)$. Then $|N(x) \cap C_j| = |g(N(x) \cap C_j)| = |N(y) \cap C_j|$. □

Example 5.23. Petersen Graph X:



Let $G = \{e, \sigma, \sigma^2, \sigma^3, \sigma^4\}$, where $\sigma = (1,2,3,4,5)(6,7,8,9,10)$. Then G acts on X with orbits $\pi = \{C_1, C_2\}$, where $C_1 = \{1,2,3,4,5\}$, $C_2 = \{6,7,8,9,10\}$. Since $A(X/\pi) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, we obtain $\det(xI - A(X/\pi)) = (x-3)(x-1)$.

5.4 Interlacing of rational functions

Lemma 5.24. *Let A be an $n \times n$ real symmetric matrix and $z \in \mathbb{R}^n$ is nonzero. Set $\phi(x) = z^t(xI - A)^{-1}z$ and $\psi(x) = 1 - z^t(xI - A)^{-1}z$. Then*

(1) *$\phi'(x) > 0$, $\psi'(x) < 0$ if $\phi(x)$, $\psi(x)$ are defined.*

(2) *Every root of $\phi(x)$ (resp. $\psi(x)$) has multiplicity 1.*

(3) *Every pole of $\phi(x)$ (resp. $\psi(x)$) is simple.*

(4) *The roots of $\phi(x)$ (resp. $\psi(x)$) interlace the poles of $\phi(x)$.*

Proof. (1) Observe

$$\begin{aligned}\phi(x) &= z^t(xI - A)^{-1}z = z^t(\sum_{\theta \in ev(A)} (x-\theta)^{-1}E_\theta)z \\ &= \sum_{\theta \in ev(A)} \frac{z^t E_\theta z}{x-\theta}.\end{aligned}$$

Hence

$$\phi'(x) = -\sum_{\theta \in ev(A)} \frac{z^t E_\theta z}{(x-\theta)^2} < 0.$$

And $\psi'(x) = (1-\phi(x))' = -\phi'(x) > 0$.

(2) From (1), we have $\phi'(x) \neq 0$, $\psi'(x) \neq 0$. Hence they have no repeated roots.

(3) Obviously from $\phi(x) = \sum_{\theta \in ev(A)} \frac{z^t E_\theta z}{x-\theta}$. Hence every pole of $\phi(x)$ is 1. Similar for $\psi(x) = 1 - \phi(x)$.

(4) Observe $\phi(x)$ is decreasing by 1, $\lim_{x \to x} \phi(x) = 0$, $\lim_{x \to -x} \phi(x) = 0$. Hence after deleting the common factors of $\phi(x)$, the roots of $\phi(x)$ interlace the poles of $\phi(x)$. Hence the roots of $\phi(x)$ interlace the poles of $\phi(x)$.

□

Chapter 6

The Laplacian of a Graph

6.1 Laplacian and incidence matrix of a graph

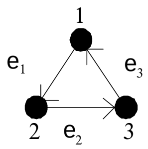
Definition 6.1. Let X be a graph (not necessary simple). An *orientation* X^σ of X is a digraph that assigns each edge e a directed edge $\sigma(e)$.

Definition 6.2. Let X^σ be an orientation of X. The *incidence matrix* D of X^σ is an $n \times m$ matrix where $n = |X|$, $m = |R|$ such that for $x \in X$ and $e = yz \in X^\sigma$,

$$D_{xe} = \begin{cases} 1, \text{ if } x = z, \ (x \text{ is the } \textit{head} \text{ of } e) \\ -1, \text{ if } x = y, \ (x \text{ is the } \textit{tail} \text{ of } e) \\ 0, \text{ if } x \neq y, x \neq z. \end{cases}$$

Note 6.3. *Each column of D has exactly one 1 entry and -1 entry.*

Example 6.4.



$$\text{For this graph, } D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \end{array}.$$

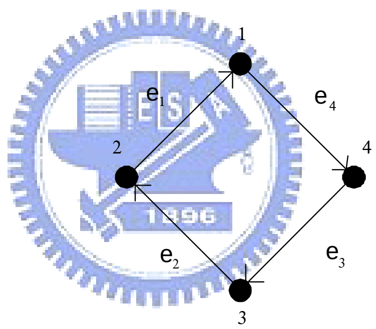
Lemma 6.5. *Let X be a graph with an orientation X^σ and let D be the incidence matrix of X^σ. Then $DD^t = \triangle(X) - A(X)$ where $\triangle(X)$ is a diagonal matrix with $(\triangle(X))_{yy}$ the degree of y. Such $Q := DD^t$ is called the Laplacian of X.*

Proof. Pick x, $y \in X$. Observe

$$\begin{aligned}(DD^t)_{xy} &= \sum_{e \in X^\sigma} D_{xe} D^t_{ey} = \sum_{e \in X^\sigma} D_{xe} D_{ye} \\ &= \begin{cases} deg(x), & \text{if } x = y, \\ -1, & \text{if } x \neq y, x \sim y, \\ 0, & \text{if } x \neq y, x \nsim y. \end{cases} \\ &= \triangle(X) - A(X).\end{aligned}$$

□

Example 6.6.



