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博 士 論 文

延遲型神經網路之多重穩定性及收斂性

Multistability and convergence in delayed neural networks

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摘　　要

本論文主要在於研究延遲型神經網路系統具有多平衡點時的動態收斂性及多重穩定性。此篇論文首先討論了延遲型微分方程的基礎理論及單調性動態系統之收斂性質，繼而研究高維度延遲型神經網路系統中的多重穩定性和擬收斂性。我們在具有飽和或非飽和 S 型活化函數的延遲型神經系統中，藉由幾何方法設定參數條件以證明多平衡點的存在性，並在擁有多平衡點的系統中建立正向不變區域以及穩定性平衡點的吸引盆。當限制抑制性延遲回饋時間夠小時，可以更進一步探討此系統的強保序性質，並得知一般解存有擬收斂性。因此、本文在高維度延遲型神經網路系統中同時建立了多平衡點的存在性及一般解的擬收斂性。我們也在文中描敘幾個數值模擬，以佐證所獲得之理論。

Multistability and convergence in delayed neural networks

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Abstract

We are interested in convergence of dynamics for delayed equations with multiple equilibria as well as multistability in delayed recurrent neural networks. This dissertation begins from reviewing basic theory of delayed differential equations, convergence theory of monotone dynamical systems. The multistability and quasiconvergence for a general n-dimensional delayed neural networks are then investigated. We present the existence of 2^n stable stationary solutions for the delayed neural networks with saturated and unsaturated sigmoidal activation functions. The theory is obtained through formulating parameter conditions based on a geometrical setting. Positively invariant regions for the flows generated by the system and the basins of attraction for these stationary solutions are also established. It is further confirmed that quasiconvergence is generic for the network through justifying the strongly order preserving property. The magnitude of delays is involved in the conditions which yield such an ordering property. Our theory on existence of multiple equilibria is then incorporated into this quasiconvergence for the system. A number of numerical simulations are presented to illustrate our theory.

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新竹這個有「風城」之稱的城市，對我來說就如人生的第二故鄉；因為有熟悉的環境、更因為有許許多多相處融洽的師長與朋友。

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Contents

Chapter 1

Introduction

This dissertation aims to contribute toward convergence of dynamics for delayed equations with multiple equilibria as well as multistability in delayed recurrent neural networks (DRNN). Multistability of a neural network is referred to coexistence of multiple stable patterns such as equilibria or periodic orbits. In general, multistability is accompanied by coexistence with unstable or saddle states. Existence of many equilibria is a necessary feature in the applications of neural networks to associative memory storage or pattern recognition [14, 18, 29, 40]. Recently, further application potentials of multistability have been found in decision making, digital selection or analogy amplification [22]. "Quasiconvergence" for a system is referred to that every solution tends to the set of stationary solutions, while "convergence" (or "complete stability") means that every solution tends to a single stationary solution, as time tends to infinity.

In general, constructing a Lyapunov function, if possible, and then applying the LaSalle's invariant principle is a typical methodology in concluding convergence of dynamics for ordinary differential equations with multiple equilibria. For the case of delayed differential equations, such a theory is still valid. However, it is more difficult to apply the theory in plenty of realistic models. In fact, to the best of our knowledge, there is no example with rigorous justification on the convergence of dynamics in multi-dimensional delayed differential equations with multiple equilibria. Recently, Pituk [42] studied the convergence to equilibria in general "scalar" functional (delayed) differential equations by using monotone dynamics theory, and the results are applicable to some biological models. Therein, he gave a necessary and sufficient condition for the convergence of all solutions in the case when a scalar functional differential equation possesses at most two equilibria. Moreover, motivated by the existence of a nonconstant periodic solution in a quasimonotone delayed differential equation with three equilibria even in the case when all solutions are bounded [31], Pituk also proposed stronger conditions to guarantee the convergence of all solutions without restriction on the number of equilibria.

The convergence to multiple equilibria has been studied in the Hopfield neural networks without delays (ordinary differential equation case) [29]. Such a convergence was derived by constructing a Lyapunov function on the system, when the connecting weights are symmetric, and then applying the LaSalle's invariant principle. Similar treatment has also been adopted to derive complete stability in cellular neural networks (CNN), even for the cases of saturated and standard output functions [36, 45]. In [19, 50, 51], the authors studied the cellular neural network with and without delays and obtained the complete stability by using a scheme analogous to the Gauss-Seidel method or M-matrix theory. However, each of these works contains some gaps and rigorousness remains to be justified. Even in a single neural model with the standard piecewise linearity, how the whole picture of dynamics depending on parameters has not been pieced together [20]. Moreover, all the aforementioned results on complete stability of the delayed models were rigidly restricted to the standard, piecewise linear, activation functions. Furthermore, convergence dynamics has not been declared to coexist with multiple equilibria in multi-dimensional delayed neural network with general nonlinear activation functions.

To be in possession of both comprehension in basic theory and applications for practical models, this dissertation comprises two parts. The first one contains basic existence and uniqueness theory of delayed differential equations and convergence theory of monotone dynamical systems which has been widely applied in studying mathematical models in biology. The global convergence and quasiconvergence theory of a monotone dynamical systems are addressed for further study of DRNN systems. The second part contains global dissipativity, comparison of investigations on neural networks with and without delays, and several dynamical results of the delayed recurrent neural networks such as stability of multiple equilibria, basins of attraction and guasiconvergent dynamics. Moreover, monotonicity of the DRNN system is derived in a special partial order, and thus generic quasiconvergence is certified.

The model equation we mainly consider in this presentation is

$$\frac{dx_i(t)}{dt} = -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t-\tau_{ij})) + I_i, \qquad (1.1)$$

where $i = 1, \cdots, n$; n corresponds to the number of neurons in the neural network system, $x_i(t)$ describes the state of the ith neuron at time t, the constant $\mu_i > 0$ denotes the rate with which the ith neuron will reset its potential to the resetting state in isolation when disconnected from the network and external inputs. $g_j(\cdot)$ is the activation function and $g_j(x_j(t))$ denotes the output of the jth neuron at time t. The constant $0 \leq \tau_{ij} \leq \tau$, $\tau := \max_{1\leq i,j\leq n} \tau_{ij}$, corresponds to the transmission delay along the axon and I_i stands for an independent bias current source. The constants α_{ij}, β_{ij} are connection weights from jth neuron to ith neuron. The outputs of all neurons are sensed by another synapse whose weighted sum $\sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t))$ and $\sum_{j=1}^{n} \beta_{ij} g_j(x_j(t-\tau_{ij}))$ contribute to determine the state of the ith neuron of the system. The outputs $g_j(x_j(t))$ and $g_j(x_j(t-\tau))$ are generated by the dynamics of jth neuron, and fed back to all neurons in this system, including itself. We refer (1.1) as a feedback system and call α_{ii}, β_{ii} as *self-feedback weights* and α_{ij}, β_{ij} as *nonself-feedback weights* for $i \neq j$. When all the activation functions are increasing, the positivity and negativity of α_{ij} mean excitatory and inhibitory effect, respectively. Same interpretation applies to the delay feedback weights β_{ij}. System (1.1) reduces to the classical and delayed Hopfield neural networks [29, 37], as $\beta_{ij} = 0$ for all i, j, and $\alpha_{ij} = 0$ for all i, j, respectively. It also represents the cellular neural networks without delays [14] and with delays [43]. Indeed, a CNN system built in a multi-dimensional coupling fashion can always be rewritten in a one-dimensional coupling form, by renaming the indices [13]. Such an arrangement, however, suppresses the local connection representation.

In electronic implement, time delays of neural network systems are unavoidable due to axonal conduction times, distances of interneurons and the finite switching speeds of amplifiers. The dynamics for differential equations with delays can be rather complicated. Although the stationary equations are identical for system (1.1) without delay ($\tau_{ij} = 0$ for all i, j) and with delay ($\tau_{ij} > 0$), the stability for the equilibrium points and dynamical behaviors of the systems can be very different. There have been literatures [2, 3, 4, 41, 44] exploring the effects of delays in differential equations and neural network systems. For system (1.1), the theory on unique equilibrium and global convergence to the equilibrium have been studied extensively in [5, 6, 7, 8, 17, 30, 33, 34, 39, 43, 55]. These studies indicate a coincidence between the systems with delays and without delays. The presentation moves up the investigations in this direction by establishing the existence of multiple stationary solutions for system (1.1). More specifically, we construct 2^n stable stationary solutions for system (1.1) with two classes of activation functions. The theory is obtained through formulating parameter conditions based on a geometrical setting. We first derive conditions for the existence of 3^n equilibria for Eq (1.1) with sigmoidal activation functions and saturated activation functions. Some regions containing these stationary solutions are shown to be positively invariant under the flows generated by Eq (1.1). In the issue of exponential stability of the equilibria, we also estimate basins of attraction for these stationary solutions. Therein, the basins of attraction of stationary solutions were derived from a criterion concerning the slope of the activation functions. The ranges of the basins depend on the parameters therein. We further extend the basins of attraction of 2^n stable stationary solutions to the confirmed positively invariant regions.

