國 立 交 通 大 學

應用數學系

碩 士 論 文

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The Study of Chordal Ring Networks

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摘　要

「弦環式網路」是一種常被討論的區域網路架構 [1, 3, 8, 10, 11]。一個「無向的弦環式網路」是一個無向的三正則圖。在文獻[8, 10, 11]中，黃光明老師、陳尚寬學長、以及 Wright，將「無向的弦環式網路」推廣成「有向的弦環式網路」，並給出計算「有向的弦環式網路」的直徑的方法。在文獻[3]中，陳尚寬學長、黃光明老師、以及劉昱綺學姊又推廣「有向的弦環式網路」來提出另一種有向的網路的連法，稱為「混合的弦環式網路」。雖然「無向的弦環式網路」的直徑已被完整地研究、並且可以運用公式得出，但是截至目前為止，「有向的弦環式網路」的直徑、以及「混合的弦環式網路」的直徑卻還未被完全找出來。在這篇論文裡，我們首先推導「有向的弦環式網路」以及「混合的弦環式網路」的同構性質；我們接著得出某些特殊的「有向的弦環式網路」以及「混合的弦環式網路」的直徑，與之前文獻不同的是，我們並不需要先計算出對應的「雙環式網路」的直徑來得出這些直徑。

關鍵詞：弦環式網路、有向的弦環式網路、混合的弦環式網路、雙環式網路、直徑、同構。

中 華 民 國 九 十 三 年 六 月

The Study of Chordal Ring Networks

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Abstract

Chordal ring networks have been proposed as a popular architecture for local area networks [1, 3, 8, 10, 11]. An undirected chordal ring network is an undirected regular graph of degree 3. In [8, 10, 11], Hwang, Chen, and Wright proposed the directed version of the undirected chordal ring network and derived the diameter of a directed chordal ring network. Furthermore, in [3], Chen et al. proposed the mixed chordal ring network. While the diameter of an undirected chordal ring network has been well studied [1], the diameter of a directed chordal ring network and the diameter of a mixed chordal ring network are not known. In this thesis, we shall study the isomorphism property of chordal ring networks and we shall find out the diameter of some directed chordal ring networks and the diameter of some mixed chordal ring networks.

Keywords: Chordal ring network, directed chordal ring network, mixed chordal ring network, double-loop network, diameter, isomorphism.

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「只要給我一個立足點，我將能移動地球。」這是希臘數學家阿基米德曾說過的一句話。對我而言，本論文的完成就好比地球一般的巨大，單憑一己之力，絕無法以蜉蟻之力來撼動。而幸運的是生活周遭遇到了一群關心我的師長、同學與及朋友們。在生論文的過程中，給了我無數的幫助及建議。特別要感謝同門師兄唐文祥，因為有你提供不少寶貴的意見讓我的論文才能順利地完成。同時也要感謝博士班學長郭君逸及張飛黃和我同研究室的同學，貴弘、致維、正傑及宏嘉、三樓研究室的抮君、昭芳、桀丰、啟賢、喻培、嘉文與及二樓研究室的學妹們。對我來說，生命中能和這些人相遇，真的是我的福氣。

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1 Introduction

Chordal ring networks were first proposed by Arden and Lee [1]. An *undirected chordal ring network* $UCR(N, h)$ is an undirected graph with N nodes $0, 1, \cdots, N-1$ and $3N/2$ edges of two types:

$$i \leftrightarrow i+1 \pmod{N} \quad \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$i \leftrightarrow i+h \pmod{N} \quad \forall\, i = 1, 3, 5, \cdots, N-1,$$

where N is even, and h is odd. See Figure 1.

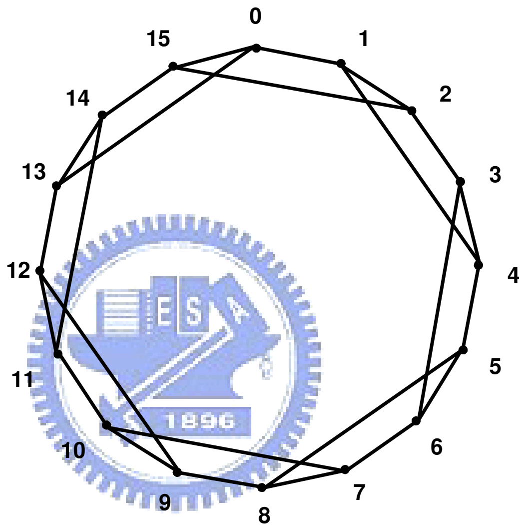


Figure 1: The undirected chordal ring network $UCR(16, 3)$.

Hwang and Wright [11] proposed the directed version of the undirected chordal ring network. A *directed chordal ring network* $DCR(N, 1, \overrightarrow{h})$ is a directed graph with N nodes $0, 1, \cdots, N-1$ and $3N/2$ links (i.e., directed edges) of two types:

$$i \rightarrow i+1 \pmod{N} \quad \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$i \rightarrow i+h \pmod{N} \quad \forall\, i = 1, 3, 5, \cdots, N-1,$$

where N is even and h is odd.

Hwang [8] generalized the directed chordal ring network $DCR(N, 1, \overrightarrow{h})$ to the *directed chordal ring network* $DCR(N, s, \overrightarrow{h})$. $DCR(N, s, \overrightarrow{h})$ is a directed graph with N nodes $0, 1, \cdots, N-1$ and $3N/2$ links of two types:

$$i \to i + s \pmod N \ \ \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$i \to i + h \pmod N \ \ \forall\, i = 1, 3, 5, \cdots, N-1,$$

where N is even, s is odd, and h is odd. See Figure 2.

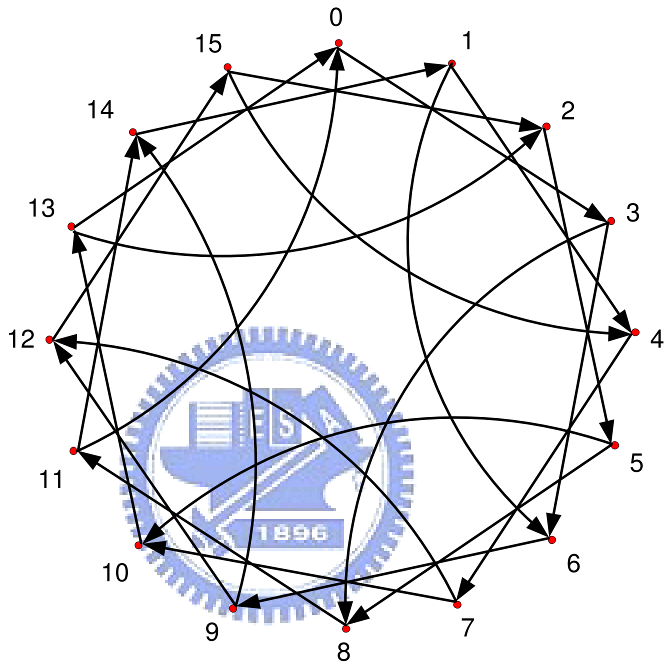


Figure 2: The directed chordal ring network $DCR(16, 3, \vec{5})$.

Chen et al. [3] proposed another type of directed chordal ring networks and denoted it as $DCR(N, s, \overleftarrow{h})$. A *directed chordal ring network* $DCR(N, s, \overleftarrow{h})$ is a directed graph with N nodes $0, 1, \cdots, N-1$ and $3N/2$ links of two types:

$$i \to i + s \pmod N \ \ \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$i + h \pmod N \to i \ \ \forall\, i = 1, 3, 5, \cdots, N-1,$$

where N is even, s is odd, and h is odd. Chen et al. [3] combined $DCR(N, s, \overrightarrow{h})$ and $DCR(N, s, \overleftarrow{h})$ and proposed the mixed chordal ring network $MCR(N, s, h)$.

More precisely, a *mixed chordal ring network* $MCR(N, s, h)$ is a directed graph with N nodes $0, 1, \cdots, N-1$ and $2N$ links types:

$$i \to i + s \quad (\text{mod } N) \;\; \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$i \to i + h \quad (\text{mod } N) \;\; \forall\, i = 1, 3, 5, \cdots, N-1,$$

$$i + h \quad (\text{mod } N) \to i \;\; \forall\, i = 1, 3, 5, \cdots, N-1,$$

where N is even, s is odd, and h is odd. See Figure 3.