For this graph, $D = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$ and

$$Q(X) = DD^t = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

Note 6.7. (1) *$Q(X)$ is symmetric.*

(2) *$Q(X)$ is independent of the orientation σ.*

$$(3)\ \ Q(X)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=0,\ and\ D^t\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}=0$$

Lemma 6.8. *Let $Q := Q(X)$ be the Laplacian of X. Then all eigenvalues of Q are nonnegative.*

Proof. Let λ be an eigenvalue of Q with eigenvector x. Then $Qx = DD^tx = \lambda x$. Observe $x^tDD^tx = x^t\lambda x$. Hence $||D^tx||^2 = \lambda||x||^2$. The result follows. □

Lemma 6.9. *For any matrix D, the nullspace of DD^t equals the nullspace of D^t.*

Proof. Observe nullspace$(D^t) \subseteq$ nullspace(DD^t). Suppose $DD^tU = 0$. Hence $U^tDD^tU = 0$. Hence $||D^tU||^2 = 0$. Hence $D^tU = 0$. Hence nullspace$(D^t) \supseteq$ nullspace(Q). Hence the result follows. □

Theorem 6.10. *Suppose X has c connected components. Then 0 is an eigenvalue of Q with multiplicity c.*

Proof. Suppose $X = X_1 \cup X_2 \cup \cdots \cup X_c$, where X_i are connected components. We claim the nullspace of D^t has dimension c. For $1 \le i \le c$, let U_i be a column vector such that

$$U_i(x) = \begin{cases} 1, \text{ if } x \in X_i, \\ 0, \text{ if } x \notin X_i. \end{cases}$$

Then $D^tU_i = 0$. In fact, $D^tU = 0$ for $U \in span\{U_1, U_2, \cdots, U_c\}$. Hence the nullspace of D^t has dimension at least c. On the other hand, suppose $D^tU = 0$ for some vector U. Then by the construction of D, $U(x) = U(y)$ for $x \sim y$. Hence $U(x) = U(y)$ for any x, y in the same component. Then $U \in span\{U_1, \cdots, U_c\}$. Hence the nullspace of D^t has dimension c. The theorem follows from this and Lemma 6.9. □

Lemma 6.11. *Let X be a regular graph of order n with valency k, and $\theta_1 \ge \theta_2 \ge \cdots \ge \theta_n$ be eigenvalues of $A(X)$. Suppose $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are eigenvalues of $Q(X)$. Then $\lambda_1 = 0$ and $\lambda_i = k - \theta_i$ for $i = 1, 2, \cdots, n$.*

Proof. $Q = \triangle(X) - A(X) = kI - A(X)$ since X is k-regular. Thus every eigenvector of A with eigenvalue θ_i is an eigenvector of Q with eigenvalue $k - \theta_i$. $\lambda_1 = 0$ by Theorem 6.10. □

Definition 6.12. The *complement* $\overline{X}$ of a graph X is the graph with vertex set X and edge set $\overline{R} = \{e = xy \mid x \neq y, e \notin E\}$.

Lemma 6.13. *Let X be a graph. Then $Q(X) + Q(\overline{X}) = Q(K_n)$ where $n = |X|$.*

Proof. Observe

$$\begin{aligned} Q(X) &= \triangle(X) - A(X), \\ Q(\overline{X}) &= \triangle(\overline{X}) - A(\overline{X}). \end{aligned}$$

Hence

$$\begin{aligned} Q(X) + Q(\overline{X}) &= \triangle(X) + \triangle(\overline{X}) - (A(X) + A(\overline{X})) \\ &= (n-1)I - (J - I) = nI - J \\ &= Q(K_n). \end{aligned}$$

□

Lemma 6.14. *Let X be a graph with n vertices. Then $\lambda_i(\overline{X}) = n - \lambda_{n-i+2}(X)$ for $2 \leq i \leq n$.*

Proof. Let $U_1, U_2, \cdots, U_n$ be orthogonal eigenvectors of $Q(X)$ corresponding to $\lambda_1(X), \lambda_2(X), \cdots, \lambda_n(X)$ respectively, and $U_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Observe

$$\begin{aligned} Q(\overline{X})U_i &= (Q(K_n) - Q(X))U_i = (nI - J - Q(X))U_i \\ &= (n - \lambda_i(X))U_i, \end{aligned}$$

since $JU_i = 0$ for $2 \leq i \leq n$. □

Corollary 6.15. *Let X be a graph with n vertices. Then*

(1) $\lambda_i(X) \leq n$.

(2) $\{i \mid \lambda_i(X) = n\} = \overline{c}(X) - 1$ *where $\overline{c}(X)$ is the number of connected components in $\overline{X}$.*

Proof. This is clear from Theorem 6.10 and Lemma 6.14. □

Lemma 6.16. *Let U be a column vector. Then* $U^tQU = \sum_{xy \in R} (U_x - U_y)^2$.

Proof. Observe

$$\begin{aligned} U^tQU &= U^tDD^tU = (D^tU)^t(D^tU) = ||D^tU||^2 \\ &= \sum_{e \in R} (D^tU)_e^2 = \sum_{e \in R} (\sum_{x \in X} D^t_{ex} U_x)^2 \\ &= \sum_{e \in R} (\sum_{x \in X} D_{xe} U_x)^2 = \sum_{e = xy \in R} (U_x - U_y)^2. \end{aligned}$$

□

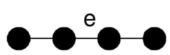
6.2 The number of spanning trees of a graph

Definition 6.17. A tree is a connected simple graph without cycles.

Definition 6.18. Let X be a graph. A *spanning tree* T of X is a subgraph of X that is a tree and contains all vertices of X.

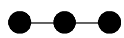
Definition 6.19. Let X be a multigraph and $e = uv$ is an edge in X. Then $X \backslash e$ is the graph with vertex set X and edge set $R \backslash \{e\}$. X/e is the multigraph obtained by identifying the vertices u and v and deleting the edge e. X/e is the graph obtained by *contracting* the edge e.

Example 6.20. X:



Then $X \setminus e$:

And X/e:



Lemma 6.21. *Let X be a multigraph. Let $\tau(X)$ denote the number of spanning tree in X. Then*

$$\tau(X) = \tau(X \backslash e) + \tau(X/e).$$

Proof. Pick an edge e. Then every spanning tree either contains e or does not contain e. Observe $\tau(X\backslash e)$ counts the number of spanning trees in X that do not contain the edge e, and $\tau(X/e)$ counts the number of spanning trees in X that contain the edge e. The result follows. □

Definition 6.22. Let M be a square matrix and S is a subset of its index set. Then $M[S]$ denote the submatrix of M obtained by deleting the rows and columns indexed by S.