The existence of multiple equilibria and their attractive domains have been studied for Eq (1.1) with the standard activation function in [56]. The result therein is about locally exponential stability of multiple equilibria; and the argument strongly relies on the piecewise linearity, saturations of the standard activation function and subsequent partition of phase space. Besides, some of the arguments therein need modifications to meet rigorousness, and the global dynamics remains as an unsolved problem. Our geometrical approach can be applied to Eq (1.1) with more general sigmoidal activation functions. In addition, larger positively invariant sets and basins of attraction have been established. Moreover, the criteria in our theory is weaker than those in [56].

In order to approach the convergence results of multi-dimensional delayed neural networks, we further discuss the strongly order preserving property, hence quasiconvergence behaviors for Eq (1.1), by the theory of Smith and Thieme [48]. The magnitude of delays is involved in the conditions which yield such an ordering property. The dynamics scenario for system (1.1) is thus composed of quasiconvergence (or convergence) with multiple equilibria. A number of numerical simulations are also performed to demonstrate our theory.

The remaining part of this dissertation is organized as follows. In Chapter 2, we introduce some notions and basic theory of delayed differential equations such as the existence and uniqueness of solution. We also identify these basic properties for delayed neural networks. In Chapter 3, we recall some notions and basic theory of monotone dynamical systems from [47], including several dynamical properties of a strongly order preserving semiflow, generic quasiconvergence and global convergence property of a monotone dynamical system. In Chapter 4, we specifically study the dynamics of neural networks with delays and present several numerical simulations on the dynamics. Some analytic methodology such as characteristic equations, Lyapunov function and Lyapunov functional are compared for the neural networks with and without delays. We consider two classes of activation functions which are commonly employed in neural network theory. Global dissipativity and several dynamical results of the delayed recurrent neural networks such as stability of multiple equilibria, basins of attraction are studied. Finally, in Chapter 5, we investigate the guasiconvergent dynamics of the DRNN system. Specifically, strongly order preserving property is derived in a special partial order, and thus generic quasiconvergence is confirmed.

Chapter 2

Basic Notions of Delayed Differential Equations

In this section, we introduce some notions and basic theory of delayed differential equations including the existence and the uniqueness of solution. In addition, we apply these theory to delayed neural networks in Section 2.2.

2.1 Fundamental Theorems in Delayed Equations

Let $\tau > 0$ be a given positive number (the delay time) and denote by $\mathcal{C}$ the Banach space $C([-\tau, 0], \mathbb{R}^n)$ endowed with the norm $\|\phi\| = \sup_{\theta\in[\tau,0]} |\phi(\theta)|$. $\mathcal{C}$ is the phase space when we deal with delayed differential equations. Let $\ell \geq 0$ and $\mathbf{x} \in C([-\tau, \ell], \mathbb{R}^n)$, then for any $t \in [-\tau, \ell]$, we denote the element $\mathbf{x}_t$ in $\mathcal{C}$ given by

$$\mathbf{x}_t(\theta) = \mathbf{x}(t + \theta),\ \ \theta \in [-\tau, 0]. \tag{2.1}$$

Assume $\mathcal{S}$ is a subset of $\mathcal{C}$ and $F : \mathcal{S} \to \mathbb{R}^n$ is a given function. We call

$$\frac{d\mathbf{x}(t)}{dt} = F(\mathbf{x}_t) \tag{2.2}$$

a *delayed differential equation* (DDE) or *functional differential equation* (FDE) on $\mathcal{S}$, comparing with an ordinary differential equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathcal{F}(\mathbf{x}), \tag{2.3}$$

where $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.1.1. *(i) A function* $\mathbf{x} = \mathbf{x}(t)$ *is called a solution of Eq* (2.2) *on* $[t_0 - \tau, t_0 + \ell)$ *if* $\mathbf{x} \in C([t_0 - \tau, t_0 + \ell), \mathbb{R}^n)$*,* $\mathbf{x}_t$ *defined as* (2.1) *lies in* $\mathcal{S}$ *and satisfies* (2.2) *for* $t \in [t_0, \ell)$*. (ii) For given* $t_0 \in \mathbb{R}$ *and* $\phi \in \mathcal{C}$*, we say* $\mathbf{x}(t_0, \phi)$ *is a solution of Eq* (2.2) *with initial value* ϕ *at* t_0 *if there is an* $\ell > 0$ *such that* $\mathbf{x}(t_0, \phi)$ *is a solution of Eq* (2.2) *on* $[t_0 - \tau, t_0 + \ell)$ *and* $\mathbf{x}_{t_0}(t_0, \phi) = \phi$*.*

For given subset $\mathcal{S}$ of $\mathcal{C}$, we denote the class of all continuous functions from $\mathcal{S}$ to $\mathbb{R}^n$ by $C(\mathcal{S}, \mathbb{R}^n)$ and the class of all bounded continuous functions from $\mathcal{S}$ to $\mathbb{R}^n$ by $C^0(\mathcal{S}, \mathbb{R}^n)$. We recall some well-known results from [25].

Theorem 2.1.2. (Existence of solution) *Suppose* $\mathcal{S}$ *is an open subset in* $\mathcal{C}$*. If* $\mathcal{W} \subseteq \mathcal{S}$ *is compact and* $F^0 \in C(\mathcal{S}, \mathbb{R}^n)$ *is given, then there exist a neighborhood* $\mathcal{V} \subseteq \mathcal{S}$ *of* $\mathcal{W}$ *with* $F^0 \in C^0(\mathcal{V}, \mathbb{R}^n)$*, a neighborhood* $\mathcal{U} \subseteq C^0(\mathcal{V}, \mathbb{R}^n)$ *of* F^0 *and a constant* $\kappa > 0$ *such that for any* $\phi \in \mathcal{W}$*,* $F \in \mathcal{U}$*, there is a solution* $\mathbf{x}(t; \phi)$ *of Eq* (2.2) *with initial condition* $\mathbf{x}_{t_0} = \phi$ *that exists on* $[t_0 - \tau, t_0 + \kappa]$*.*

Theorem 2.1.3. (Uniqueness of solution) *Suppose* $\mathcal{S}$ *is an open subset in* $\mathcal{C}$ *and* F *is Lipschitizian in each compact set in* $\mathcal{S}$*. If* $\phi \in \mathcal{S}$*, then there is a unique solution of Eq* (2.2) *with initial condition* ϕ *at* t_0*.*

Theorem 2.1.4. (Extending domain of existence) *Suppose* $\mathcal{S}$ *is an open subset in* $\mathcal{C}$ *and* F *is Lipschitizian in each compact set in* $\mathcal{S}$*. Then for each* $\phi \in \mathcal{S}$*, there is a maximal interval* I *on which Eq* (2.2) *has a unique solution,* $\mathbf{x}(t_0, \phi)$*; i.e., if Eq* (2.2) *has a solution* $\mathbf{y}(t_0, \phi)$ *on an interval* J *then* $J \subset I$ *and* $\mathbf{y}_t = \mathbf{x}_t$ *for all* $t \in J$*. Furthermore, the maximal interval* J *is open.*

In contrast to that the phase space for the ordinary differential equations (2.3) is $\mathbb{R}^n$, the one for the delayed differential equation (2.2) is a infinite dimensional Banach space. Although these two equations have the same existence and uniqueness criteria, it is more complicated to check the criteria for the function F which is defined on a Banach space.

2.2 Fundamental Theorem for Delayed Neural Networks

The property of the solution in (1.1) is strongly relevant to the activation functions $g_j(\cdot)$. Based on the above basic theory of delayed differential equations, we have the following result.

Theorem 2.2.1. *Suppose that each of the activation function g_j is a Lipschitz function with Lipschitz constant L_j, then DRNN* (1.1) *has a unique solution for every given initial condition.*

Proof: From (1.1), for all $\phi \in \mathcal{C}$ with $\tau := \max_{1\le i,j\le n} \tau_{ij}$, $F = (F_1, F_2, \cdots, F_n)$ is defined as

$$F_i(\phi) = -\mu_i\phi_i(0) + \sum_{j=1}^{n} \alpha_{ij} g_j(\phi_j(0)) + \sum_{j=1}^{n} \beta_{ij} g_j(\phi_j(-\tau_{ij})) + I_i.$$

So, we have

$$\begin{aligned} |F_i(\phi) - F_i(\psi)| &= |-\mu_i[\phi_i(0) - \psi_i(0)] + \sum_{j=1}^{n} \alpha_{ij}[g_j(\phi_j(0)) - g_j(\psi_j(0))] \\ &\quad + \sum_{j=1}^{n} \beta_{ij}[g_j(\phi_j(-\tau_{ij})) - g_j(\psi_j(-\tau_{ij}))]| \\ &\le \{|\mu_i| + \sum_{j=1}^{n} |\alpha_{ij}|L_j + \sum_{j=1}^{n} |\beta_{ij}|L_j\}\|\phi - \psi\|, \end{aligned}$$

Hence, each F_i is Lipschitz, and then F in Theorem 2.1.3 is Lipschitz. Consequently, (1.1) has a unique solution for any given initial condition. □

2.3 Lyapunov Functional and Lyapunov-Razumikhin Theorem

In the case of ordinary differential equations (2.3), complete stability (convergence) and quasiconvergence could be based on applying the LaSalle's invariant principle to the Lyapunov functions. Let us review this principle quoted from [23].