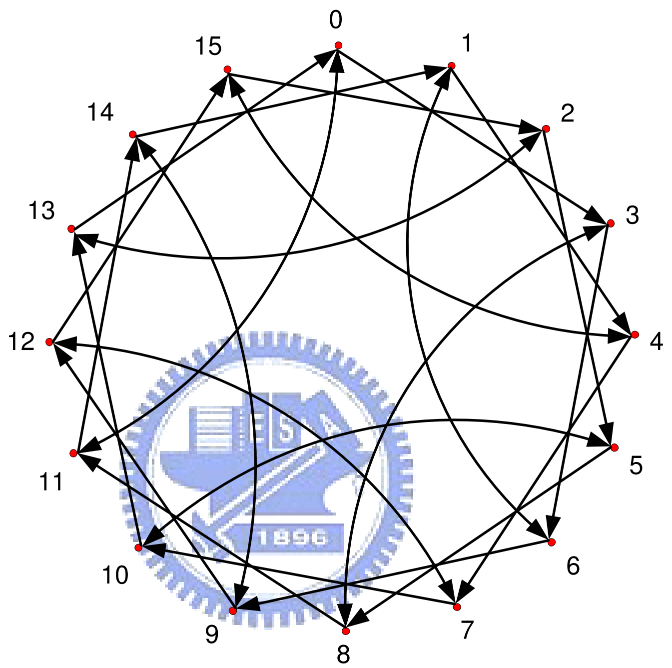


Figure 3: The mixed chordal ring network $MCR(16, 3, 5)$.

While the diameter of an undirected chordal ring network $UCR(N, h)$ has been well studied [1], the diameter of a directed chordal ring network $DCR(N, s, \overrightarrow{h})$ and the diameter of a mixed chordal ring network $MCR(N, s, h)$ are not known. In this thesis, we try to find the diameter of a chordal ring network.

This thesis is organized as follows. In Section 2, we describe previous results of the chordal ring networks. In section 3, we discuss the isomorphism properties of chordal ring networks. In section 4, we derive the diameter of some directed chordal ring networks and the diameter of some mixed chordal ring networks.

2 Previous results

In this section, we will briefly review previous results of chordal ring networks. Since most of these results depend on double-loop networks, we first define what is a double-loop network. A *double-loop network* $DL(N, a, b)$ is a directed graph with N nodes $0, 1, \cdots, N-1$ and $2N$ links:

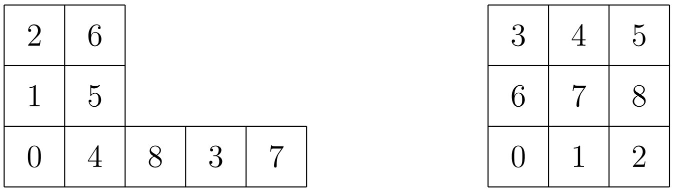
$$i \to i + a \pmod N \quad \forall\, i = 0, 1, 2, \cdots, N-1,$$
$$i \to i + b \pmod N \quad \forall\, i = 0, 1, 2, \cdots, N-1.$$

Fiol et al. [5] proved that $DL(N, a, b)$ is strongly connected if and only if gcd(N, a, b) $= 1$. For surveys of the double-loop networks, please refer to [7, 9].

When $DL(N, a, b)$ is strongly connected, then we can talk about a minimum distance diagram. This diagram gives a shortest path from node u to node v for any u, v. Since a double-loop network is node-symmetric, it suffices to give a shortest path from node 0 to any other node. Let 0 occupy cell (0,0). Then v occupies cell (i, j) if and only if $ia + jb \equiv v \pmod N$ and $i + j$ is the minimum among all (i', j') satisfying the congruence, where $\equiv$ means congruent modulo N. Namely, a shortest path from 0 to v is through taking i a-links and j b-links (in any order). Note that in a cell (i, j), i is the column index and j is the row index. A minimum distance diagram includes every node exactly once (in case of two shortest paths, the convention is to choose the cell with the smaller row index, i.e., the smaller j). Wong and Coppersmith [14] proved that the minimum distance diagram is always an L-shape (a rectangle is considered a degeneration). See Figure 4 for two examples.

An L-shape is determined by four parameters l, h, p, n as shown in Figure 5. These four parameters are the lengths of four of the six segments on the boundary of the L-shape. For example, $DL(9, 4, 1)$ in Figure 4 has $l = 5$, $h = 3$, $p = 3$, and $n = 2$.

The *diameter* of a network is the maximum distance over all node-pairs; it is the maximum transmission delay between two stations. Arden and Lee [1] derived the



$DL(9,4,1)$ $DL(9,1,6)$

Figure 4: Two examples of L-shapes.

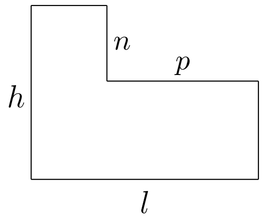


Figure 5: An L-shape with parameters.

diameter of an undirected chordal ring network $UCR(N,h)$ and proposed a routing algorithm. Without loss of generality, they assumed that $h \leq N/2$.

Theorem 1 [1] *Let $UCR(N,h)$ be an undirected chordal ring network and $i = \lceil \frac{N}{2(h+1)} \rceil$, $\triangle = \frac{N}{2} \pmod{h+1}$. Then the diameter D of $UCR(N,h)$ is given by*

When $\boxed{i < \frac{h-3}{2}}$ *and*

- $\triangle = 0$ $\Rightarrow D = i + \frac{h-1}{2}$;
- $1 \leq \triangle \leq \frac{h+1}{2} - 1$ $\Rightarrow D = i + \frac{h-3}{2}$;
- $\triangle = \frac{h+3}{2} - i$ $\Rightarrow D = i + \frac{h-1}{2}$;
- $\frac{h+5}{2} - i \leq \triangle \leq h - i$ $\Rightarrow D = i + \frac{h-3}{2}$;
- $h - i + 1 \leq \triangle \leq h$ $\Rightarrow D = i + \frac{h-1}{2}$.

When $\boxed{i = \frac{h-3}{2}}$ *and*

- $\triangle = 0$ $\Rightarrow D = h - 2$;

- $1 \leq \triangle \leq 2$ $\Rightarrow D = h - 3;$

- $3 \leq \triangle \leq h$ $\Rightarrow D = h - 2.$

When $\boxed{i \geq \frac{h-1}{2}}$ *and*

- $\triangle = 0$ $\Rightarrow D = 2i + 1;$

- $\triangle = 1$ $\Rightarrow D = 2i - 1;$

- $2 \leq \triangle \leq \frac{h+3}{2}$ $\Rightarrow D = 2i;$

- $\frac{h+5}{2} \leq \triangle \leq h$ $\Rightarrow D = 2i + 1.$

Hwang and Wright [11] proposed the directed chordal ring network $DCR(N, 1, \overrightarrow{h})$. They observed that by combining two nodes in $DCR(N,\ 1,\ \overrightarrow{h})$ as a supernode, $DCR(N,\ 1,\ \overrightarrow{h})$ is reduced to the double-loop network $DL(\frac{N}{2},\ 1,\ \frac{1+h}{2})$.

Hwang [8] generalized the directed chordal ring network $DCR(N, 1, \overrightarrow{h})$ to the *directed chordal ring network* $DCR(N, s, \overrightarrow{h})$. In [8], Hwang called $DCR(N, s, \overrightarrow{h})$ a *1.5 loop network*. The 1.5 loop network is derived by allowing the full ring of $DCR(N, 1, \overrightarrow{h})$ to consist of several subrings instead of a hamiltonian circuit. Hwang proved that

Lemma 2 [8] *A necessary condition for* $DCR(N, s, \overrightarrow{h})$ *to be strongly connected is* s *is an odd integer.*

Hwang [8] observed that by combining two nodes in $DCR(N, s, \overrightarrow{h})$ as a supernode, $DCR(N, s, \overrightarrow{h})$ is reduced to the double-loop network $DL(\frac{N}{2}, s, \frac{s+h}{2})$; he used this to prove

Theorem 3 [8] *The diameter of* $DCR(N, s, \overrightarrow{h}) = 1 + 2\times$ *the diameter of* $DL(\frac{N}{2},\ s,\ \frac{s+h}{2})$.

Hwang [8] also proved that

Theorem 4 [8] *$DCR(N, s, \overrightarrow{h})$ is strongly connected if and only if* $\gcd(N, s, h) = 1$.