Note 6.23. *Let $Q = Q(X)$ be the Laplacian of X and uv be an edge in X. Then $Q[u,v] = Q(X/e)[v]$.*

Theorem 6.24. *Let $Q = Q(X)$ be the Laplacian of a graph X. Then for any $u \in X$, $\det(Q[u]) = \tau(X)$.*

Proof. We prove this theorem by induction on the number of edges of X. Fix an edge $e = uv$. Observe $Q[u] = Q(X\backslash e)[u]+E$, where E is the $(n-1)\times(n-1)$ matrix with $E_{vv} = 1$ and all other entries equal to 0. Then

$$\begin{aligned}\det(Q[u]) &= \det(Q(X\backslash e)[u]) + \det(Q[u,v]) \\ &= \det(Q(X\backslash e)[u]) + \det(Q(X/e)[v]).\end{aligned}$$

By induction, $\det(Q(X\backslash e)[u]) = \tau(X\backslash e)$ and $\det(Q(X/e)[v]) = \tau(X/e)$. Hence the result follows. □

Corollary 6.25. *The number of spanning trees of K_n is n^{n-2}.*

Proof. Observe $Q(K_n) = (n-1)I - A(K_n) = (n-1)I - (J-I) = nI - J$. Hence $Q(K_n)[1] = nI - J$ with size $(n-1)\times(n-1)$. Observe the eigenvalues of J are $0, 0, 0, 0, \cdots, 0$, ($n-2$ times), $n-1$ and then the eigenvalues of $Q(K_n)[1] = n, n, n, n, \cdots$, ($n-2$ times), 1. Hence $\det(Q(K_n)[1]) = n^{n-2}$. □

Definition 6.26. Let M be an $n \times n$ matrix. The *adjugate* of M ($adjM$) is a $n \times n$ matrix such that $(adjM)_{ij} = (-1)^{i+j}\det(M[j;i])$ where $M[j;i]$ is the submatrix of M that deletes row j and column i.

Note 6.27. (1) *$M \cdot adj(M) = \det(M) \cdot I$.*

(2) *If $\det(M) \neq 0$ then $M \cdot \dfrac{adj(M)}{\det(M)} = I$.*

(3) $(adj(Q))_{uu} = \tau(X)$ *for all* $u \in X$.

Example 6.28. Let $M = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$. Then $adj(M) = \begin{pmatrix} 0 & -2 & 1 \\ -5 & 1 & 2 \\ 5 & 0 & -5 \end{pmatrix}$ and $M \cdot adj(M) = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{pmatrix} = \det(M) \cdot I$.

Theorem 6.29. *Let X be a graph and Q be its Laplacian. Then $adj(Q) = \tau(X)J$.*

Proof. We consider two cases.

Case1: X is not connected. Observe 0 is an eigenvalue of $Q(X)$ with multiplicity at least 2 by Theorem 6.10. Hence $rank(Q(X)) \leq n-2$. Let $Q[i;j]$ be the submatrix of Q obtained by deleting the row i and the column j of Q. Hence $rank(Q[i;j]) \leq n-2$. Since the size of $Q[i;j]$ is $(n-1)\times(n-1)$. Hence $\det(Q[i;j]) = 0$. Then $adj(Q) = 0 = \tau(X)J$. Note $\tau(X) = 0$ since X has no spanning tree.

Case2: X is connected. From Theorem 6.10, 0 is an eigenvalue of Q and all eigenvectors corresponding to 0 has the form

$$c\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Observe $Qadj(Q) = \det(Q)I = 0 \cdot adj(Q)$. Hence each column of $adj(Q)$ is an eigenvector of Q corresponding to 0. Then $adj(Q)$ has the form

$$\begin{pmatrix} t_1 & t_2 & \dots & t_n \\ t_1 & t_2 & \dots & t_n \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \dots & t_n \end{pmatrix}.$$

But the diagonals of Q are all the same number $\tau(X)$ by Note 6.27(3). Hence $adj(Q) = \tau(X)J$.

□

Theorem 6.30. *Let* $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ *be eigenvalues of* $Q(X)$*. Then* $\tau(X) = \frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n$.

Proof. The result clearly follows if X is not connected. So we consider X is connected. Observe the characteristic polynomial of the Laplacian of X is

$$\begin{aligned} \det(xI - Q) &= (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \\ &= x(x-\lambda_2)\cdots(x-\lambda_n) \\ &= (-1)^{n-1}(\lambda_2\cdots\lambda_n)x + \cdots . \end{aligned}$$

and on the other hands,

$$\begin{aligned} \det(xI - Q) &= \sum_u \det(-Q[u])x + \cdots \\ &= n(-1)^{n-1}\tau(X)x + \cdots . \end{aligned}$$

Hence the result follows from comparing the coefficients. □

6.3 The representation of a graph and its energy

Definition 6.31. A *representation* ρ of a graph X in $\mathbb{R}^k$ is a map ρ from X into $\mathbb{R}^k$.

Suppose $|X| = n$ and identify $x \in X$ to be a column vector $x = (0, \cdots, 0, 1, 0, \cdots, 0)^t$ with xth position is 1. A representation $\rho : X \to \mathbb{R}^k$ is linear if $\rho(X) = LX$ for some $k \times n$ matrix L. Let $w : R \to \mathbb{R}^{>0}$ be a function that gives each edge e of X a weight $w(e)$.

Definition 6.32. Let $\rho : X \to \mathbb{R}^k$ be representation. Then

$$\mathcal{E}(\rho) := \sum_{e=uv\in R} w(e)||\rho(u) - \rho(v)||^2$$

is called the *energy* of ρ with respect to the weight function w.