Suppose the vector field $\mathcal{F}$ in (2.3) is locally Lipschitzian. Let L be a scalar function defined and continuous on $\mathbb{R}^n$ and $\varphi(t, \mathbf{x})$ be the flow map of (2.3). To determine if L decreases along the orbit of (2.3), we can consider

$$\dot{L}(\mathbf{x}) := \limsup_{h \to 0^+} \frac{1}{h}[L(\varphi(h, \mathbf{x})) - L(\mathbf{x})]. \tag{2.4}$$

If L is locally Lipschitz continuous, (2.4) is equal to

$$\limsup_{h \to 0^+} \frac{1}{h}[L(\mathbf{x} + h\mathcal{F}(\mathbf{x})) - L(\mathbf{x})]. \tag{2.5}$$

Suppose L is bounded in $\mathbb{R}^n$ and $\dot{L}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Let $E := \{\mathbf{x} \in \mathbb{R}^n | \dot{L}(\mathbf{x}) = 0\}$ and let M be the largest invariant set of (2.3) in E. LaSalle's invariant principle says that if $\varphi(t, \mathbf{x})$ is bounded for $t \geq 0$, then the ω-limit set of $\varphi(t, \mathbf{x})$ belongs to M.

There also exists an analogous theory in delayed equations. Consider the DDE (2.2)

$$\frac{d\mathbf{x}(t)}{dt} = F(\mathbf{x}_t),$$

where $F : \mathcal{C} \to \mathbb{R}^n$ is completely continuous.

Definition 2.3.1. *We say $W : \mathcal{C} \to \mathbb{R}$ is a* Lyapunov functional *on a set $\mathcal{S}$ in $\mathcal{C}$ relative to* (2.2) *if W is continuous on $\bar{\mathcal{S}}$, the closure of $\mathcal{S}$, and $\dot{W} \leq 0$ on $\mathcal{S}$, where*

$$\dot{W}(\phi) := \limsup_{h \to 0^+} \frac{1}{h}[W(\mathbf{x}_h(\phi)) - W(\phi)]. \tag{2.6}$$

For the given $\mathcal{S}$, let

$$E(\mathcal{S}) := \{\phi \in \bar{\mathcal{S}} | \dot{W}(\phi) = 0\}$$

and let $M(\mathcal{S})$ denote the largest subset of $E(\mathcal{S})$ that is invariant under the flow generated by Eq (2.2). The following theorem is an invariant principle for autonomous delayed differential equations.

Theorem 2.3.2. [25] *If W is a Lyapunov functional on $\mathcal{S}$ and $\mathbf{x}_t(\phi)$ is a bounded solution of Eq* (2.2) *that remains in $\mathcal{S}$, then $\mathbf{x}_t(\phi)$ tends to $M(\mathcal{S})$ as $t \to \infty$.*

The following result is concerned with the stability of a system with a single equilibrium.

Corollary 2.3.3. [25] *Suppose* $W : \mathcal{C} \to \mathbb{R}$ *is continuous and there exist nonnegative continuous functions* $a(\cdot)$ *and* $b(\cdot)$, $a(0) = b(0) = 0$, $\lim_{r\to+\infty} a(r) = +\infty$ *and*

$$a(|\phi(0)|) \le W(\phi), \quad \dot{W}(\phi) \le -b(|\phi(0)|).$$

Then the trivial solution is stable and every solution is bounded. If, in addition, $b(\cdot)$ *is positive definite, then every solution approaches the trivial solution as* $t \to \infty$.

Another approach for studying the stability of steady states in a delayed differential equations is constructing an appropriate Lyapunov "function" for the given system.

We say $V : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function (or Razumikhin function) if V has continuous first partial derivatives. For a Lyapunov function, we define the upper right-hand derivative of V with respect to (2.2) is defined as

$$\begin{aligned}\dot{V}(\phi) &:= \limsup_{h\to 0^+} \frac{1}{h}\{V(\phi(0) + hF(\phi)) - V(\phi(0))\} \\ &= \sum_{i=1}^{n} \frac{\partial V(\phi(0))}{\partial x_i} F_i(\phi). \end{aligned} \tag{2.7}$$

The second equality holds when V has continuous first partial derivatives. For a given set $\mathcal{S} \subseteq \mathcal{C}$, define

$$\tilde{E}(\mathcal{S}) := \{\phi \in \bar{\mathcal{S}} | \max_{-\tau\le\theta\le 0} V(\mathbf{x}_t(\phi)(\theta)) = \max_{-\tau\le\theta\le 0} V(\phi(\theta)) \text{ for all } t \ge 0\}$$

and let $\tilde{M}(\mathcal{S})$ denote the largest subset of $\tilde{E}(\mathcal{S})$ that is invariant under the flow generated by Eq (2.2). The following theorem is an invariance principle for autonomous delayed differential equations.

Theorem 2.3.4. [21] *Suppose there exist a Lyapunov function* V *and a closed set* $\mathcal{S}$ *in* $\mathcal{C}$ *that is positively invariant under Eq* (2.2) *such that*

$$\dot{V}(\phi) \le 0, \text{ for all } \phi \in \mathcal{S} \text{ with } V(\phi(0)) = \max_{-\tau\le\theta\le 0} V(\phi(\theta)).$$

Then for any $\phi \in \mathcal{S}$ *such that* $\mathbf{x}(\phi)(\cdot)$ *is defined and bounded on* $[-\tau, \infty)$, $\omega(\phi) \subseteq \tilde{M}(\mathcal{S}) \subseteq \tilde{E}(\mathcal{S})$. *Hence* $\mathbf{x}_t(\phi) \to \tilde{M}(\mathcal{S})$ *as* $t \to \infty$.

As an consequence of Theorem 2.3.4, the following is an asymptotic stability of an equilibrium for autonomous delayed differential equations.

Corollary 2.3.5. [21] *Let* $F(0) = 0$ *and suppose there exist a Lyapunov function* V *and a constant* $\alpha > 0$ *such that*

$$
\begin{aligned}
&(i) && V(0) = 0 \text{ and } V(\phi) > 0 \text{ for all } 0 \neq \|\phi\| < \alpha,\\
&(ii) && \dot{V}(0) = 0, \text{ and}\\
&(iii) && \dot{V}(\phi) < 0 \text{ for all } 0 \neq \|\phi\| < \alpha \text{ with } \max_{-\tau \leq \theta \leq 0} V(\phi(\theta)) = V(\phi(0)).
\end{aligned}
$$

Then the solution $\mathbf{x} = 0$ *of Eq* (2.2) *is asymptotically stable.*

The LaSalle's invariant principle is an effective methodology to investigate the stability of steady states and global dynamics. However, suitable Lyapunov functions or Lyapunov functionals need to be constructed to fit the practical models. Moreover, let us recall that the functional W is defined on the infinite dimensional Banach space $\mathcal{C}$ and the definition (2.7) is concerned with the functional F. From the definitions (2.5), (2.6) and (2.7), we know that it is more difficult to propose a Lyapunov functional W or a Lyapunov function V with negative derivative along solutions of the delayed differential equation (2.2) than to construct a Lyapunov function in the ordinary differential equation (2.3).

Chapter 3

Monotone Dynamical Systems

In [27, 28], Hirsch developed a theory on almost quasiconvergence in continuous time networks. In such a dynamical scenario, there may exist cycles or other kinds of non-convergent orbits, but they cannot be stable. We will employ the monotone dynamics theory to explore the almost quasiconvergence of delayed recurrent neural networks in Chapter 5. Monotone dynamics theory has been widely applied in systems including reaction-diffusion systems, semilinear diffusion equations and various biological systems. Matano introduced the important idea of strongly order preserving semiflows [38], which is more flexible than strong monotonicity, proposed by Hirsch. The work of Smith and Thieme [47, 48] represents a synthesis of the approaches of Hirsch and Matano that attempts to simplify and streamline the arguments. Significant improvements in the theory was obtained therein with additional compactness hypotheses that are often satisfied in the applications.

In this chapter, we recall some notations and basic theory of monotone dynamical systems from [47]. In Chapter 5, we will further confirm that quasiconvergence is generic for the networks through justifying the strongly order preserving property as the self-feedback time lags are small by using the theory of Smith and Thieme [48].

3.1 Preliminary

In this section, we introduce the basic theory of monotone dynamical systems which will be applied to study the convergence of dynamics in the topic of neural networks.

Consider an ordered metric space Ω with metric d and *partial order* relaton $\leq$ which means that:

$$\begin{aligned} (i) &\quad x \leq x \text{ for all } x \in \Omega \text{ (reflexive);} \\ (ii) &\quad x \leq y \text{ and } y \leq z \text{ implies } x \leq z \text{ (transitive);} \\ (iii) &\quad x \leq y \text{ and } y \leq x \text{ implies } x = y \text{ (antisymmetric).} \end{aligned}$$

Definition 3.1.1. *(i) We write $x < y$ if $x \leq y$ and $x \neq y$.*
(ii) Given subsets U and V of Ω, we write $U \leq V (U < V)$ when $x \leq y (x < y)$ holds for each choice of $x \in U$ and $y \in V$.