Theorem 5 [8] *$DCR(N, s, \overrightarrow{h})$ has a hamiltonian circuit if and only if its corresponding double-loop network $DL(\frac{N}{2}, s, \frac{s+h}{2})$ does.*

Chen et al. [3] combined $DCR(N, s, \overrightarrow{h})$ and $DCR(N, s, \overleftarrow{h})$ and proposed the mixed chordal ring network $MCR(N, s, h)$. They proved that the mixed chordal ring network also has the above two properties since

Theorem 6 [3] *$MCR(N, s, h)$ is strongly connected if and only if gcd$(N, s, h) = 1$.*

Theorem 7 [3] *$MCR(N, s, h)$ has a hamiltonian circuit if and only if $DL(\frac{N}{2}, s, \frac{s+h}{2})$ or*
$DL(\frac{N}{2}, s, \frac{s-h}{2})$ does.

Chen et al. [3] also proved

Theorem 8 [3] *Let D be the diameter of $MCR(N, s, h)$. Then $D \geq (2N)^{\frac{1}{2}} + o(N)$.*

Chen et al. [3] observed that by combining two nodes in $MCR(N, s, h)$ as a supernode, $MCR(N, s, h)$ is reduced to the double-loop network $DL(\frac{N}{2}, \frac{s-h}{2}, \frac{s+h}{2})$; they used this to prove

Theorem 9 [3] *Let D be the diameter of $MCR(N, s, h)$. Let $DL(\frac{N}{2}, \frac{s-h}{2}, \frac{s+h}{2})$ be the corresponding double-loop network of $MCR(N, s, h)$ and assume that the L-shape of $DL(\frac{N}{2}, \frac{s-h}{2}, \frac{s+h}{2})$ has lengths l, h, p, n. Then $D \leq 2\max\{l, h\} - 1$.*

3 Isomorphism

Two directed graphs G_1 and G_2 are *isomorphic* if there is a bijection function f from $V(G_1)$ to $V(G_2)$ such that $u \to v$ is a link in $E(G_1)$ if and only if $f(u) \to f(v)$ is a link in $E(G_2)$. When G_1 and G_2 are isomorphic, we will write $G_1 \cong G_2$. Note that unless otherwise specified, all the nodes in this thesis are considered to be taken modular N. That is, node $i+1$ is the node $i+1 \pmod N$ and node $i+h$ is the node $i+h \pmod N$. We now prove

Theorem 10 $DCR(N, s, \overleftarrow{h}) \cong DCR(N, s, \overrightarrow{N-h})$.

Proof. By definition, $DCR(N, s, \overleftarrow{h})$ is a directed graph with N nodes $0, 1, \cdots, N-1$ and $3N/2$ links of two types:

$$i \rightarrow i+s \pmod N \;\; \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$i+h \pmod N \rightarrow i \;\; \forall\, i = 1, 3, 5, \cdots, N-1,$$

where N is even, s is odd, and h is odd. By definition, $DCR(N, s, \overrightarrow{N-h})$ is a directed graph with N nodes $0, 1, \cdots, N-1$ and $3N/2$ links of two types:

$$i \rightarrow i+s \pmod N \;\; \forall\, i = 0, 1, 2, \cdots, N-1,$$

$$i \rightarrow i+N-h \pmod N \;\; \forall\, i = 1, 3, 5, \cdots, N-1,$$

where N is even, s is odd, and h is odd. Let f be a function from the nodes of $DCR(N, s, \overleftarrow{h})$ to the nodes $DCR(N, s, \overrightarrow{N-h})$ such that

$$f(i) = i+1 \pmod N \;\; \forall\, i = 0, 1, 2, \cdots, N-1.$$

First consider the following type of links in $DCR(N, s, \overleftarrow{h})$:

$$i \rightarrow i+s \pmod N \;\; \forall\, i = 0, 1, 2, \cdots, N-1.$$

Since $f(i) = i+1 \pmod N$ and $f(i+s) = i+1+s \pmod N$, it is clear that $f(i) \rightarrow f(i+s)$ is a link in $DCR(N, s, \overrightarrow{N-h})$. Now consider the following type of links in $DCR(N, s, \overleftarrow{h})$:

$$i+h \pmod N \rightarrow i \;\; \forall\, i = 1, 3, 5, \cdots, N-1.$$

Note that $f(i+h) = i+h+1 \pmod N$ and $f(i) = i+1 \pmod N$. Since i is odd and h is odd and N is even, $i+h+1 \pmod N$ is odd. By the definition of $DCR(N, s, \overrightarrow{N-h})$, the node $i+h+1 \pmod N$ has a link to $(i+h+1 \pmod N) + N-h \pmod N$, which is the node $i+1 \pmod N$. Thus $f(i+h) \rightarrow f(i)$ is a link in $DCR(N, s, \overrightarrow{N-h})$. From the above, we have proved that if $u \rightarrow v$ is a link in $DCR(N,s,\overleftarrow{h})$, then $f(u) \to f(v)$ is a link in $DCR(N,s,\overrightarrow{N-h})$. Note that $DCR(N,s,\overrightarrow{N-h})$ has the same number of links as $DCR(N,s,\overleftarrow{h})$. Thus if $f(u) \to f(v)$ is a link in $DCR(N,s,\overrightarrow{N-h})$, then $u \to v$ is a link in $DCR(N,s,\overleftarrow{h})$. Hence $DCR(N,s,\overleftarrow{h}) \cong DCR(N,s,\overrightarrow{N-h})$. ∎

Similarly, we have

Theorem 11 $DCR(N,s,\overrightarrow{h}) \cong DCR(N,s,\overleftarrow{N-h})$.

We now prove

Theorem 12 $MCR(N,s,h) \cong MCR(N,s,N-h)$.

Proof. By definition, $MCR(N,s,h)$ is the combination of $DCR(N,s,\overrightarrow{h})$ and $DCR(N,\ s,\ \overleftarrow{h})$. Also, $MCR(N,s,N-h)$ is the combination of $DCR(N,s,\overrightarrow{N-h})$ and $DCR(N,\ s,\ \overleftarrow{N-h})$. By Theorem 11, $DCR(N,\ s,\ \overrightarrow{h}) \cong DCR(N,s,\overleftarrow{N-h})$. By Theorem 10, $DCR(N,\ s,\ \overleftarrow{h}) \cong DCR(N,s,\overrightarrow{N-h})$. Thus $MCR(N,s,h) \cong MCR(N,\ s,\ N-h)$. ∎

Theorem 13 *Suppose* $\gcd(N,s)=1$. *Then*

$$DCR(N,s,\overrightarrow{h}) \cong DCR(N,1,\overrightarrow{h_1}),$$

where h_1 *is the unique integer in* $\{1,2,\cdots,N-1\}$ *satisfying*

$$h_1 s \equiv h \pmod{N}.$$

Proof. Since $\gcd(N,s)=1$, we have

$$\{i \times s \pmod{N} : i = 0,1,2,\cdots,\ N-1\} = \{0,\ 1,\ 2,\cdots,\ N-1\}.$$

Consider the nodes s and $s+h$ in $DCR(N,s,\overrightarrow{h})$. Suppose

$$s + h = k \times s \pmod{N} \tag{3.1}$$

for some integer k in $\{0, 1, , \cdots, N-1\}$. Let

$$h_1 = k - 1 \pmod{N}.$$

Since $s \neq s + h$, we have $k \neq 1$ and $h_1 \neq 0$. Since $h_1 s \equiv (k-1) \times s \pmod{N}$, by (3.1), we have $h_1 s \equiv h \pmod{N}$. From the above, h_1 is the unique integer in $\{1, 2, \cdots, N-1\}$ such that

$$h_1 s \equiv h \pmod{N}.$$

Let f be a function from the nodes of $DCR(N, s, \overrightarrow{h})$ to the nodes $DCR(N$, 1, $\overrightarrow{h}_1)$ such that

$$f(i \times s) = i \ \ \forall \ i = 0, 1, 2, \cdots, N-1.$$

First consider the following type of links in $DCR(N, s, \overrightarrow{h})$:

$$i \to i + s \pmod{N} \ \forall \ i = 0, 1, 2, \cdots, N-1.$$

Suppose $i = m \times s \pmod{N}$ for some integer m in $\{0, 1, , \cdots, N-1\}$. Then $f(i) = m$ and $f(i+s) = f(ms+s) = f((m+1)s) = m+1$. Since $m \to m+1$ is a link in $DCR(N, 1, \overrightarrow{h}_1)$, it is clear that $f(i) \to f(i+s)$ is a link in $DCR(N, 1, \overrightarrow{h}_1)$.