Lemma 6.33. *Let X be a graph and $\rho(X) = LX$ be a representation of X in $\mathbb{R}^k$. Fix an orientation X^σ of X with incidence matrix D. Then for $e = uv$, $||\rho(u) - \rho(v)||^2 = ((LD)^t LD)_{ee}$.*

Proof. Observe

$$\begin{aligned}((LD)^t LD)_{ee} &= \sum_{f\in\{1,2,\cdots,k\}} (LD)^t_{ef}(LD)_{fe} = \sum_{f\in\{1,2,\cdots,k\}} (LD)^2_{fe} \\ &= \sum_{f\in\{1,2,\cdots,k\}} (\sum_{x\in X} L_{fx}D_{xe})^2 = \sum_{f\in\{1,2,\cdots,k\}} (L_{fu} - L_{fv})^2 \\ &= \sum_{f\in\{1,2,\cdots,k\}} ((\rho(u) - \rho(v))_f)^2 = ||\rho(u) - \rho(v)||^2.\end{aligned}$$

□

Suppose $|R| = m$. The *weight matrix* W of w is $m \times m$ diagonal (indexed by $e \in R$) such that $W_{ee} = w(e)$.

Lemma 6.34. *As notation above, $\mathcal{E}(\rho) = trace(W(LD)^t LD)$.*

Proof. Observe

$$\begin{aligned}trace(W(LD)^t LD) &= \sum_{e\in R} (W(LD)^t LD)_{ee} = \sum_{e\in R} W_{ee}((LD)^t(LD))_{ee} \\ &= \sum_{e=uv\in R} w(e)||\rho(u) - \rho(v)||^2 = \mathcal{E}(\rho).\end{aligned}$$

□

We recall some facts in linear algebra.

Note 6.35. *Let M be an $n \times n$ matrix.*

(1) $trace(M) = M_{11} + M_{22} + \cdots + M_{nn}$.

(2) $trace(MM') = trace(M'M)$.

Theorem 6.36. *Let X be a graph. Suppose $\rho : X \to \mathbb{R}^k$ represented by an $k \times n$ matrix L. Let W be a weight matrix. Then $\mathcal{E}(\rho) = trace(LDW(LD)^t)$ for any incident matrix D of any orientation X^σ of X.*

Proof. From Note 6.35 and Lemma 6.34, we have

$$\mathcal{E}(\rho) = trace(W(LD)^t LD) = trace(LDW(LD)^t).$$

□

6.4 Weighted Laplacian

Lemma 6.37. *Let X be a graph with a weight matrix W and an orientation X^σ and let D be the incidence matrix of X^σ. Set $Q := DWD^t$. Then*

$$Q_{xy} = \begin{cases} 0, \ if \ x \nsim y, x \neq y, \\ -w(e), \ if \ e = xy, \\ \sum_{z \sim x} w(zx), \ if \ x = y. \end{cases}$$

In particular, Q is independent of the orientation σ. Such $Q := DWD^t$ is called the weighted Laplacian of X.

Proof. Observe

$$\begin{aligned} Q_{xy} &= (DWD^t)_{xy} = \sum_{e \in R} D_{xe} W_{ee} D^t_{ey} \\ &= \begin{cases} 0, \ if \ x \nsim y, x \neq y, \\ -w(e), \ if \ e = xy, \\ \sum_{z \sim x} w(zx), \ if \ x = y. \end{cases} \end{aligned}$$

□

Note 6.38. (1) $Q = DWD^t =$

$$D \begin{pmatrix} \sqrt{w(1)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \sqrt{w(m)} \end{pmatrix} \begin{pmatrix} \sqrt{w(1)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \sqrt{w(m)} \end{pmatrix} D^t$$

$=$

$$D \begin{pmatrix} \sqrt{w(1)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \sqrt{w(m)} \end{pmatrix} (D \begin{pmatrix} \sqrt{w(1)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \sqrt{w(m)} \end{pmatrix})^t.$$

We use the notation

$$\sqrt{W} := \begin{pmatrix} \sqrt{w(1)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & \sqrt{w(m)} \end{pmatrix}.$$

(2) $Q\begin{pmatrix}1\\ \vdots\\ \vdots\\ 1\end{pmatrix} = 0$ *by Note* 6.7(3).

(3) *If Q' is an $n \times n$ matrix satisfying*

(*i*) $Q'_{xy} < 0$ *if* $x \neq y$, $x \sim y$,

(*ii*) $Q'_{xy} = 0$ *if* $x \neq y$, $x \nsim y$,

(*iii*) $Q'\begin{pmatrix}1\\ \vdots\\ 1\\ \vdots\\ 1\end{pmatrix} = 0$

Then Q' is a weighted Laplacian for some weight function w. In fact, this W satisfies $W_{xy} = -Q'_{xy}$ for $x \sim y$.

Lemma 6.39. *Let X be a graph of n vertices. Let Q be a weighted Laplacian of X with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_n$. Let c denote the number of connected components in X. Then*

(1) $\lambda_1 \geq 0$,

(2) $c = \max\{i \mid \lambda_i = 0\}$. *In particular,* $\lambda_1 = 0$.

Proof. (1) Observe λ_1 is an eigenvalue of Q and $Q = DWD^t = D\sqrt{W}\sqrt{W}D^t$. Then $Q = (D\sqrt{W})(D\sqrt{W})^t$. Let U_1 be the eigenvector of Q corresponding to λ_1. Then $\lambda_1 U_1 = QU_1 = (D\sqrt{W})(D\sqrt{W})^t U_1$. Hence

$$U_1^t \lambda_1 U_1 = U_1^t (D\sqrt{W})(D\sqrt{W})^t U_1.$$

Hence $||(D\sqrt{W})^t U_1||^2 = \lambda_1 ||U_1||^2$. Hence λ_1 is nonnegative, the result follows.

(2) The proof is similar to Theorem 6.10.

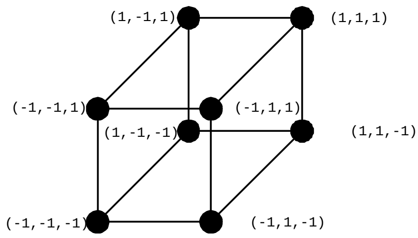
□

Definition 6.40. A representation $\rho : X \to \mathbb{R}^k$ is *balanced* if $\sum_{x \in X} \rho(x) = 0$.

Definition 6.41. A linear representation $\rho : X \to \mathbb{R}^k$ is *orthonormal* if $LL^t = I_{k\times k}$ where L is an $k \times n$ matrix that represents ρ.