We assume that the partial order relation is *closed*; it means that the order relation and the topology on Ω are compatible in the sense that $x \leq y$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and $x_n \leq y_n$ for all n. For $A \subset \Omega$ we write $\overline{A}$ for the closure of A and $IntA$ for the interior of A.

In the applications, the order relation usually comes from a *positive cone*. It means that Ω is typically a subset of a Banach space $\tilde{\Omega}$ with a nonempty closed subset, positive cone, K possessing the properties :

$$\begin{aligned} (i) &\quad \mathbb{R}_+ \cdot K \subset K, \\ (ii) &\quad K + K \subset K, \\ (iii) &\quad K \cap (-K) = \{0\}, \end{aligned}$$

where $\mathbb{R}_+ := (0, +\infty)$ and $-K := \{-k | k \in K\}$. In this case, the relation defined by $x \leq y$ if and only if $y - x \in K$ is a closed partial order relation.

Definition 3.1.2. *(I) A semiflow on Ω is a continuous map $\Phi : \Omega \times \mathbb{R}^+ \to \Omega$ which satisfies :*

$$\begin{aligned} (i) &\quad \Phi_0 = id_\Omega \\ (ii) &\quad \Phi_t \circ \Phi_s = \Phi_{t+s} \ for \ t, s \geq 0. \end{aligned}$$

Here, $\Phi_t(x) := \Phi(x, t)$ for $x \in \Omega$ and id_Ω is the identity map on Ω.
(II) The orbit of x is denoted by

$$\mathcal{O}(x) := \{\Phi_t(x) | t \geq 0\}.$$

Definition 3.1.3. *Let $\mathcal{E}$ be the set of all equilibrium points for Φ. (i) The omega limit set, $\omega(x)$, of $x \in \Omega$ is defined by*

$$\omega(x) = \cap_{t\geq 0}\overline{\cup_{s\geq t}\Phi_s(x)}.$$

(ii) A point $x \in \Omega$ is called a quasiconvergent point if $\omega(x) \subset \mathcal{E}$. The set of such points is denoted by Q.
(iii) A point $x \in \Omega$ is called a convergent point if $\omega(x)$ consists of a single point of $\mathcal{E}$. The set of such points is denoted by C.

Definition 3.1.4. *(i) The semiflow Φ is said to be monotone provided*

$$\Phi_t(x) \leq \Phi_t(y) \text{ whenever } x \leq y \text{ and } t \geq 0.$$

(ii) Φ is called strongly order preserving, SOP, if it is monotone and whenever $x < y$ there exist open subsets U, V of Ω with $x \in U$ and $y \in V$ and $t_0 > 0$ such that

$$\Phi_{t_0}(U) \leq \Phi_{t_0}(V).$$

Note that monotonicity of Φ implies that $\Phi_t(U) \leq \Phi_t(V)$ for all $t \geq t_0$. A dynamical system on Ω is monotone if it preserves the ordering of initial data. A SOP system has stronger ordering preserving about the neighborhoods of two points, $x < y$. The order relation between these two points, $x < y$, will be kept forever.

3.2 The Convergence Criterion

Hereafter, we assume that Φ is monotone and $\overline{\mathcal{O}(x)}$ is a compact subset of Ω for each $x \in \Omega$. In the remainder of this chapter, all theorems and propositions are quoted from [47]. We will also give some remarks to catch the key points of the monotone dynamical theory.

Theorem 3.2.1. (Convergence Criterion) *Let $\Phi_T(x) \geq x$ for some $T > 0$. Then $\omega(x)$ is a T-periodic orbit. If $\Phi_t(x) \geq x$ for t belonging to some nonempty open subset of $(0, \infty)$ then $\Phi_t(x) \to p \in \mathcal{E}$ as $t \to \infty$. In particular, if Φ is SOP and $\Phi_T(x) > x$ for some $T > 0$ then $\Phi_t(x) \to p \in \mathcal{E}$ as $t \to \infty$.*

In the previous theorem, since $\Phi_T(x) \geq x$, monotonicity implies that $\Phi_{(m+1)T}(x) \geq \Phi_{mT}(x)$ for $m = 1, 2, \cdots$. Thus, by the compactness of the orbit closure, $\Phi_{mT}(x) \to p$ as $m \to \infty$ for some p. By continuity of Φ, it could be proved that $\Phi_{t+T}(p) = \Phi_t(p)$ and $\omega(x) = \mathcal{O}(p)$. The next result describes how an omega limit set is imbedded in the space Ω. It is fundamental to the monotone dynamics theory.

Theorem 3.2.2. (Nonordering of Limit Sets) *An omega limit set cannot contain distinct points x and y with the property that there exists neighborhoods U of x and V of y such that $U \leq V$. If Φ is SOP then a limit set cannot contain two points x and y with $x < y$.*

To interpret the nonordering property, we suppose $\omega(z)$ contains distinct points x and y possessing neighborhood U and V, respectively, such that $U \leq V$. Then $\Phi_{t_1}(z) \in U$ for some $t_1 > 0$; in addition, there is a constant $t_2 > t_1$ such that $\Phi_{t_2}(z) \in V$ and then $\Phi_t(z) \in V$ for all t sufficiently near to t_2. For these t we have $\Phi_t(z) \geq \Phi_{t_1}(z)$ by the fact $U < V$. The Convergence Criterion implies that $\Phi_t(z) \to p \in \mathcal{E}$ as $t \to \infty$. Therefore, $\omega(z) = \{p\}$, a contradiction.

Hereafter, we assume that Φ is *SOP*. Since the fundamental nonordering property of limit sets, we have the following proposition which will imply the important *Limit Set Dichotomy*.

Proposition 3.2.3. *Whenever the semiflow is SOP, the dynamics has the following properties:*

- (Colimiting Principle) *If $x < y$, $t_k \to \infty$, $\Phi_{t_k}(x) \to p$ and $\Phi_{t_k}(y) \to p$ as $k \to \infty$ then $p \in \mathcal{E}$.*
- (Intersection Principle) *If $x < y$ then $\omega(x) \cap \omega(y) \subset \mathcal{E}$.*
- *Let x, y satisfy $x < y$. If $t_k \to \infty$, $\Phi_{t_k}(x) \to a$, $\Phi_{t_k}(y) \to b$ as $k \to \infty$ and $a < b$ then $\mathcal{O}(a) < b$.*
- (Absorption Principle) *Let $u, v \in \Omega$. If there exists $x \in \omega(u)$ such that $x < \omega(v)$, then $\omega(u) < \omega(v)$. Similarly, if there exists $x \in \omega(u)$ such that $\omega(v) < x$, then $\omega(v) < \omega(u)$.*

- (Limit Set Separation Principle) *Let x, y satisfy $x < y$. If $t_k \to \infty$, $\Phi_{t_k}(x) \to a$, $\Phi_{t_k}(y) \to b$ as $k \to \infty$ and $a < b$ then $\omega(x) < \omega(y)$.*

Based on this proposition, the following fundamental result is derived.

Theorem 3.2.4. (Limit Set Dichotomy) *If $x < y$ then either*
(a) $\omega(x) < \omega(y)$, or
(b) $\omega(x) = \omega(y) \subset \mathcal{E}$.
If case (b) holds and $t_k \to \infty$ then $\Phi_{t_k}(x) \to p$ if and only if $\Phi_{t_k}(y) \to p$.

Limit Set Dichotomy points out two possible order relations between eventual behaviors of ordered points. For two ordered points, $x < y$, their omega limit sets either preserve the order or are totally equal; moreover, the omega limit set is consisted of equilibria in the latter case.

3.3 Generic Quasiconvergence

In this section, we discuss the convergent dynamics of a semiflow with strongly order-preserving property. Herein, a compactness assumption is required, and the main result is about the generic quasiconvergence. To start with, we give the following definition.

Definition 3.3.1. *If $x \in \Omega$, we say that x can be approximated from below (above) in Ω if there exists a sequence $\{x_n\}$ in Ω such that $x_n < x_{n+1} < x$ $(x < x_{n+1} < x_n)$ for $n \geq 1$ and $x_n \to x$ as $n \to \infty$.*

Consider a relatively weak compactness assumption :

(**T**) For each $x_0 \in \Omega$, $\mathcal{O}(x_0)$ has compact closure in Ω. Furthermore, if $\{x_i\}_{i\geq 1}$ approximates x_0 from below or from above then $\cup_{i\geq 0}\omega(x_i)$ has compact closure contained in Ω.

Remark. (1) The compactness assumption (**T**) is satisfied when the following hold: (i) The orbit $\mathcal{O}(B) := \bigcup_{x\in B} \mathcal{O}(x)$ is bounded whenever B is a bounded set in Ω. (ii) There exists some $t_0 > 0$ such that the operator Φ_{t_0} is compact.

(2) In fact, when these two conditions in (1) hold and x_0 is approximated from below by $\{x_i\}_{i\geq 1}$, then $\{x_i\}_{i\geq 0}$ is compact and therefore $\bigcup_{i\geq 0}\mathcal{O}(x_i)$ is bounded. Since Φ_{t_0} is a compact operator, $\overline{\Phi_{t_0}(\bigcup_{i\geq 0}\mathcal{O}(x_i))}$ is compact in Ω. The set $\bigcup_{i\geq 0}\omega(x_i)$ is contained in the latter closure and then has compact closure.