Now consider the following type of links in $DCR(N, s, \overrightarrow{h})$:

$$i \to i + h \pmod{N} \ \ \forall \ i = 1, 3, 5, \cdots, N-1.$$

Let i be an odd integer in $\{1, 3, 5, \cdots, N-1\}$. Suppose

$$i = m \times s \pmod{N}$$

for some integer m in $\{0, 1, , \cdots, N-1\}$. Since i is odd, m is odd. Consider the set of nodes $\{0, 1, 2, \cdots, N-1\}$ of $DCR(N, s, \overrightarrow{h})$. Since

$$\{0, \ 1, \ 2, \cdots, \ N-1\} = \{i \times s \pmod{N} : i = 0, 1, 2, \cdots, \ N-1\}$$

and $i + h$ is a node in $DCR(N, s, \overrightarrow{h})$, we have

$$i + h \equiv q \times s \pmod{N}$$

for some integer q in $\{0, 1, 2, \cdots, N-1\}$. Then

$$f(i) = m \text{ and } f(i+h) = q.$$

Since $i = m \times s \pmod N$ and $i + h \equiv q \times s \pmod N$, we have

$$(q-m) \times s \equiv h \pmod N.$$

Since $(q-m) \times s \equiv h \pmod N$ and $h_1 s \equiv h \pmod N$, we have

$$(q-m) \times s \equiv h_1 s \pmod N.$$

Thus

$$ms + h_1 s \equiv qs \pmod N.$$

Since $\gcd(N, s) = 1$,

$$m + h_1 \equiv q \pmod N.$$

Since m is odd, there is a link $m \to m + h_1$ in $DCR(N, 1, \overrightarrow{h}_1)$; i.e., $m \to q$ is a link in $DCR(N, 1, \overrightarrow{h}_1)$. That is, $f(i) \to f(i+h)$ is a link in $DCR(N, 1, \overrightarrow{h}_1)$.

¿From the above, we have proved that if $u \to v$ is a link in $DCR(N, s, \overrightarrow{h})$, then $f(u) \to f(v)$ is a link in $DCR(N, 1, \overrightarrow{h}_1)$. Note that $DCR(N, 1, \overrightarrow{h}_1)$ has the same number of links as $DCR(N, s, \overrightarrow{h})$. Thus if $f(u) \to f(v)$ is a link in $DCR(N, 1, \overrightarrow{h}_1)$, then $u \to v$ is a link in $DCR(N, s, \overrightarrow{h})$. Hence $DCR(N, s, \overrightarrow{h}) \cong DCR(N, 1, \overrightarrow{h}_1)$. ∎

Similarly, we have

Theorem 14 *Suppose* $\gcd(N, s) = 1$. *Then*

$$DCR(N, s, \overleftarrow{h}) \cong DCR(N, 1, \overrightarrow{h_1}),$$

where h_1 *is the unique integer in* $\{1, 2, \cdots, N-1\}$ *satisfying*

$$h_1 s \equiv -h \pmod N.$$

Proof. By Theorem 10, $DCR(N, s, \overleftarrow{h}) \cong DCR(N, s, \overrightarrow{N-h})$. By Theorem 13, $DCR(N, s, \overrightarrow{N-h}) \cong DCR(N, 1, \overrightarrow{h_1})$, where h_1 is the unique integer in $\{1, 2, \cdots, N-1\}$ satisfying

$$h_1 s \equiv N - h \pmod{N}.$$

Thus we have this theorem. ∎

Furthermore, we have

Theorem 15 *Suppose* $\gcd(N, s) = 1$. *Then*

$$MCR(N, s, h) \cong MCR(N, 1, h_1),$$

where h_1 *is the unique integer in* $\{1, 2, \cdots, N-1\}$ *satisfying*

$$h_1 s \equiv h \pmod{N}.$$

Proof. $MCR(N, s, h)$ is the combination of $DCR(N, s, \overrightarrow{h})$ and $DCR(N, s, \overleftarrow{h})$. By Theorem 13,

$$DCR(N, s, \overrightarrow{h}) \cong DCR(N, 1, \overrightarrow{h_1}),$$

where h_1 is the unique integer in $\{1, 2, \cdots, N-1\}$ satisfying

$$h_1 s \equiv h \pmod{N}.$$

and by Theorem 14,

$$DCR(N, s, \overleftarrow{h}) \cong DCR(N, 1, \overrightarrow{h_2}),$$

where h_2 is the unique integer in $\{1, 2, \cdots, N-1\}$ satisfying

$$h_2 s \equiv -h \pmod{N}.$$

Thus $h_2 s \equiv -h \pmod{N}$; i.e., $-h_2 s \equiv h \pmod{N}$. Hence

$$h_1 \equiv -h_2 \pmod{N}.$$

By Theorem 11, $DCR(N,1,\overrightarrow{h_2}) \cong DCR(N,1,\overleftarrow{N-h_2})$. Since $h_1 \equiv -h_2 \pmod{N}$, we have

$$DCR(N,s,\overleftarrow{h}) \cong DCR(N,1,\overleftarrow{N-h_2}) \cong DCR(N,1,\overleftarrow{N+h_1}) \cong DCR(N,1,\overleftarrow{h_1}).$$

We have this theorem. ∎

Before going further, we describe how to transform the directed chordal ring network $DCR(N,s,\overrightarrow{h})$ into the double-loop network $DL(\frac{N}{2},s,\frac{s+h}{2})$. For each node i in $DCR(N,\ s,\ \overrightarrow{h})$, there is a link from i to $i+s$. Since s is odd, when i is even, $i+s$ is odd. For each i in $\{0,2,4,\cdots,N-2\}$, merge the pair of nodes i and $i+s$ as a supernode and denote it by $(i/2)^*$. Let the set of supernodes

$$\{(i/2)^* : i = 0,2,4,\cdots,N-2\}$$

be the set of nodes of the double-loop network. See Figure 6 for an illustration. The set of links of the double-loop network is derived as follows. For each node $(i/2)^*$ in the double-loop network, since

$$i+s \rightarrow i+2s \ \text{ is a link in } DCR(N,s,\overrightarrow{h})$$

and

$$i+s \rightarrow i+s+h \ \text{ is a link in } DCR(N,s,\overrightarrow{h}),$$

in the double-loop network,

$$(i/2)^* \rightarrow (i/2+s)^* \ \text{ is a link}$$

and

$$(i/2)^* \rightarrow (i/2+(s+h)/2)^* \ \text{ is a link.}$$

From the above, $DCR(N,s,\overrightarrow{h})$ is transformed into the double-loop network $DL(\frac{N}{2},s,\frac{s+h}{2})$.

See Figure 7 for an example.

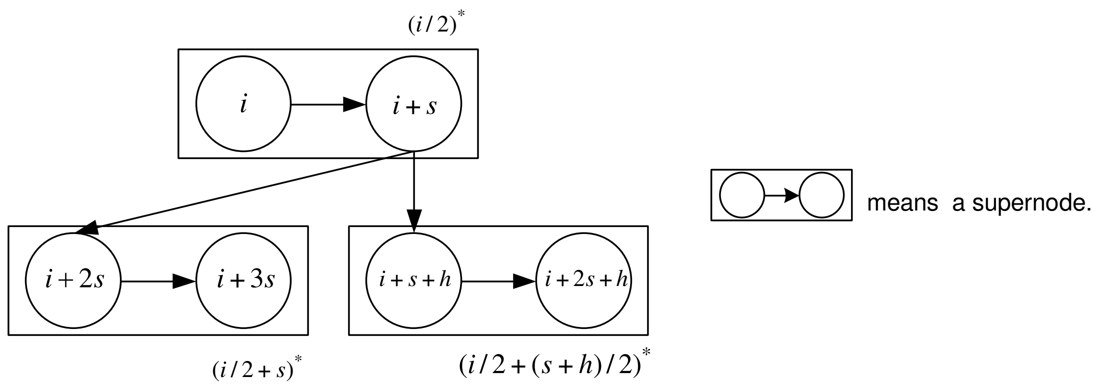


Figure 6: i is even, the links $(i/2)^* \to (i/2+s)^*$ and $(i/2)^* \to (i/2+(s+h)/2)^*$.