Note 6.42. $k \leq n$.

Example 6.43.



For above representation of a cube in R^3, we have

$$L = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix},$$

$$L(1,1,1,1,1,1,1,1)^t = 0$$

and

$$LL^t = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Hence the representation $\frac{1}{2\sqrt{2}}L$ is balanced and orthonormal.

Theorem 6.44. *Let Q be the weighted Laplacian for the weight matrix W of X and $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are eigenvalues of Q. Let ρ be an orthonormal representation $\rho : X \to \mathbb{R}^k$. Then $\mathcal{E}(\rho) \geq \lambda_2 + \cdots + \lambda_k$, where $\mathcal{E}(\rho)$ is the energy of ρ with respect to W. Furthermore, there is an orthonormal representation of X into $\mathbb{R}^k$ such that above equality holds.*

Proof. Let L be the $k \times n$ matrix represented ρ. Observe by Theorem 5.7 and

$$\begin{aligned} \mathcal{E}(\rho) &= trace(LDW(LD)^t) \\ &= trace(LQL^t) \\ &= \text{sum of eigenvalues of } LQL^t \\ &\geq \lambda_1 + \lambda_2 + \cdots + \lambda_k \\ &= \lambda_2 + \cdots + \lambda_k. \end{aligned}$$

Set

$$U_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Let $U_1, U_2, \cdots, U_k$ be orthonormal eigenvectors of Q corresponding to eigenvalues $\lambda_1, \cdots, \lambda_k$ respectively. Set

$$L^t = \begin{pmatrix} U_1 & U_2 & \cdots & U_k \end{pmatrix}.$$

Then $LL^t = I$ and

$$\begin{aligned} trace(LQL^t) &= trace(\begin{pmatrix} U_1^t \\ U_2^t \\ \vdots \\ U_k^t \end{pmatrix} Q(U_1 U_2 \cdots U_k)) \\ &= trace(\begin{pmatrix} U_1^t \\ U_2^t \\ \vdots \\ U_k^t \end{pmatrix} (\lambda_1 U_1 \ \lambda_2 U_2 \ \cdots \ \lambda_k U_k)) \\ &= trace \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_k \end{pmatrix} \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_k = \lambda_2 + \cdots + \lambda_k. \end{aligned}$$

□

Corollary 6.45. *Let Q be the weighted Laplacian for the weight matrix W of X and $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are eigenvalues of Q. Let $\rho : X \to \mathbb{R}^k$ be an orthonormal balanced representation. Then $\mathcal{E}(\rho) \geq \lambda_2 + \cdots + \lambda_{k+1}$. Furthermore, there is an orthonormal balanced representation of X into $\mathbb{R}^k$ such that above equality holds.*

Proof. Let L represent ρ. Observe

$$L \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0$$

since ρ is balanced. Set

$$L' = \begin{pmatrix} \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ & L & \end{pmatrix}.$$

Observe L' is orthonormal from $X \to \mathbb{R}^{k+1}$. Observe $\mathcal{E}(L') = \mathcal{E}(\rho)$. Hence by Theorem 6.44,

$$\mathcal{E}(\rho) = \mathcal{E}(L') \geq \lambda_2 + \cdots + \lambda_{k+1}.$$

Similarly, we can obtain the equality. □

6.5 The second least eigenvalue

Throughout this section, let X be a graph with n vertices, Q be the Laplacian of X and $\lambda_1(X) \leq \lambda_2(X) \leq \cdots \lambda_n(X)$ be the eigenvalues of $Q(X)$.

Definition 6.46. Let X be a graph. Then S is a subgraph of X if $S \subseteq X$ and $R(S) \subseteq R(X)$.

Theorem 6.47. *Let X be a graph. Suppose that S is a subset of X. Then $\lambda_2(X) \leq \lambda_2(X \backslash S) + |S|$.*

Proof. Observe $U^t Q(X) U \geq \lambda_2(X) ||U||^2$ for any U orthogonal to the all 1's column vector. Pick $U \in \mathbb{R}^X$ such that

(1) $U_x = 0$, if $x \in S$,

(2) $W = U \restriction (X \backslash S)$ is an eigenvector of $Q(X \backslash S)$ corresponding to $\lambda_2(X \backslash S)$ and orthogonal to the all 1's column vector,

(3) $||U|| = 1$.

Then from above and by Lemma 6.16,

$$\begin{aligned}
\lambda_2(X) &\leq U^t Q(X) U \\
&= \sum_{xy\in R} (U_x - U_y)^2 \\
&= \sum_{xy\in R(X\backslash S)} (U_x - U_y)^2 + \sum_{x\in S} \sum_{\substack{xy\in R \\ y\notin S}} U_y^2 \\
&= \sum_{xy\in R(X\backslash S)} (W_x - W_y)^2 + |S| \\
&= W^t Q(X\backslash S) W + |S| \\
&= \lambda_2(X\backslash S)||W||^2 + |S|,
\end{aligned}$$

where $||W|| = ||U|| = 1$. Hence $\lambda_2(X) \leq \lambda_2(X\backslash S) + |S|$. □

Corollary 6.48. *Let X be a graph. Suppose X is not complete. Then $\lambda_2(X) \leq \kappa_0(X)$ where $\kappa_0(X)$ is the vertex connectivity of X.*

Proof. We can find a subset $S \subseteq X$ such that $|S| = \kappa_0(X)$ and $X\backslash S$ is disconnected. Then $\lambda_2(X\backslash S) = 0$. Hence

$$\lambda_2(X) \leq \lambda_2(X\backslash S) + |S| = 0 + |S| = \kappa_0(X).$$

□

Corollary 6.49. *$\lambda_2(T) \leq 1$ for any tree T with at least three vertices.*

Proof. It is clear by Corollary 6.48 since $\kappa_0(T) = 1$ for any tree T. □

Note 6.50. *For any graph, $\lambda_2(X) \leq \kappa_0(X) \leq \kappa_1(X) \leq \delta(X)$ where $\delta(X)$ is the minimal degree of X.*