The assumption (**T**) is assumed to hold throughout the remainder of this section. The key to the proof that the generic point of Ω is a quasiconvergence point is the following result.

Theorem 3.3.2. (Sequential Limit Set Trichotomy) *Let $x_0 \in \Omega$ have the property that it can be approximated from below in Ω by a sequence $\tilde{x}_n$. Then there exists a subsequence x_n such that $x_n < x_{n+1} < x_0$, $n \geq 1$, with $x_n \to x_0$ satisfying one of the following.*

(a) There exists $u_0 \in \mathcal{E}$ such that

$$\omega(x_n) < \omega(x_{n+1}) < u_0 = \omega(x_0),\ \ n \geq 1$$

and

$$\lim_{n\to\infty} \operatorname{dist}(\omega(x_n), u_0) = 0.$$

(b) There exists $u_0 \in \mathcal{E}$ such that

$$\omega(x_n) = u_0 < \omega(x_0),\ \ n \geq 1.$$

(c) $\omega(x_n) = \omega(x_0) \subset \mathcal{E}$ for $n \geq 1$.

An analogous result holds if x_0 can be approximated from above in Ω.

Remark. In each of these three cases of *the Sequential Limi Set Trichotomy*, the point x_0 possesses typical dynamics. In case (a), the point x_0 is convergent. In case (b), x_0 belongs to the closure of the set of convergent points. And x_0 is a quasiconvergent point in case (c).

Under the strongly order-preserving property and the compactness assumption (**T**), a semiflow has the following generic quasiconvergent result, based on the property *Sequential Limit Set Trichotomy*.

Theorem 3.3.3. *Suppose each point of* Ω *can be approximated either from above or from below in* Ω*. If the semiflow* Φ *possesses the property (T) and has the strongly order-preserving property. Then* $\Omega = \mathrm{Int}Q \cup \overline{\mathrm{Int}C}$*. In particular,* $\mathrm{Int}Q$ *is dense in* Ω*.*

Although the result in previous theorem does not confirm the behavior of every orbit, it establishes the fact that there does not exist any non-trivial attractive periodic orbit, and this is a significant result in the applications. The following is an immediate consequence of the Limit Set Trichotomy. It describes the possibilities for the omega limit sets in the case that a point can be approximated from above and from below on Ω.

Proposition 3.3.4. *Let* Ω *be an ordered metric space and* Φ_t *be a strongly order preserving semiflow on* Ω*. Let* $x_0 \in \Omega$ *be such that it can be approximated from above in* Ω *and from below in* Ω*. Then there exists sequences* x_n *and* z_n *in* Ω *satisfying* $x_n \to x_0$*,* $z_n \to x_0$*,* $x_n < x_{n+1} < x_0 < z_{n+1} < z_n$*,* $n \geq 1$*, and one of the following holds:*

(a) There exists $u_0 \in \mathcal{E}$ *such that, for* $n \geq 1$*,*

$$\omega(x_n) < \omega(x_{n+1}) < \omega(x_0) = u_0 < \omega(z_{n+1}) < \omega(z_n) \text{ and}$$

$$\lim_{n\to\infty} \mathrm{dist}(\omega(x_n), u_0) = \lim_{n\to\infty} \mathrm{dist}(\omega(z_n), u_0) = 0.$$

(b) There exist $u_0, v_0 \in \mathcal{E}$ *such that, for* $n \geq 1$*, either*

(i) $\omega(x_n) < \omega(x_{n+1}) < \omega(x_0) = u_0 < v_0 = \omega(z_n)$*,* $\lim_{n\to\infty} \mathrm{dist}(\omega(x_n), u_0) = 0$ *and whenever* $v \in \mathcal{E}$*,* $v > u_0$ *then* $v \geq v_0$*,*

or

(ii) $\omega(x_n) = u_0 < v_0 = \omega(x_0) < \omega(z_{n+1}) < \omega(z_n)$*,* $\lim_{n\to\infty} \mathrm{dist}(\omega(z_n), v_0) = 0$ *and whenever* $u \in \mathcal{E}$*,* $u < v_0$ *then* $u \leq u_0$*.*

(c) There exists $u_0 \in \mathcal{E}$ *such that, for* $n \geq 1$*, either*

(i) $\omega(x_n) < \omega(x_{n+1}) < \omega(x_0) = u_0 = \omega(z_n)$*,* $\lim_{n\to\infty} \mathrm{dist}(\omega(x_n), u_0) = 0$

or

(ii) $\omega(x_n) = u_0 = \omega(x_0) < \omega(z_{n+1}) < \omega(z_n)$, *and* $\lim_{n\to\infty} \operatorname{dist}(\omega(z_n), u_0) = 0$.

(d) There exist equilibria u_0 *and* v_0 *such that, for* $n \geq 1$,

$$\omega(x_n) = u_0 < \omega(x_0) < v_0 = \omega(z_n).$$

If $u \in \mathcal{E}$ *and* $u < \omega(x_0)$ *then* $u \leq u_0$. *If* $v \in \mathcal{E}$ *and* $\omega(x_0) < v$ *then* $v \geq v_0$.

(e) There exists $u_0 \in \mathcal{E}$ *such that, for* $n \geq 1$, *either*

(i) $\omega(x_n) = u_0 < \omega(x_0) = \omega(z_n) \subset \mathcal{E}$ *and, whenever* $u \in \mathcal{E}$ *satisfies* $u < \omega(x_0)$, *then* $u \leq u_0$

or

(ii) $\omega(x_n) = u_0 > \omega(x_0) = \omega(x_n) \subset \mathcal{E}$ *and, whenever* $u \in \mathcal{E}$ *satisfies* $u > \omega(x_0)$, *then* $u \geq u_0$.

(f) For $n \geq 1$, $\omega(x_n) = \omega(x_0) = \omega(z_n) \subseteq \mathcal{E}$.

3.4 Global Results

When there exists only one equilibrium in Ω, the dynamic is globally attracting as the following.

Theorem 3.4.1. (Global Asymptotic Stability.) *Suppose that the semiflow is SOP,* Ω *contains exactly one equilibrium* e *and every point of* $\Omega\diagdown\{e\}$ *can be approximated from above and from below in* Ω. *Then* $\omega(x) = e$ *for all* $x \in \Omega$.

Remark. If $x \in \Omega\diagdown\{e\}$ then only alternatives (a), (e) and (f) of Proposition 3.3.4 may hold since the others imply more than a single equilibrium. In particular $x \in Q$, hence $\omega(x) = e$.

If there exists multiple equilibria in Ω, the dynamics is more complicated even in a scalar delayed differential equation. Pituk [42] considered a class of scalar DDEs generating a strongly order preserving semiflow with respect to the "*exponential ordering*",

$\leq_\mu$, generated by the closed cone

$$K_\mu := \{\phi \in C([-\tau, 0], \mathbb{R}) \mid \phi \geq 0 \text{ and } \phi(s)e^{\mu s} \text{ is nondecreasing on } [-\tau, 0]\}. \qquad (3.1)$$

Herein, $\geq$ is the standard partial order and $\mu \geq 0$ will be given in hypotheses. The main result said that, the global convergence of the solutions of scalar DDE (2.2) is equivalent (without any restriction on the number of equilibria) to the boundedness of the solutions of the relative ordinary differential equation, under hypotheses: there exist $\mu > 0$ and a bounded linear functional $\Pi : \mathcal{C} \to \mathbb{R}$ such that

$$|f(\psi) - f(\phi)| \leq \Pi(|\psi - \phi|) \text{ for all } \psi, \phi \in \mathcal{C} \qquad (3.2)$$

and

$$-\Pi(\phi) + \mu\phi(0) > 0 \text{ whenever } \phi \in \mathcal{C} \text{ and } \phi >_\mu 0. \qquad (3.3)$$

In the case of multi-dimensional delayed differential equations with multiple equilibria, the global convergence of dynamics remains as an unsolved problem.

Chapter 4

Neural Networks with Delays

4.1 Global Dissipativity

The concept of dissipativity has been applied in diverse areas of neural networks such as stability theory, chaos and synchronization theory, and robust control. A flow on a complete metric space is said to be *dissipative* if there is a bounded subset, of the metric space, which attracts each point of the whole space under the flow [24]. In [35], the global dissipation and global exponential dissipation of delayed neural networks (1.1) with several activation functions were analyzed. We give the explicit definitions for delayed recurrent neural networks.