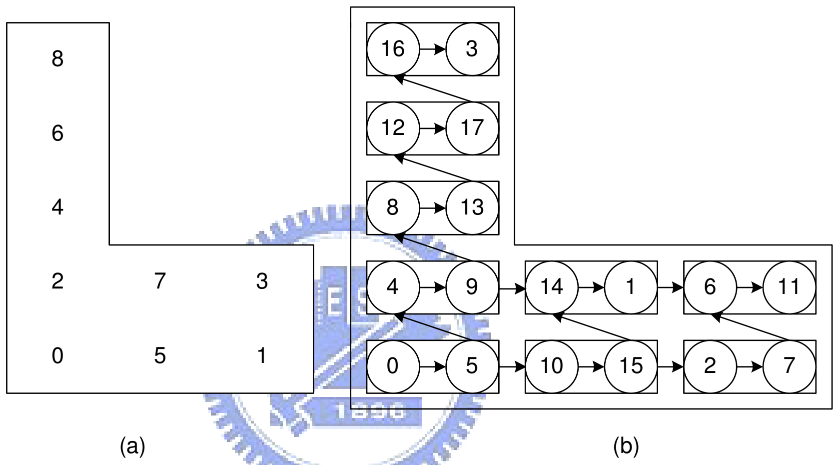


Figure 7: (a) The L-shape of $DL(9,5,2)$. (b) Transforming $DCR(18,5,\vec{17})$ into $DL(9,5,2)$.

We now describe how to transform the directed chordal ring network $DCR(N, s, \overrightarrow{h})$ into the double-loop network $DL(\frac{N}{2}, s, \frac{h-s}{2})$. For each node i in $DCR(N, s, \overrightarrow{h})$, there is a link from i to $i+s$. Since s is odd, when i is odd, $i+s$ is even. For each i in $\{1,3,5,\cdots,N-1\}$, merge the pair of nodes i and $i+s$ as a supernode and denote it by $((i-1)/2)^*$. Let the set of supernodes

$$\{((i-1)/2)^* : i = 1,3,5,\cdots,N-1\}$$

be the set of nodes of the double-loop network. See Figure 8 for an illustration. The set of links of the double-loop network is derived as follows. For each node

$((i-1)/2)^*$ in the double-loop network, since

$$i+s \to i+2s \text{ is a link in } DCR(N, s, \overrightarrow{h})$$

and

$$i \to i+h \text{ is a link in } DCR(N, s, \overrightarrow{h}),$$

in the double-loop network,

$$((i-1)/2)^* \to ((i-1)/2+s)^* \text{ is a link}$$

and

$$((i-1)/2)^* \to ((i-1)/2+(h-s)/2)^* \text{ is a link.}$$

From the above, $DCR(N, s, \overrightarrow{h})$ is transformed into the double-loop network $DL(\frac{N}{2}$, s, $\frac{h-s}{2})$. See Figure 9 for an example.

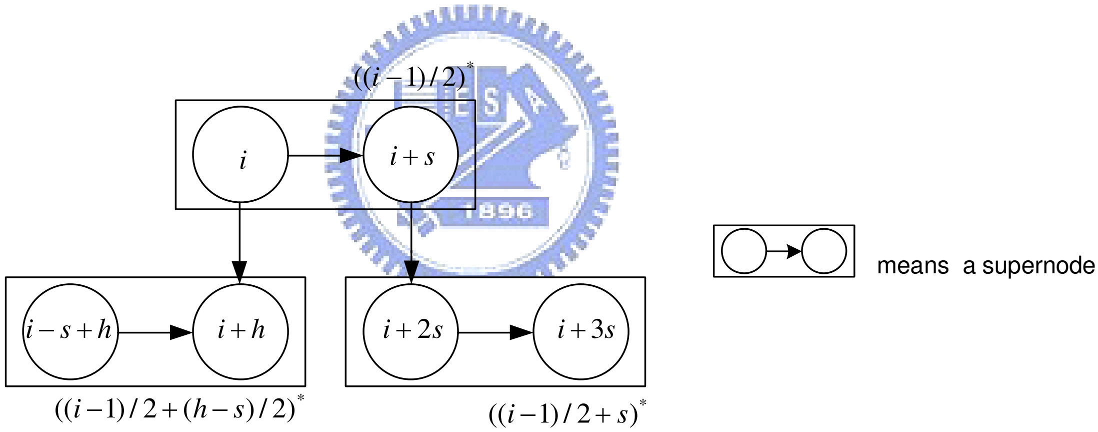


Figure 8: i is odd, the links $((i-1)/2)^* \to ((i-1)/2+s)^*$ and $((i-1)/2)^* \to ((i-1)/2+(h-s)/2)^*$.

Furthermore, since $DCR(N, s, \overleftarrow{h}) \cong DCR(N, s, \overrightarrow{N-h})$ and $DCR(N, s, \overrightarrow{N-h})$ can be transformed into $DL(\frac{N}{2}, s, \frac{s-h}{2})$ (when i is even) and into $DL(\frac{N}{2}, s, \frac{-s-h}{2})$ (when i is odd), $DCR(N, s, \overleftarrow{h})$ can be transformed into the double-loop networks $DL(\frac{N}{2}, s, \frac{s-h}{2})$ and $DL(\frac{N}{2}, s, \frac{-s-h}{2})$.

As for $MCR(N, s, h)$, we describe how to transform $MCR(N, s, h)$ into the double-loop network $DL(\frac{N}{2}, \frac{s-h}{2}, \frac{s+h}{2})$. For each node i in $MCR(N, s, h)$ where i

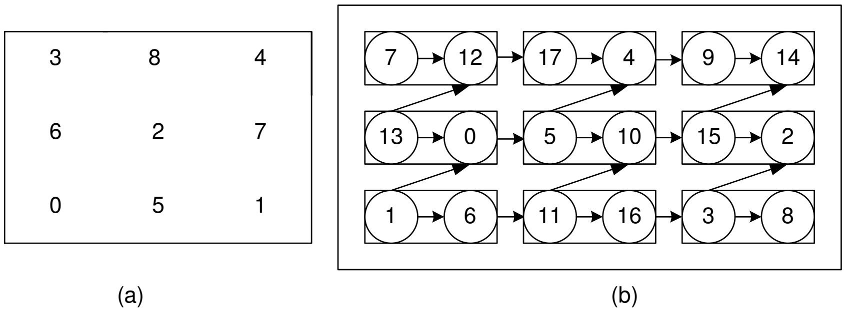


Figure 9: (a) The L-shape of $DL(9, 5, 6)$. (b) Transforming $DCR(18, 5, \vec{17})$ into $DL(9, 5, 6)$.

is odd, there is a link from i to $i + h$. Since h is odd, $i + h$ is even. For each i in $\{1, 3, 5, \cdots, N - 1\}$, merge the pair of nodes i and $i+h$ as a supernode and denote it by $((i - 1)/2)^*$. Let the set of supernodes

$$\{((i-1)/2)^* : i = 1, 3, 5, \cdots, N-1\}$$

be the set of nodes of the double-loop network. The set of links of the double-loop network is derived as follows. For each node $((i-1)/2)^*$ in the double-loop network, since

$$i \to i + s \quad \text{is a link in } MCR(N, s, h)$$

and

$$i + h \to i + h + s \quad \text{is a link in } MCR(N, s, h),$$

in the double-loop network,

$$((i-1)/2)^* \to ((i-1)/2 + (s-h)/2)^* \quad \text{is a link}$$

and

$$((i-1)/2)^* \to ((i-1)/2 + (s+h)/2)^* \quad \text{is a link.}$$

From the above, $MCR(N, s, h)$ is transformed into the double-loop network $DL(\frac{N}{2}, \frac{s-h}{2}, \frac{s+h}{2})$.