Note 6.51. *For any graph X, the Laplacian of $Q(X)$ has $|rank(Q(X)) - rank(Q(X\backslash e))| \leq |rank(Q(X) - Q(X\backslash S))| \leq 2$.*

Lemma 6.52. *Let X be a graph and $e = uv$ be an edge of X. Then*

$$\lambda_2(X\backslash e) \leq \lambda_2(X) \leq \lambda_2(X\backslash e) + 2.$$

Proof. For any $z \in \mathbb{R}^X$,

$$\begin{aligned} z^tQ(X)z &= \sum_{\substack{i\sim j \\ i,j\in X}} (z_i - z_j)^2 = \sum_{\substack{i\sim j \\ i,j\in X\backslash e}} (z_i - z_j)^2 + (z_u - z_v)^2 \\ &= z^tQ(X\backslash e)z + (z_u - z_v)^2 \end{aligned}$$

by Lemma 6.16. Let $z = U_2(X\backslash e)$ be the eigenvector of $Q(X\backslash e)$ corresponding to $\lambda_2(X\backslash e)$ and orthogonal to the all 1's column vector. Then

$$\begin{aligned} \lambda_2(X)||z||^2 &\le z^tQ(X)z && (6.1) \\ &= z^tQ(X\backslash e)z + (z_u - z_v)^2 \\ &= \lambda_2(X\backslash e)||z||^2 + (z_u - z_v)^2 \\ &\le \lambda_2(X\backslash e)||z||^2 + 2(z_u^2 + z_v^2) && (6.2) \\ &\le \lambda_2(X\backslash e)||z||^2 + 2||z||^2. && (6.3) \end{aligned}$$

Hence $\lambda_2(X) \le \lambda_2(X\backslash e) + 2$. Let $z = U_2(X)$ be the eigenvector of $Q(X)$ corresponding to $\lambda_2(X)$ and orthogonal to the all 1's column vector. Then

$$\begin{aligned} \lambda_2(X)||z||^2 &= z^tQ(X)z \\ &= z^tQ(X\backslash e)z + (z_u - z_v)^2 \\ &\ge \lambda_2(X\backslash e)||z||^2 + (z_u - z_v)^2 \\ &\ge \lambda_2(X\backslash e)||z||^2. \end{aligned}$$

Hence $\lambda_2(X) \ge \lambda_2(X\backslash e)$. □

Lemma 6.53. *Let X be a graph. Then for any proper nonempty subset $S \subsetneq X$*

$$\lambda_2(X) \le \frac{n|\partial S|}{|S|(n - |S|)}$$

where $n = |X|$ and ∂S is the boundary of S.

Proof. Set Z be a column vector and

$$Z_x = \begin{cases} n - |S|, & \text{for } x \in S, \\ -|S|, & \text{otherwise.} \end{cases}$$

Observe $(1, 1, \cdots, 1)Z = (n - |S|)|S| - |S|(n - |S|) = 0$. Hence

$$\lambda_2(X)||Z||^t \le Z^tQ(X)Z = \sum_{uv\in R} (Z_u - Z_v)^2 = |\partial S|n^2.$$

Note that

$$\begin{aligned} ||Z||^2 &= (n-|S|)^2|S| + (n-|S|)|S|^2 \\ &= (n-|S|)|S|(n-|S|+|S|) = n(n-|S|)|S|. \end{aligned}$$

Hence

$$\lambda_2(X) \le \frac{n|\partial S|}{|S|(n-|S|)}.$$

□

Definition 6.54. $\phi(X) := \min\limits_{\substack{S \subsetneq X \\ S \neq \emptyset}} \frac{|\partial S|}{|S|}$ is called the *conductance* of a graph X.

Corollary 6.55. *For a graph X, $\lambda_2(X) \le 2\phi(X)$.*

Proof. Note that $\partial S = \partial \overline{S}$. The Corollary is from previous Lemma. □

6.6 Interlacing of eigenvalues

Lemma 6.56. *Let C, D be $s \times t$, $t \times s$ matrices respectively. Then* $\det(I - CD) = \det(I - DC)$.

Proof. Let

$$X = \left(\begin{array}{c|c} I & C \\ \hline D & I \end{array}\right), Y = \left(\begin{array}{c|c} I & \mathrm{O} \\ \hline -D & I \end{array}\right).$$

Observe

$$\begin{aligned} \det(XY) &= \det(\begin{pmatrix} I-CD & C \\ O & I \end{pmatrix}) \\ &= \det(I-CD)\det(I) = \det(I-CD). \end{aligned}$$

Similarly, $\det(YX) = \det(I - DC)$. Since $\det(XY) = \det(YX)$, $\det(I - CD) = \det(I - DC)$. □

Theorem 6.57. *Let X be a graph with a fixed edge e. Then*

$$\begin{cases} \lambda_i(X\backslash e) \le \lambda_i(X) \le \lambda_{i+1}(X\backslash e), \ \textit{for } i = 1, 2, \cdots, n-1, \\ \lambda_n(X\backslash e) \le \lambda_n(X). \end{cases}$$

Proof. Suppose $e = uv$, and u, v are the first two vertices of X. Set

$$Z = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Observe

$$ZZ^t = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

By the construction of $Q(X)$, we obtain $Q(X) = Q(X\backslash e) + ZZ^t$. Observe

$$\lambda I - Q(X) = \lambda I - Q(X\backslash e) - ZZ^t = (\lambda I - Q(X\backslash e))(I - (\lambda I - Q(X\backslash e))^{-1} ZZ^t).$$

Hence by Lemma 6.56 with $C = (\lambda I - Q(X\backslash e))^{-1} Z$, $D = Z^t$,

$$\begin{aligned} \det(\lambda I - Q(X)) &= \det(\lambda I - Q(X\backslash e)) \det(I - (\lambda I - Q(X\backslash e))^{-1} ZZ^t) \\ &= \det(\lambda I - Q(X\backslash e)) \det(1 - Z^t(\lambda I - Q(X\backslash e))^{-1} Z). \end{aligned}$$