Definition 4.1.1. *(i)The system* (1.1) *is said to be a dissipative system, if there exists a compact set* $\mathcal{U} \subset \mathcal{C}$ *such that for each* $\phi \in \mathcal{C}$ *there exists* $T > 0$ *with* $\mathbf{x}(t;\phi) \in \mathcal{U}$ *whenever* $t \geq T$*. In this case,* $\mathcal{U}$ *is called a globally attractive set. (ii)A set* $\mathcal{U}$ *is called positive invariant if for each* $\phi \in \mathcal{U}$*,* $\mathbf{x}(t;\phi) \in \mathcal{U}$ *whenever* $t \geq 0$*.*

Definition 4.1.2. *The system* (1.1) *is said to be a globally exponentially dissipative system, if it is a dissipative system with a globally attractive set* $\mathcal{U}$ *and there exists a compact set* $\tilde{\mathcal{U}} \supset \mathcal{U}$ *such that for each* $\phi \in \mathcal{C}\diagdown\tilde{\mathcal{U}}$*, there exists constants* $r(\phi) > 0$ *and* $s > 0$ *such that*

$$\inf_{\phi\in\mathcal{C}\setminus\tilde{\mathcal{U}}}\{\| \mathbf{x}_t(\phi) - \tilde{\mathbf{x}} \| | \tilde{\mathbf{x}} \in \tilde{\mathcal{U}}\} \leq r(\phi)e^{-st} \text{ for all } t \geq 0.$$

The set $\tilde{\mathcal{U}}$ *is called a globally exponentially attractive set.*

In [35], the dissipative property was discussed in DRNN system with several classes of activation functions as following:

- The set of bounded activation functions is defined as

$$\mathcal{G}_1 := \{g_i \| g_i(\xi)| \le \rho_i, \quad 0 \le D_r g_i(\xi) \le \gamma_i \text{ for all } \xi \in \mathbb{R}\},$$

  where D_r denotes the right-hand derivative of a function and $0 \le \rho_i, \gamma_i < \infty$.

- The set of Lipschitz activation functions is defined as

$$\mathcal{G}_2 := \{g_i | 0 \le \frac{g_i(\xi_1) - g_i(\xi_2)}{\xi_1 - \xi_2} \le \gamma_i < \infty \text{ for all } \xi_1, \xi_2 \in \mathbb{R}\}.$$

- The general set of continuous nondecreasing activation functions is denoted as

$$\mathcal{G}_3 := \{g_i | g_i \in C(\mathbb{R}, \mathbb{R}), \ D_r g_i(\xi) \ge 0 \text{ for all } \xi \in \mathbb{R}\}.$$

By constructing Lyapunov functions and using certain matrix theory, the authors in [35] demonstrated that the DRNN (1.1) is a dissipative system. Particularly, the dissipative property of DRNN system with activation functions in each of the previous three class is summarized as following: (i) If $g_i \in \mathcal{G}_1$, for all $i = 1, \cdots, n$, the system (1.1) is dissipative with a positive invariant and globally attractive set. Furthermore, it is also globally exponentially dissipative with another globally exponentially attractive set. (ii) If $g_i \in \mathcal{G}_2$, $g_i(0) = 0$ and $|g_i(\xi)| \to \infty$ as $|\xi| \to \infty$, for all $i = 1, \cdots, n$, the system (1.1) is dissipative under additional conditions on interconnection weights. (iii) Finally, if $g_i \in \mathcal{G}_3$ and $g_i(0) = 0$, for all $i = 1, \cdots, n$, the system (1.1) is also dissipative under suitable conditions on interconnection weights. Herein, we recall one dissipative results of the system with activation functions in class $\mathcal{G}_1$.

Theorem 4.1.3. [35] *With each activation function $g_i \in \mathcal{G}_1$, the delayed recurrent neural network* (1.1) *is a dissipative system. The set $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ is a positive invariant* *and globally attractive set, where*

$$
\begin{aligned}
\mathcal{U}_1 \quad &:= \{\phi \in \mathcal{C} | \sum_{i=1}^{n} \mu_i [|\phi_i(\theta)| - \frac{1}{2\mu_i} (\sum_{j=1}^{n} (|\alpha_{ij}| + |\beta_{ij}|)\rho_j + |I_i|)]^2 \\
&\qquad \leq \sum_{i=1}^{n} \frac{1}{4\mu_i} [\sum_{j=1}^{n} (|\alpha_{ij}| + |\beta_{ij}|)\rho_j + |I_i|]^2, \text{ for all } \theta \in [-\tau, 0]\}, \\
\mathcal{U}_2 \quad &:= \{\phi \in \mathcal{C} || \phi_i(\theta)| \leq \frac{1}{\mu_i} [\sum_{j=1}^{n} (|\alpha_{ij}| + |\beta_{ij}|)\rho_j + |I_i|], \text{ for all } \theta \in [-\tau, 0]\}.
\end{aligned}
$$

The dynamics of the DRNN system depends on the characteristic of the activation functions, interconnection weights and the amount of the delay time. The last theorem declared the delay independent dissipation property of the DRNN system (1.1) with bounded activation functions, and the globally attractive set is estimated by this bound and parameters. In case of $g_i \in \mathcal{G}_2$, Lipschitz activation functions, the globally attractive set concerning Lipschitz constant γ_i and parameters was obtained. As for $g_i \in \mathcal{G}_3$, general continuous nondecreasing activation functions, the condition on parameter is crucial for dissipative property.

Remark. When we consider (1.1) with activation functions in one of the previous classes, this system has the global dissipativity property. Then $\{\mathbf{x}_t | t \geq 0\}$ has compact closure in $\mathcal{C}$, and therefore each solution exists for all $t > 0$ by Theorem 4.1.3.

4.2 Comparison of Neural Networks with and without Delays

In this section, we summarize some difference between neural networks with and without delays, involving characteristic equations, Lyapunov function and Lyapunov functional theory and some different dynamics induced from delay.

4.2.1 Characteristic Equations

Utilizing the linearized theory and computing roots of the associated characteristic equations are often effective in studying local behaviors of the system. In the recurrent neural network without delay, the characteristic equation of the linearized system at an equilibrium $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ is

$$\begin{aligned} &\det(\Delta(\lambda)) = 0, \\ &\Delta(\lambda) := \lambda I_n + D - A - B, \end{aligned} \tag{4.1}$$

where $D = \mathrm{diag}[-\mu_1, \cdots, -\mu_n]$, $A = [a_{ij}]$, and $B = [b_{ij}]$ are $n \times n$ matrices with $a_{ij} = \alpha_{ij} g_j^{'}(\bar{x}_j)$, $b_{ij} = \beta_{ij} g_j^{'}(\bar{x}_j)$. On the other hand, for the delayed case, even if $\tau_{ij} = \tau$ for all $i,\ j$, with τ positive, the characteristic equation is

$$\begin{aligned} &\det(\Delta(\lambda)) = 0, \\ &\Delta(\lambda) := \lambda I_n + D - A - e^{-\lambda\tau} B. \end{aligned} \tag{4.2}$$

Thus, the characteristic equations corresponding to the linearized delayed differential equations are no longer ordinary polynomials; instead, they are exponential polynomials. It is well known [49] that the equilibrium of DRNN (1.1) is asymptotically stable if all the roots of the transcendental function (4.2) have negative real parts. Analysis on zeros in (4.2) is much more complicated than the situation in (4.1). A standard result [32] tells us that the equilibrium of (1.1) can only lose stability as parameters vary in a way that the characteristic equation has a root passing through the imaginary axis. By Rouché's Theorem [16], the bifurcations could be determined at the points in parameter space or delay values, see [4], for example.

4.2.2 Lyapunov Functionals and Lyapunov Functions

In neural network systems without delays, complete stability (convergence) and quasiconvergence could be obtained by applying the LaSalle's invariant principle to their respective Lyapunov functions. Such functions are originated from the studies of Cohan and Grossberg [15]. We recall some Lyapunov functions and functional from the literatures.

- In a classical neural network (β_{ij}=0 for all i, j in Eq (1.1)), if the connection weights matrix $[\alpha_{ij}]$ is symmetric, the activation functions are bounded, differential and $g_i'(\xi) > 0$ for all $\xi \in \mathbb{R}$ and $i = 1, 2, \cdots, n$, the authors in [54] proposed the Lyapunov function

$$L(\mathbf{x}) = -\frac{1}{2}\langle \mathbf{g}(\mathbf{x}), \mathbf{A}\mathbf{g}(\mathbf{x}) + 2\mathbf{I}\rangle + \sum_{i=1}^{n} \int_{g_i(0)}^{g_i(x_i)} g_i^{-1}(\xi)d\xi, \tag{4.3}$$

where $\mathbf{g} = (g_1, \cdots, n)$, $\mathbf{A}$ is the matrix $[\alpha_{ij}]$, and $\mathbf{I} = (I_1, \cdots, I_n)$.