Theorem 16 *Given a directed chordal ring network* $DCR(N, s, \overrightarrow{h})$*, if* $\gcd(k, N) = 1$*, then*

$$DCR(N, s, \overrightarrow{h}) \cong DCR(N, ks, k\overrightarrow{h}).$$

Proof. From previous discussion, $DCR(N, s, \overrightarrow{h})$ corresponds to $DL(\frac{N}{2}, s, \frac{s+h}{2})$. Note that if $\gcd(k, N) = 1$, then $DL(N, a, b) \cong DL(N, ka, kb)$; see [6]. Since $\gcd(k, N) = 1$, $\gcd(k, N/2) = 1$. Thus $DL(\frac{N}{2}, s, \frac{s+h}{2}) \cong DL(\frac{N}{2}, ks, k(\frac{s+h}{2}))$. Since $DL(\frac{N}{2}, ks, k(\frac{s+h}{2}))$ is exactly $DL(\frac{N}{2}, ks, \frac{ks+kh}{2})$ and $DL(\frac{N}{2}, ks, \frac{ks+kh}{2})$ corresponds to $DCR(N, ks, \overrightarrow{kh})$, we have this theorem. ∎

Similarly, we have

Theorem 17 *Given a directed chordal ring network* $DCR(N, s, \overleftarrow{h})$*, if* $\gcd(k, N) = 1$*, then*

$$DCR(N, s, \overleftarrow{h}) \cong DCR(N, ks, \overleftarrow{kh}).$$

Theorem 18 *Given a mixed chordal ring network* $MCR(N, s, h)$*, if* $\gcd(k, N) = 1$*, then*

$$MCR(N, s, h) \cong MCR(N, ks, kh).$$

Proof. Since $MCR(N, s, h)$ is the combination of $DCR(N, s, \overrightarrow{h})$ and $DCR(N, s, \overleftarrow{h})$. By Theorem16 and Theorem17, we have

$$MCR(N, s, h) \cong MCR(N, ks, kh).$$

∎

Corollary 19 $MCR(N, s, h) \cong MCR(N, -s, -h)$*.*

Proof. Take $k = -1$. Then $\gcd(k, N) = 1$. By Theorem 18, $MCR(N, s, h) \cong MCR(N, -s, -h)$. ∎

Corollary 20 $MCR(N, s, h) \cong MCR(N, -s, h)$*.*

Proof. By Theorem 12 and Corollary 19, $MCR(N,s,h) \cong MCR(N,s,N-h) \cong MCR(N,-s,-N+h)$. Since $MCR(N,-s,-N+h) \cong MCR(N,-s,h)$, we have this corollary. ∎

Let's look at an example. By Corollary 20, $MCR(16,3,5) \cong MCR(16,13,5)$. Since $\gcd(16,13) = 1$, by Theorem 15, we have $MCR(16,13,5) \cong MCR(16,1,9)$. Thus the mixed chordal ring networks in Figure 3 and Figure 10 are isomorphic.

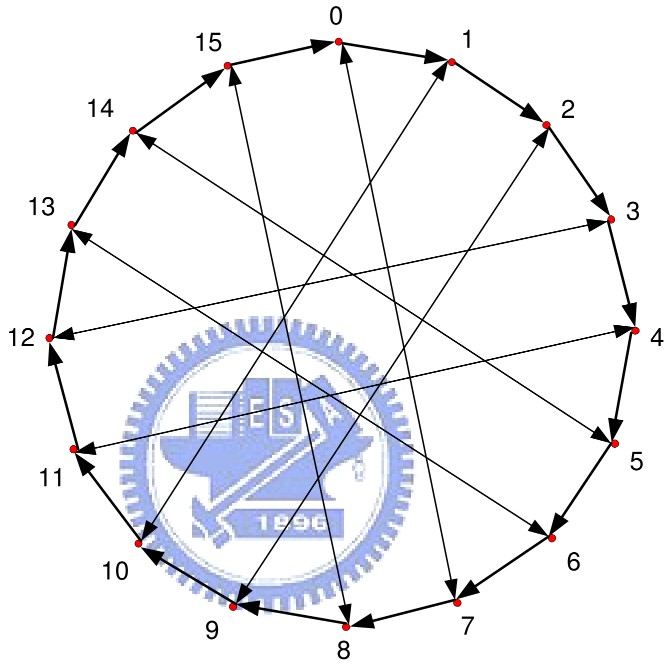


Figure 10: $MCR(16,1,9)$; it is isomorphic to Figure 3.

4 The diameter of $DCR(N,s,\vec{h})$ and $MCR(N,s,h)$

Note that the diameter of a double-loop network $DL(N,a,b)$ can be computed in $O(\log N)$ time using the Cheng-Hwang algorithm [4]. Therefore, by Theorem 3 the diameter of $DCR(N,s,\overrightarrow{h})$ can be derived in $O(\log N)$ time. However, unless we perform the Cheng-Hwang algorithm, the diameter of $DCR(N,s,h)$ is not known.

In this section, we will derive of the diameter of some $DCR(N, s, \overrightarrow{h})$ directly. We will also derive the diameter of some $MCR(N, s, h)$ directly.

In the previous section, we have shown how to transform $DCR(N, s, \overrightarrow{h})$ into $DL(\frac{N}{2},\ s,\ \frac{s+h}{2})$. Recall that Wong and Coppersmith [14] proved that the minimum distance diagram of a double-loop network is an L-shape. Let $d(k)$ denote the number of cells (i, j) in an L-shape of a double-loop network with $i + j = k$. Hwang and Xu [12] defined two double-loop networks to be *equivalent* if they have the same $d(k)$ for every k. Note that two equivalent double-loop networks have the same diameter. In [13], Rödseth proved that $DL(N, a, b)$ is equivalent to $DL(N, N - a, b - a)$. In [2], Chen and Hwang proved that $DL(N, N - a, b - a)$ is equivalent to $DL(N, a, a - b)$ and thus $DL(N, a, b)$ is equivalent to $DL(N, a, a - b)$. For example, $DL(9, 1, 7)$ is equivalent to $DL(9, 1, -6)$, which is $DL(9, 1, 3)$. See Figure 11.

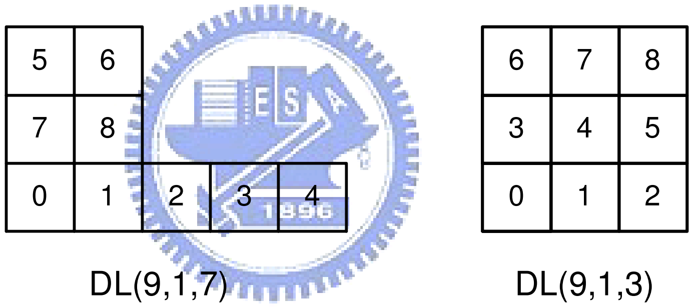


Figure 11: Two equivalent L-shapes.

We have the following theorem.

Theorem 21 *Let D_1 be the diameter of $DCR(N, s, \overrightarrow{h})$ and D_2 be the diameter of $DCR\ (N, s, \overleftarrow{h})$. Then $D_1 = D_2$.*

Proof. The corresponding double-loop network of $DCR(N, s, \overrightarrow{h})$ is $DL(\frac{N}{2}, s, \frac{s+h}{2})$ and the corresponding double-loop network of $DCR(N,\ s,\ \overleftarrow{h})$ is $DL(\frac{N}{2}, s, \frac{s-h}{2})$. Note that $DL(\frac{N}{2}, s, \frac{s+h}{2})$ is equivalent to $DL(\frac{N}{2}, s, s - \frac{s+h}{2})$. Since $DL(\frac{N}{2}, s, s - \frac{s+h}{2})$ is exactly $DL(\frac{N}{2}, s, \frac{s-h}{2})$, $DL(\frac{N}{2}, s, \frac{s+h}{2})$ is equivalent to $DL(\frac{N}{2}, s, \frac{s-h}{2})$. Thus

$DL(\frac{N}{2}, s, \frac{s+h}{2})$ and $DL(\frac{N}{2}, s, \frac{s-h}{2})$ have the same diameter. By Theorem 3, we can have $D_1 = D_2$. ∎

Let's see an application of the above theorems. It can be seen from Figure 11 that the diameter of $DL(9, 1, 7)$ is 4. By Theorem 3, the diameter of $DCR(18, 1, \overrightarrow{13})$ is 9. By Theorem 21, the diameter of $DCR(18, 1, \overleftarrow{13})$ is also 9. By Theorem 11, the diameter of $DCR(18, 1, \overrightarrow{5})$ is also 9.