Then

$$\begin{aligned} \frac{\det(\lambda I - Q(X))}{\det(\lambda I - Q(X\backslash e))} &= \det(I - Z^t(\lambda I - Q(X\backslash e))^{-1} Z) \\ &= 1 - Z^t(\lambda I - Q(X\backslash e))^{-1} Z. \end{aligned}$$

Hence the roots of $\det(\lambda I - Q(X))$ interlaces the roots of $\det(\lambda I - Q(X\backslash e))$ by Lemma 5.24(4). Hence the results follow. □

Chapter 7

Matroids

7.1 Rank functions

Definition 7.1. Let Ω be a finite set. A *rank function* on Ω is a function $rk : \mathcal{P}(\Omega) \to \mathbb{N} \cup \{0\}$ such that

(1) If A and B are subsets of Ω and $A \subseteq B$, then $rk(A) \leq rk(B)$;

(2) For all subsets A and B of Ω,

$$rk(A \cap B) + rk(A \cup B) \leq rk(A) + rk(B);$$

(3) If $A \subseteq \Omega$, then $rk(A) \leq |A|$,

where $\mathcal{P}(A)$ is the set of all subsets of A.

Lemma 7.2. *Fix an $m \times n$ matrix D and let $\Omega = \{1, 2, \cdots, n\}$. For $A \subseteq \Omega$, $rk(A) :=$ the dimension of the subspace in $\mathbb{R}^m$ spanned by those columns of D indexed by A. Then rk is a rank function on Ω.*

Proof. The first and third conditions are clear. We check the second condition. Let $D_1, \cdots, D_n$ be the columns of D. Set $V = Span_{a \in A} D_a$, $W = Span_{b \in B} D_b$. Observe $rk(A \cap B) \leq \dim(V \cap W)$ and $rk(A \cup B) = \dim(V + W)$. Hence

$$\begin{aligned} rk(A \cap B) + rk(A \cup B) &\leq \dim(V \cap W) + \dim(V + W) \\ &= \dim(V) + \dim(W) = rank(A) + rank(B). \end{aligned}$$

□

Definition 7.3. Let Ω be a finite set with a rank function rk. Then $M := (\Omega, rk)$ is called a *matroid*.

Definition 7.4. Let $M := (\Omega, rk)$ be a matroid. Then $A \subseteq \Omega$ is *independent* if $rk(A) = |A|$. $A \subseteq \Omega$ is *dependent* if $rk(A) < |A|$. A *basis* of M is a maximal independent subset of Ω.

Example 7.5. Let $\Omega = \{1, 2, \cdots, n\}$ be a finite set and $rk(A) := |A|$ for any $A \subseteq \Omega$. Then for any subset $A \subseteq \Omega$ is independent. $\{1, 2, \cdots, n\}$ is a basis.

Example 7.6. Let $\Omega = \{1, 2, \cdots, n\}$ be a finite set and $rk(B) = 0$ for all $B \subseteq \Omega$. Hence $\emptyset$ is the only independent set and $\emptyset$ is a basis.

Theorem 7.7. *Let (Ω, rk) be a matroid and $A \subseteq \Omega$. Suppose $B \subseteq A$ is a maximal independent set in A. Then $rk(B) = rk(A) = |B|$.*

Proof. We prove the theorem by induction on $|A - B|$. If $A = B$, then $rk(A) = rk(B) = |B|$ is clear. In general, suppose $B \subsetneq A$. Pick $x \in A - B$. Consider $C := B \cup \{x\}$ and $D := A - \{x\}$. Then

$$rk(C \cap D) + rk(C \cup D) \le rk(C) + rk(D).$$

Observe $B = C \cap D$ and $A = C \cup D$. Hence

$$rk(A) + rk(B) \le rk(C) + rk(D). \tag{7.1}$$

Note

$$rk(C) \le |C| = |B| + 1$$

and

$$|B| = rk(B) \le rk(C).$$

Observe C is dependent since $B \subsetneq C$. Hence $rk(C) < |B| + 1$. Hence $rk(C) = |B| = rk(B)$. Thus we obtain $rk(A) \le rk(D)$ by equation(7.1). Hence $rk(A) = rk(D)$. Observe $B \subseteq D$ is a maximal independent set in D and $|D - B| < |A - B|$. By induction, $rk(B) = rk(D)$. Hence $rk(B) = rk(A)$. □

Corollary 7.8. *Let $M = (\Omega, rk)$ is a matroid. Then all bases of M have the same size $rk(\Omega)$.*

Proof. This is obvious by above Theorem. □

Lemma 7.9. *Let* (Ω, rk) *be a matroid. Then*

$$rk(\overline{A}) + |A| \leq rk(\overline{B}) + |B|$$

for any $A \subseteq B \subseteq \Omega$.

Proof. Observe

$$\begin{aligned} rk(\overline{A}) + |A| &= rk(\overline{B} \cup (B - A)) + |A| \\ &\leq rk(\overline{B}) + rk(B - A) + |A| \\ &\leq rk(\overline{B}) + |B - A| + |A| = rk(\overline{B}) + |B|. \end{aligned}$$

□

7.2 The dual

Definition 7.10. Let $M := (\Omega, rk)$ be a matroid. Define $rk^{\perp} : \mathcal{P}(\Omega) \to \mathbb{N} \cup \{\emptyset\}$ by

$$rk^{\perp}(A) = rk(\overline{A}) + |A| - rk(\Omega).$$

$rk^{\perp}$ is called the *dual* of rk.

Note 7.11. $rk^{\perp}(\emptyset) = rk(\Omega) + |\emptyset| - rk(\Omega) = 0$.

Lemma 7.12. *Let* $M := (\Omega, rk)$ *be a matroid. Then* $(rk^{\perp})^{\perp} = rk$.