- In the delayed Hopfield neural network (α_{ij}=0 for all i, j in Eq (1.1)), the authors in [52] addressed Lyapunov functionals for this system with a unique equilibrium. If $\bar{\mathbf{x}}$ is the unique equilibrium and $\mathbf{x}_t$ is a solution of the system, by defining $\mathbf{u}(t) := \mathbf{x}(t) - \bar{\mathbf{x}}$, Eq (1.1) becomes

$$\frac{du_i(t)}{dt} = -\mu_i u_i(t) + \sum_{j=1}^{n} \beta_{ij}\tilde{g}_j(u_j(t-\tau_{ij})), \quad i = 1, \cdots, n, \tag{4.4}$$

where $\tilde{g}_j(u_j) = g_j(u_j + \bar{x}_j) - g_j(\bar{x}_j)$. For a solution $\mathbf{u}_t$ of (4.4), the authors used the following Lyapunov functional

$$W_1(\mathbf{u}_t) = \sum_{i=1}^{n} \frac{1}{\mu_i} u_i(t)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{|\beta_{ij}|}{\mu_i} \int_{t-\tau_{ij}}^{t} \tilde{g}_j^2(u_j(s))ds. \tag{4.5}$$

- In the general delayed neural network Eq (1.1), the author in [5] used the Lyapunov functional

$$W_2(t) = W_2(\mathbf{u})(t) = \sum_{i=1}^{n} \left(\frac{1}{2}u_i^2(t) + \frac{1}{2}\sum_{j=1}^{n} |\beta_{ij}| L_j^{2\zeta_j} \int_{t-\tau_{ij}}^{t} u_j^2(s)ds \right), \tag{4.6}$$

where $\mathbf{u}(t) := \mathbf{x}(t) - \mathbf{x}^*$, L_j is the Lipschitz constant of $g_j(\cdot)$ and $\{\zeta_j\}$ are constants chosen in the proof. The globally asymptotical stability is concluded by using the LaSalle's invariant principle.

The global stability of the unique equilibrium was demonstrated in delayed Hopfield neural networks and delayed cellular neural networks in many literatures (see the aforementioned [5, 52] for example), by constructing an appropriate Lyapunov function or Lyapunov functional. It seems that the Lyapunov functionals used in DRNNs are usually more complicated, involving the delay terms, than those analogously used in neural networks without delays. Besides, to the best of our knowledge, the global Lyapunov function or functional has not been proposed to deal with the delayed neural network system with multiple equilibria. However, we will justify the stability of multiple equilibria by constructing a local Lyapunov functional latter.

4.3 Activation Functions and Multiple Equilibria

Existence and stability of stationary patterns for neural networks certainly depend on properties of activation functions. We shall consider general sigmoidal activation functions $g_i(\cdot)$ as well as the standard activation function for Eq (1.1):

- class $\mathcal{A}$: $g_i \in C^2, \begin{cases} g_i'(\xi) > 0, \ (\xi - \sigma_i)g_i''(\xi) < 0, \ \text{for all } \xi \in \mathbb{R}, \\ \lim_{\xi \to +\infty} g_i(\xi) = v_i, \lim_{\xi \to -\infty} g_i(\xi) = u_i; \end{cases}$
- class $\mathcal{B}$: $g_i \in C, \ g_i(\xi) = \begin{cases} u_i & \text{if } -\infty < \xi < p_i, \\ \tilde{g}_i(\xi) & \text{if } p_i \le \xi \le q_i, \\ v_i & \text{if } q_i < \xi < \infty, \end{cases}$

where, u_i, v_i, p_i, q_i and σ_i are constants with $u_i < v_i$ and $p_i < q_i$, $\tilde{g}_i(\cdot)$ are increasing functions, $i = 1, \cdots, n$. Class $\mathcal{A}$ contains general bounded smooth sigmoidal functions, and class $\mathcal{B}$ consists of nondecreasing functions with saturation. Typical configurations of these functions are depicted in Figure 4.1. Class $\mathcal{B}$ contains the piecewise linear functions with two corner points at p_i, q_i:

$$\tilde{g}_i(\xi) = u_i + \frac{v_i - u_i}{q_i - p_i}(\xi - p_i), \ \xi \in [p_i, q_i]; \tag{4.7}$$

and in particular, the standard activation function for the CNN:

$$\overline{g}(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|), \ \xi \in \mathbb{R}, \tag{4.8}$$

as depicted in Figure 4.2 (a). Notably, in practical implementation, the transition from the linear regime to the saturated regime in the standard activation function is smooth. Thus, the theory developed for the dynamics of Eq(1.1) should be also valid for the activation function with smooth corners at $\xi = \pm 1$, as demonstrated in Figure 4.2 (b). Our investigations have provided theoretical basis for all these activation functions. In Section 4.7, we will see some differences between the dynamics for Eq (1.1) with activation functions of classes $\mathcal{A}$ and the ones of class $\mathcal{B}$. Let $\rho_i = \max\{|u_i|,\ |v_i|\}$, $\gamma_i = \sup_{\xi\in\mathbb{R}} g_i'(\xi)$, $i = 1, \cdots, n$.

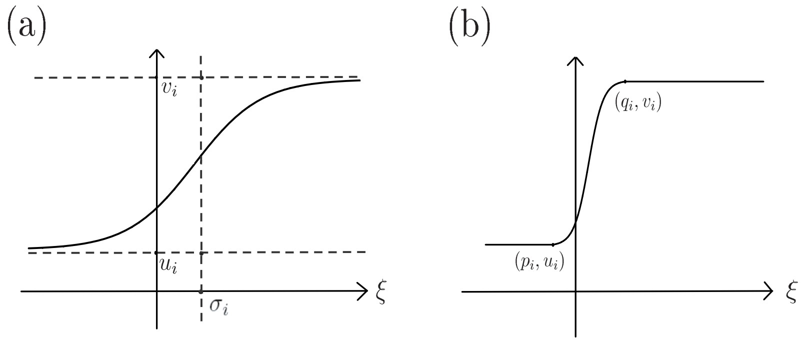

Figure 4.1: The configurations of (a) typical smooth sigmoidal activation functions in class $\mathcal{A}$ and (b) saturated activation functions in class $\mathcal{B}$.

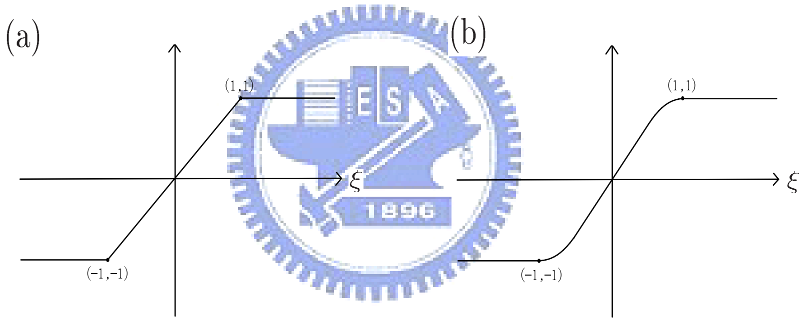

Figure 4.2: The graphs for (a) the standard activation function $\overline{g}(\xi) = \frac{1}{2}(|\xi+1|-|\xi-1|)$, (b) saturated activation functions with smooth corners.

Let us review some basic notion of delayed differential equations. We set $\tau = \max_{1\leq i,j\leq n} \tau_{ij}$. The initial condition for Eq (1.1) is $x_i(\theta) = \phi_i(\theta),\ -\tau \leq \theta \leq 0,\ i = 1, \cdots, n$ with $\phi = (\phi_1, \cdots,\ \phi_n) \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Recall that the norm of ϕ is defined as $\|\phi\| = \max_{1\leq\ i\leq\ n}\{\sup_{s\in[-\tau,\ 0]} |\phi_i(s)|\}$. Let us denote $F = (F_1, \cdots, F_n)$, where F_i is the right hand side of system (1.1),

$$F_i(\mathbf{x}_t) \;=\; -\mu_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} \beta_{ij} g_j(x_j(t-\tau_{ij})) + I_i.$$

A function $\mathbf{x}(\cdot)$ is called a solution of Eq (1.1) on $[-\tau,\ell)$ if $\mathbf{x}(\cdot)\in\mathcal{C}([-\tau,\ell),\mathbb{R}^n)$, and $\mathbf{x}_t$ defined as (2.1) lies in the domain of F and satisfies Eq (1.1) for $t\in[0,\ell)$. For a given $\phi\in\mathcal{C}([-\tau,0],\mathbb{R}^n)$, let us denote by $\mathbf{x}(t;\phi)$ the solution of Eq (1.1) with $\mathbf{x}_0(\theta;\phi)=\mathbf{x}(0+\theta;\phi)=\phi(\theta)$, for $\theta\in[-\tau,0]$.

Notably, the stationary equation for Eq (1.1) is

$$\tilde{F}_i(\mathbf{x}) \;=\; -\mu_i x_i+\sum_{j=1}^{n}(\alpha_{ij}+\beta_{ij})g_j(x_j)+I_i=0,\; i=1,\cdots,n. \tag{4.9}$$

We introduce an analogue of single neuron equation $d\xi/dt=f_i(\xi)=-\mu_i\xi+(\alpha_{ii}+\beta_{ii})g_i(\xi)+I_i,\ \xi\in\mathbb{R}$. Next, we shall consider the above activation functions and formulate sufficient conditions for existence of multiple stationary solutions for Eq (1.1). Our approach is based on a geometrical observation. The first condition for Eq (1.1) with activation functions in classes $\mathcal{A}$ and $\mathcal{B}$ is, respectively,

$$\begin{aligned}(\mathrm{H}_1^{\mathcal{A}}) &: \; 0=\inf_{\xi\in\mathbb{R}} g_i'(\xi)<\frac{\mu_i}{\alpha_{ii}+\beta_{ii}}<\max_{\xi\in\mathbb{R}} g_i'(\xi)\ (=g_i'(\sigma_i)),\\ (\mathrm{H}_1^{\mathcal{B}}) &: \; (\alpha_{ii}+\beta_{ii})\max \tilde{g}_i'(\xi)>\mu_i,\end{aligned}$$

for $i=1,\cdots,n$. Condition $(\mathrm{H}_1^{\mathcal{B}})$ reduces to $(\alpha_{ii}+\beta_{ii})\frac{v_i-u_i}{q_i-p_i}>\mu_i$, if piecewise linear activation functions (4.7) are adopted, and reduces to

$$\alpha_{ii}+\beta_{ii}>\mu_i,\; i=1,\cdots,n, \tag{4.10}$$

if the standard activation function $\overline{g}(\cdot)$ in Eq (4.8) is employed, with $p_i=u_i=-1, q_i=v_i=1$.