Theorem 22 *Let D be the diameter of $DCR(N, s, \overrightarrow{h})$. If $s = h$, then $D = N - 1$.*

Proof. When $s = h$, the corresponding double-loop network of $DCR(N, s, \overrightarrow{h})$ is $DL(N/2, s, s)$, whose diameter is $N/2 - 1$. Thus by Theorem 3, $D = 1 + 2(N/2 - 1) = N - 1$. ∎

Theorem 23 *Let D be the diameter of $DCR(N, s, \overrightarrow{h})$. If $s + h = N$, then $D = N - 1$.*

Proof. When $s + h = N$, the corresponding double-loop network of $DCR(N, s, \overrightarrow{h})$ is $DL(N/2, s, 0)$, whose diameter is $N/2 - 1$. Thus by Theorem 3, $D = 1 + 2(N/2 - 1) = N - 1$. ∎

Corollary 24 *Let D be the diameter of $DCR(N, s, \overleftarrow{h})$. If $s = h$ or $s + h = N$, then $D = N - 1$.*

Proof. This corollary follows from Theorem 21, Theorem 22, and Theorem 23. ∎

In the following, we try to derive the diameter of $DCR(N, 1, \overrightarrow{h})$. Recall that h is odd. Let D be the diameter of $DCR(N, 1, \overrightarrow{h})$ and let $d(u, v)$ be the length of the shortest path from u to v. Let

$$D_i = \max\{d(i, v) : v \in \{0, 1, \cdots, N - 1\}\}$$

for $i = 0, 1, \cdots, N - 1$. We have following properties.

Lemma 25 $D = D_0$.

Proof. In a directed chordal ring network, all even numbered nodes are symmetric, and all odd numbered nodes are symmetric, too. Thus $D = \max\{D_0, D_1\}$. Suppose $D_1 > D_0$. Let i be a node such that $d(1, i) = D_1$. Then $d(1, i) > d(0, i)$. Note that node 0 has only one link $0 \rightarrow 1$ going out from it. Therefore the shortest path from node 0 to node i consists of the link $0 \rightarrow 1$ and a shortest path from node 1 to node i. Thus $d(0, i) = 1 + d(1, i)$; this contradicts with the assumption that $d(1, i) > d(0, i)$. Hence $D_1 \leq D_0$ and therefore $D = D_0$. ∎

We divide the N nodes of $DCR(N, 1, \overrightarrow{h})$ into $\lceil N/(h+1) \rceil$ groups, each group contains $h + 1$ nodes (except possibly the last group). For $i = 1, 2, \cdots, \lceil N/(h+1) \rceil - 1$, the i-th group contains nodes

$$\{(i-1)(h+1)+1, (i-1)(h+1)+2, \cdots, i(h+1)\}.$$

The last group (i.e., the $\lceil N/(h+1) \rceil$-th group) contains nodes

$$\{(\lceil N/(h+1) \rceil - 1)(h+1)+1, (\lceil N/(h+1) \rceil - 1)(h+1)+2, \cdots, N-1, 0\}.$$

For convenience, we will say that $(i-1)(h+1)+1$ is the first node of the i-th group. See Figure 12 for an illustration.

Lemma 26 *Let x be a node of $DCR(N, 1, \overrightarrow{h})$ such that $x \neq 0$. Let x' be the first node of the group containing x (i.e., the first node of the $\lceil x/(h+1) \rceil$-th group). Then there exists a shortest path P from node 0 to x that passes through x'.*

Proof. Let P be an arbitrary shortest path from 0 to x. If P passes through x', then we are done. In the following, assume that P does not pass through x'. Suppose P contains i 1-links and j h-links. Clearly, $i \geq j$. Let P' be a path from 0 to x derived by rearranging the links in P so that every 1-link follows immediately an h-link unless there is no more h-links. (For example, if P contains five 1-links

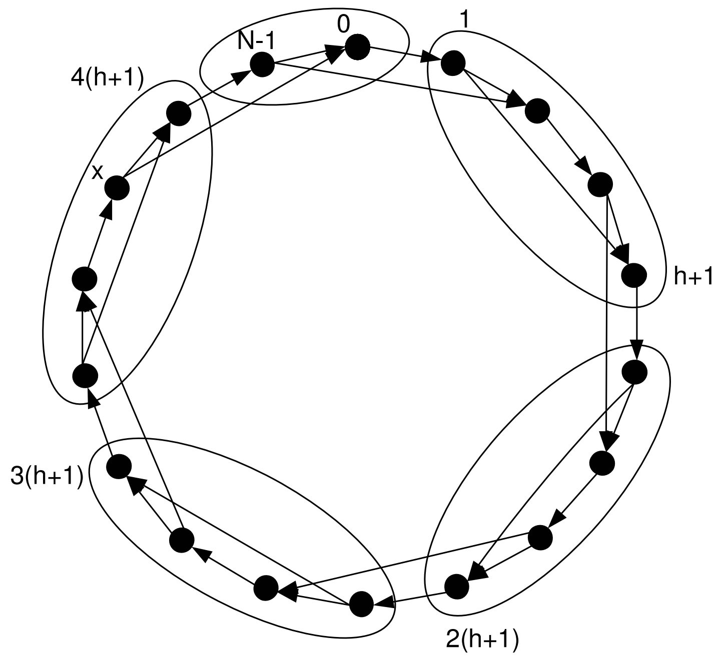


Figure 12: The nodes of $DCR(18, 1, \vec{3})$ are divided into $\lceil 18/(3+1) \rceil$ groups.

and three h-links, then the links in P' are: a 1-link, an h-link, a 1-link, an h-link, a 1-link, an h-link, a 1-link, and a 1-link.) Note that P' is also a path from 0 to x. Furthermore, P' is also shortest since it has the same number of links as P. If $j \geq \lceil x/(h+1) \rceil$, then clearly P' passes through x'. If $j < \lceil x/(h+1) \rceil$, then after P' passes through the j-th group, all the remaining links in P' are 1-links; thus P' will also pass through x'. We have this lemma. ∎

The following lemma is obvious and we omit its proof.

Lemma 27 *Let x' be the first node of the i-th group. Then $d(0, x') = 2i - 1$.*

Lemma 28 *Let $X = \{x : d(0, x) = D_0\}$. Then all the elements of X belong to the last two groups.*

Proof. Suppose this lemma is not true and there is an $x \in X$ such that x does not belong to the last two groups. Choose t such that $x + t(h+1)$ is in the last two groups. Then $d(0, x) \geq d(0, x + t(h+1))$. Let x' be the first node of the group containing x and let $(x+t(h+1))'$ be the first node of the group containing $x+t(h+1)$. Set $y = x+t(h+1)$ and set $y' = (x+t(h+1))'$ for easy writing. By Lemma 26, there is a shortest path from 0 to x that passes through x'. Thus

$$d(0,x) = d(0,x') + d(x',x).$$

Also by Lemma 26, there is a shortest path from 0 to y that passes through y'. Thus

$$d(0,y) = d(0,y') + d(y',y).$$

Since x does not belong to the last two groups and y belongs to the last two groups, by Lemma 27, $d(0,y') \geq d(0,x') + 2$. Note that $d(y',y) = d(x',x)$. Thus $d(0,y) \geq d(0,x)+2$, i.e., $d(0,x+t(h+1)) \geq d(0,x)+2$. This contradicts with the assumption that $d(0,x) \geq d(0,x+t(h+1))$. ∎

Lemma 29 *If* $3 \leq h \leq \frac{N}{2} - 1$ *and* $h+1 \mid N$, *then* $D = 2(\frac{N}{h+1} - 1) + h$.

Proof. Let $X = \{x : d(0,x) = D_0\}$. By Lemma 28, all the elements of X belong to the last two groups. Since $h+1 \mid N$, each group contains exactly $h+1$ nodes. Let $\{y, y+1, y+2, \cdots, y+h\}$ be the set of nodes in the previous group of the last group and let $\{x, x+1, x+2, \cdots, x+h\}$ be the set of nodes in the last group. Note that the node $x+h$ is node 0 and the node $x+h-1$ is node $N-1$.

By Lemma 27, we have $d(0,x) > d(0,y)$. Since $h+1 \mid N$, we have $d(0,x+1) > d(0,y+1)$, $d(0,x+2) > d(0,y+2)$, $\cdots$, $d(0,x+h-1) > d(0,y+h-1)$. Moreover, $d(0,x) > d(0,y+h)$. Since $h+1 \mid N$, we have $d(0,x) < d(0,x+1) < d(0,x+2) < \cdots < d(0,x+h-1)$. From the above, $D_0 = d(0,N-1)$. By Lemma 25, $D = d(0,N-1)$. Since $d(0,N-1) = 2(\frac{N}{h+1} - 1) + h$, we have this lemma. ∎

Theorem 30 *Let* D *be the diameter of* $DCR(N,1,\overrightarrow{h})$. *Then*

$$D = \begin{cases} N-1 & \text{if } h=1 \text{ or } h=N-1, \\ 2(\frac{N}{h+1}-1)+h & \text{if } 3 \leq h \leq \frac{N}{2}-1 \text{ and } h+1 \mid N. \end{cases}$$

Proof. The case that $h = 1$ follows from Theorem 22; the case $h = N - 1$ follows from Theorem 23. The case that $3 \le h \le \frac{N}{2} - 1$ and $h + 1 \mid N$ follows from Lemma 29. ∎

Let D be the diameter of $DCR(N, s, \overrightarrow{h})$. By Theorem 21, D is also the diameter of $DCR(N, s, \overleftarrow{h})$. Since $MCR(N, s, h)$ is derived by combining $DCR(N, s, \overrightarrow{h})$ and $DCR(N, s, \overleftarrow{h})$, the diameter of $DCR(N, s, \overrightarrow{h})$ is an upper bound for the diameter of $MCR(N, s, h)$. One might suspect that the diameter of $MCR(N, s, h)$ is also D. Unfortunately, this is not true. For example, the diameter of $DCR(18, 1, \overrightarrow{5})$ is 9 and the diameter of $MCR(18, 1, 5)$ is 5.