Proof. Choose any subset $A \subseteq \Omega$. Observe

$$\begin{aligned} (rk^{\perp})^{\perp}(A) &= rk^{\perp}(\overline{A}) + |A| - rk^{\perp}(\Omega) \\ &= (rk(\overline{\overline{A}}) + |\overline{A}| - rk(\Omega)) + |A| - (rk(\overline{\Omega}) + |\Omega| - rk(\Omega)) \\ &= rk(A) \end{aligned}$$

□

Theorem 7.13. *Let* $M = (\Omega, rk)$ *be a matroid. Then* $(\Omega, rk^{\perp})$ *is a matroid.*

Proof. We check three conditions in Definition 7.1. The first condition is clear by Lemma 7.9. Choose two subsets A, $B \subseteq \Omega$. Observe

$$\begin{aligned} rk^{\perp}(A\cap B)+rk^{\perp}(A\cup B) &= (rk(\overline{A\cap B})+|A\cap B|-rk(\Omega)) \\ &\quad +(rk(\overline{A\cup B})+|A\cup B|-rk(\Omega)) \\ &= (rk(\overline{A}\cup\overline{B})+|A\cap B|-rk(\Omega)) \\ &\quad +(rk(\overline{A}\cap\overline{B})+|A\cup B|-rk(\Omega)) \\ &\leq rk(\overline{A})+rk(\overline{B})+|A|+|B|-rk(\Omega)-rk(\Omega) \\ &= rk^{\perp}(A)+rk^{\perp}(B). \end{aligned}$$

Hence the second condition holds. Observe

$$rk^{\perp}(A) = rk(\overline{A})+|A|-rk(\Omega) \leq |A|.$$

Hence the third condition holds. □

Definition 7.14. Let $M := (\Omega, rk)$ be a matroid. Then $M^{\perp} := (\Omega, rk^{\perp})$ is called the *dual* matroid of M.

Lemma 7.15. *The bases of $M^{\perp}$ are the complements of the bases of M.*

Proof. Let A be a basis of M. Then $rk(A) = |A| = rk(\Omega)$. Observe

$$\begin{aligned} rk^{\perp}(\overline{A}) &= rk(\overline{\overline{A}})+|\overline{A}|-rk(\Omega) \\ &= |A|+|\overline{A}|-rk(\Omega) = |\Omega|-rk(\Omega)+rk(\emptyset) = rk^{\perp}(\Omega). \end{aligned}$$

We also showed in the second equality, $rk^{\perp}(\overline{A}) = |\overline{A}|$. Hence $\overline{A}$ is a basis in $M^{\perp}$. □

7.3 The restriction and contraction

Definition 7.16. Let $M = (\Omega, rk)$ be a matroid and $T \subseteq \Omega$. Then $M \restriction T := (T, rk \restriction \mathcal{P}(T))$ is called the *restriction* of M into T.

Lemma 7.17. *Let $M = (\Omega, rk)$ be a matroid. Then $M \restriction T$ is a matroid.*

Proof. Let $A \subseteq B \subseteq T$ and $\varphi = rk \restriction \mathcal{P}(T)$. Then $\varphi(A) = rk(A) \leq rk(B) = \varphi(B)$. Hence the first condition holds. Similarly the second and third conditions hold. Hence the result follows. □

Definition 7.18. Let $M = (\Omega, rk)$ be a matroid and $T \subseteq \Omega$. Define $M/T := (\overline{T}, rk/T)$ where $rk/T : \mathcal{P}(\overline{T}) \to \mathbb{N} \cup \{\emptyset\}$ such that $rk/T(A) := rk(T \cup A) - rk(T)$. Then M/T is called the *contraction* of T on M.

Lemma 7.19. *Let $M = (\Omega, rk)$ be a matroid and $T \subseteq \Omega$. Then rk/T is a rank function on $\overline{T}$ and $(M/T)^{\perp} = M^{\perp} \restriction \overline{T}$.*

Proof. Define $\psi : \mathcal{P}(\overline{T}) \to \mathbb{N} \cup \{\emptyset\}$ by $\psi(A) = rk/T(\overline{T} - A) + |A| - rk/T(\overline{T})$. Observe

$$\begin{aligned} \psi(A) &= rk((\overline{T} - A) \cup T) - rk(T) + |A| - (rk(\overline{T} \cup T) - rk(T)) \\ &= rk(\overline{A}) + |A| - rk(\Omega) = rk^{\perp}(A) \end{aligned}$$

for all $A \subseteq \overline{T}$. Hence $\psi = rk^{\perp} \restriction \mathcal{P}(\overline{T})$ is a rank function on $\overline{T}$. Observe $\psi^{\perp} = rk/T$. Hence $rk/T = (rk^{\perp} \restriction \mathcal{P}(\overline{T}))^{\perp}$ is a rank function. □

Note 7.20. *We proved $rk/T = (rk^{\perp} \restriction \mathcal{P}(\overline{T}))^{\perp}$.*

Example 7.21. Let

$$D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let $\Omega = \{1, 2, 3\}$ and $T = \{1\}$, $\overline{T} = \{2, 3\}$. Define

$$rk(A) = \begin{cases} 1, & \text{if } A \neq \emptyset, \\ 0, & \text{if } A = \emptyset. \end{cases}$$

Observe

$$rk/T(A) = rk(T \cup A) - rk(T) = 1 - 1 = 0$$

for all $A \subseteq \overline{T}$. Hence for any $A \subseteq \overline{T}$,

$$rk^{\perp}(A) = rk(\overline{A}) + |A| - rk(\Omega) = 1 + |A| - 1 = |A|,$$

and

$$\begin{aligned} (rk^{\perp} \restriction \mathcal{P}(\overline{T}))^{\perp}(A) &= rk^{\perp} \restriction \mathcal{P}(\overline{T})(\overline{T} - A) + |A| - rk^{\perp} \restriction \mathcal{P}(\overline{T})(\overline{T}) \\ &= |\overline{T} - A| + |A| - |\overline{T}| = 0. \end{aligned}$$

Hence $rk/T = (rk^{\perp} \restriction \mathcal{P}(\overline{T}))^{\perp}$.

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