Recall that Hwang [8] proved that the diameter of $DCR(N, s, \overrightarrow{h}) = 1 + 2\times$ the diameter of $DL(\frac{N}{2}, s, \frac{s+h}{2})$. In the following, we give an example to show that this is not true for a mixed chordal ring network. The corresponding double-loop network of $MCR(N, s, h)$ is $DL(\frac{N}{2}, \frac{s-h}{2}, \frac{s+h}{2})$. Consider $MCR(18, 1, 5)$; its corresponding double-loop network is $DL(9, 7, 3)$. The diameter of $DL(9, 7, 3)$ is 4, but the diameter of $MCR(18, 1, 5)$ is 5, which is not equal to $1 + 2 \times 4$.

In the remaining part of this thesis, we shall derive the diameter of some mixed chordal ring networks.

Theorem 31 *Let D be the diameter of $MCR(N, s, h)$. If $s = h$ or $s + h = N$, then $D = N - 1$.*

Proof. First consider the case that $s = h$. Let $d(u, v)$ denote the length of the shortest path from u to v. Since $\gcd(N, s, h) = 1$, we have $\gcd(N, s) = 1$. Therefore $MCR(N, s, s)$ has a hamiltonian circuit $0, s, 2s, 3s, \cdots, (N-1)s$ and hence $d(u, v) \le N - 1$ for every u and v. Thus $D \le N - 1$. On the other hand, the shortest path from s to 0 is $s, 2s, 3s, \cdots, (N-1)s, 0$, which is of length $N - 1$. Thus $D \ge N - 1$ and therefore $D = N - 1$.

Now consider the case that $s+h = N$. From the above discussion, the diameter of $MCR(N, h, h)$ is $N-1$. When $s+h = N$, $MCR(N, -s, h)$ is exactly $MCR(N, h, h)$.

By Corollary 20, $MCR(N, -s, h) \cong MCR(N, s, h)$. Thus when $s + h = N$, the diameter of $MCR(N, s, h)$ is $N - 1$. ∎

Theorem 32 *Let D be the diameter of $MCR(N, s, h)$. If $h = N/2$, then $D = N/2$.*

Proof. Let $d(u, v)$ be the length of the shortest path from u to v. Let

$$D_i = \max\{d(i, v) : v \in \{0, 1, \cdots, N-1\}\}$$

for $i = 0, 1, \cdots, N-1$. In a mixed chordal ring network, all even numbered nodes are symmetric, and all odd numbered nodes are symmetric, too. Thus $D = \max\{D_0, D_1\}$.

First consider D_0. For every even node i, since $i \to i - h$ and $i - h \to i$, we can view the two nodes i and $i-h$ as a supernode. Thus there are total $N/2$ supernodes:

$$\{(i(h+s), i(h+s) - h) : i = 0, 1, \cdots, N/2 - 1\}.$$

See Figure 13 for an illustration. Consider the i-th supernode $(i(h+s), i(h+s)-h)$ and the two s-links going out from this supernode: $i(h+s) \to i(h+s)+s$ and $i(h+s)-h \to i(h+s)-h+s$. Note that node $i(h+s)+s$ is node $(i+1)(s+h)-h$. Moreover, node $i(h+s)-h+s$ is node $(i+1)(s+h)-2h$; since $h = N/2$, node $(i+1)(s+h)-2h$ is node $(i+1)(s+h)$. The two nodes $(i+1)(s+h)$ and $(i+1)(s+h)-h$ are in the $(i+1)$-th supernode. Thus both of the two s-links going out from the i-th supernode go to the $(i+1)$-th supernode. Now consider the distance from node 0 to the two nodes in the i-th supernode $(i(h+s), i(h+s)-h)$. Then

$$d(0, i(h+s)) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i & \text{if } i \text{ is even} \end{cases}$$

and

$$d(0, i(h+s)-h) = \begin{cases} i & \text{if } i \text{ is odd} \\ i+1 & \text{if } i \text{ is even} \end{cases}.$$

Since h is odd and $h = N/2$, it is impossible that $2 \mid \frac{N}{2}$. Hence $2 \nmid \frac{N}{2}$ and

$$D_0 = \max\{d(0, (N/2-1)(h+s)-h), d(0, (N/2-1)(h+s))\} = (N/2-1)+1 = N/2.$$

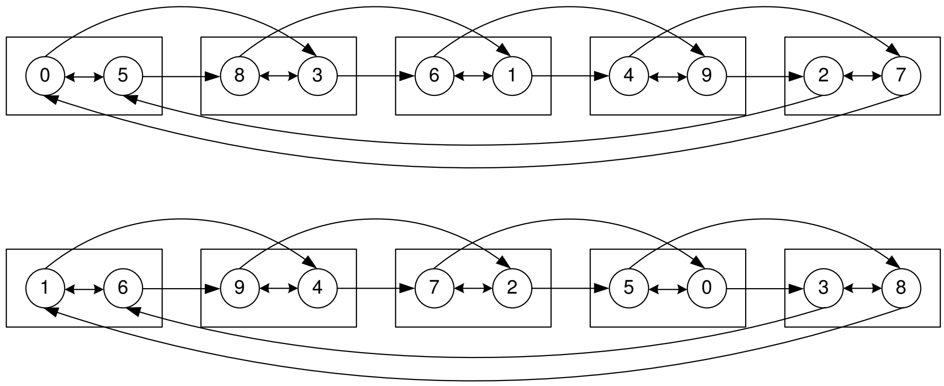


Figure 13: The diameter of $MCR(10, 3, 5)$ is 5.

Now consider D_1. For every odd node i, since $i \to i+h$ and $i+h \to i$, we can view the two nodes i and $i+h$ as a supernode. Thus there are total $N/2$ supernodes:

$$\{(1+i(h+s), 1+i(h+s)+h) : i = 0, 1, \cdots, N/2-1\}.$$

See Figure 13 for an illustration. Consider the i-th supernode $(1+i(h+s), 1+i(h+s)+h)$ and the two s-links going out from this supernode: $1+i(h+s) \to 1+i(h+s)+s$ and $1+i(h+s)+h \to 1+i(h+s)+h+s$. Note that node $1+i(h+s)+s$ is node $1+(i+1)(s+h)-h$. Moreover, node $1+i(h+s)+h+s$ is node $1+(i+1)(s+h)$. Since $h = N/2$, node $1+(i+1)(s+h)-h$ is node $1+(i+1)(s+h)+h$. The two nodes $1+(i+1)(s+h)$ and $1+(i+1)(s+h)+h$ are in the $(i+1)$-th supernode. Thus both of the two s-links going out from the i-th supernode go to the $(i+1)$-th supernode. Now consider the distance from node 1 to the two nodes in the i-th supernode $(1+i(h+s), 1+i(h+s)-h)$. Then

$$d(1, 1+i(h+s)) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i & \text{if } i \text{ is even} \end{cases}$$

and

$$d(1, 1+i(h+s)+h) = \begin{cases} i & \text{if } i \text{ is odd} \\ i+1 & \text{if } i \text{ is even} \end{cases}.$$

Since h is odd and $h = N/2$, it is impossible that $2 \mid \frac{N}{2}$. Hence $2 \nmid \frac{N}{2}$ and $D_1 = \max\{d(1, 1+(N/2-1)(h+s)+h), d(1, 1+(N/2-1)(h+s)\} = (N/2-1)+1 = N/2$.

From the above, we have $D = \max\{D_0, D_1\} = N/2$. ∎

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