Lemma 4.3.1. *(i) For activation functions in class $\mathcal{A}$, there exist two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i<\sigma_i<\tilde{q}_i$, such that $f_i'(\tilde{p}_i)=0$ and $f_i'(\tilde{q}_i)=0$, $i=1,\cdots,n$, under condition $(\mathrm{H}_1^{\mathcal{A}})$. (ii) For activation functions in class $\mathcal{B}$, there exist two points $\tilde{p}_i$ and $\tilde{q}_i$ with $\tilde{p}_i\ge p_i$ and $\tilde{q}_i\le q_i$, such that $f_i'(\tilde{p}_i)=0$, $f_i'(\tilde{q}_i)=0$, $i=1,\cdots,n$, under condition $(\mathrm{H}_1^{\mathcal{B}})$.*

Proof. We only prove case (i). For each i, since $f_i'(\xi)=-\mu_i+(\alpha_{ii}+\beta_{ii})g_i'(\xi)$, we have $f_i'(\xi)=0$ if and only if $g_i'(\xi)=\mu_i/(\alpha_{ii}+\beta_{ii})$. The graph of function $g_i'(\xi)$ is concave down and has its maximal value at σ_i. Note that $\lim_{\xi\rightarrow\pm\infty} g_i'(\xi) = 0$. Hence, since each g_i' is continuous, if

$$0 = \inf_{\xi\in\mathbb{R}} g_i'(\xi) < \frac{\mu_i}{\alpha_{ii}+\beta_{ii}} < \max_{\xi\in\mathbb{R}} g_i'(\xi)\ (= g_i'(\sigma_i)),\ i = 1,\cdots,n,$$

there exist two points $\tilde{p}_i$, $\tilde{q}_i$, with $\tilde{p}_i < \sigma_i < \tilde{q}_i$, such that $g_i'(\tilde{p}_i) = g_i'(\tilde{q}_i) = \mu_i/(\alpha_{ii}+\beta_{ii})$. This completes the proof. □

For Eq (1.1) with piecewise linear activation functions, f_i attains its local minimum at $\tilde{p}_i = p_i$, and local maximum at $\tilde{q}_i = q_i$, under assumption $(\mathrm{H}_1^{\mathcal{B}})$. In particular, for the standard activation function $\overline{g}$, $\tilde{p}_i = -1, \tilde{q}_i = 1, i = 1,\cdots,n$. A consequence of Lemma 4.3.1 is that f_i is strictly increasing on $(-\infty,\tilde{p}_i]$, decreasing on $[\tilde{q}_i,\infty)$, under condition (H_1^*).

Note that condition $(\mathrm{H}_1^*), * = \mathcal{A},\mathcal{B}$, implies $\alpha_{ii}+\beta_{ii} > 0$ for each $i = 1,\cdots,n$, since μ_i is already assumed positive. We define, for $i = 1,\cdots,n$,

$$\begin{aligned}\hat{f}_i(\xi) &= -\mu_i\xi + (\alpha_{ii}+\beta_{ii})g_i(\xi) + k_i^+\\ \check{f}_i(\xi) &= -\mu_i\xi + (\alpha_{ii}+\beta_{ii})g_i(\xi) + k_i^-,\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}k_i^+ &= \sum_{j=1,j\neq i}^{n} \rho_j(|\alpha_{ij}|+|\beta_{ij}|) + I_i\\ k_i^- &= -\sum_{j=1,j\neq i}^{n} \rho_j(|\alpha_{ij}|+|\beta_{ij}|) + I_i.\end{aligned}$$

It follows that $\check{f}_i(x_i) \le \tilde{F}_i(\mathbf{x}) \le \hat{f}_i(x_i)$, for all $\mathbf{x} = (x_1,\cdots,x_n)$ and $i = 1,\cdots,n$, since $u_j \le g_j \le v_j$ for all j, in each class of activation functions.

We consider the second parameter condition which is used to establish existence of multiple equilibria for Eq (1.1) :

$$(\mathrm{H}_2)\ :\ \hat{f}_i(\tilde{p}_i) < 0,\ \check{f}_i(\tilde{q}_i) > 0,\ i = 1,\cdots,n.$$

The configuration that motivates (H_2) is depicted in Figure 4.3 and 4.4. Under assumptions (H_1^*) and (H_2), $* = \mathcal{A}$, $\mathcal{B}$, there exist points $\hat{a}_i, \hat{b}_i, \hat{c}_i$ with $\hat{a}_i < \hat{b}_i < \hat{c}_i$ such that $\hat{f}_i(\hat{a}_i) = \hat{f}_i(\hat{b}_i) = \hat{f}_i(\hat{c}_i) = 0$ as well as points $\check{a}_i, \check{b}_i, \check{c}_i$ with $\check{a}_i < \check{b}_i < \check{c}_i$, such that $\check{f}_i(\check{a}_i) = \check{f}_i(\check{b}_i) = \check{f}_i(\check{c}_i) = 0$. Then, by applying the Brouwer's fixed point theorem, we could derive multiequilibria as following.

Theorem 4.3.2. *There exist* 3^n *equilibria for system* (1.1) *with activation functions in class* $*$, $* = \mathcal{A}, \mathcal{B}$, *under conditions* (H_1^*) *and* (H_2).

Proof. We only prove the case of class $\mathcal{A}$, i.e., under conditions $(\mathrm{H}_1^{\mathcal{A}})$ and (H_2). The equilibria of system (1.1) are zeros of Eq (4.9). According to the configurations in Figure 4.3 and 4.4, there are 3^n disjoint closed regions in $\mathbb{R}^n$, namely, $\mathbf{\Omega}^{\mathbf{w}} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \in \Omega_i^{w_i}\}$ with $\mathbf{w} = (w_1, \cdots, w_n)$, $w_i =$"l", "m" or "r", where

$$
\begin{aligned}
\Omega_i^{\mathrm{l}} &= \{\xi \in \mathbb{R} |\ \check{a}_i \le \xi \le \hat{a}_i\}, \\
\Omega_i^{\mathrm{m}} &= \{\xi \in \mathbb{R} |\ \hat{b}_i \le \xi \le \check{b}_i\}, \\
\Omega_i^{\mathrm{r}} &= \{\xi \in \mathbb{R} |\ \check{c}_i \le \xi \le \hat{c}_i\}.
\end{aligned} \tag{4.12}
$$

Herein, "l", "m", "r" mean respectively "left", "middle" and "right". Let $\mathbf{\Omega}^{\mathbf{w}}$ be one of these regions. For any given $\tilde{\mathbf{x}} = (\tilde{x}_1, \cdots, \tilde{x}_n) \in \mathbf{\Omega}^{\mathbf{w}}$, we solve for x_i in

$$
h_i(x_i) = -\mu_i x_i + (\alpha_{ii} + \beta_{ii}) g_i(x_i) + \sum_{j=1, j \ne i}^{n} (\alpha_{ij} + \beta_{ij}) g_i(\tilde{x}_j) + I_i = 0, \tag{4.13}
$$

$i = 1, \cdots, n$. Note that h_i is a vertical shift of $\hat{f}_i$ or $\check{f}_i$, due to Eq (4.11). Accordingly, one can always find three solutions to Eq (4.13) and each of them lies in one of the regions in Eq (4.12), for each i. We define a mapping $\mathbf{H}_{\mathbf{w}} : \mathbf{\Omega}^{\mathbf{w}} \to \mathbf{\Omega}^{\mathbf{w}}$ by

$$
\mathbf{H}_{\mathbf{w}}(\tilde{\mathbf{x}}) = \underline{\mathbf{x}} = (\underline{x}_1, \cdots, \underline{x}_n),
$$

where $\underline{x}_i$ is the solution of Eq (4.13) lying in $\Omega_i^{w_i}$. The mapping $\mathbf{H}_{\mathbf{w}}$ as defined is continuous, since g_i is continuous. It follows from the Brouwer's fixed point theorem that there exists one fixed point $\bar{\mathbf{x}} = (\bar{x}_1, \cdots, \bar{x}_n)$ of $\mathbf{H}_{\mathbf{w}}$ in $\mathbf{\Omega}^{\mathbf{w}}$, which is also a zero of $\tilde{F}$ in Eq (4.9). Consequently, there exist 3^n equilibria for system (1.1) and each of them lies in one of the 3^n regions $\mathbf{\Omega}^{\mathbf{w}}$. This completes the proof. $\square$

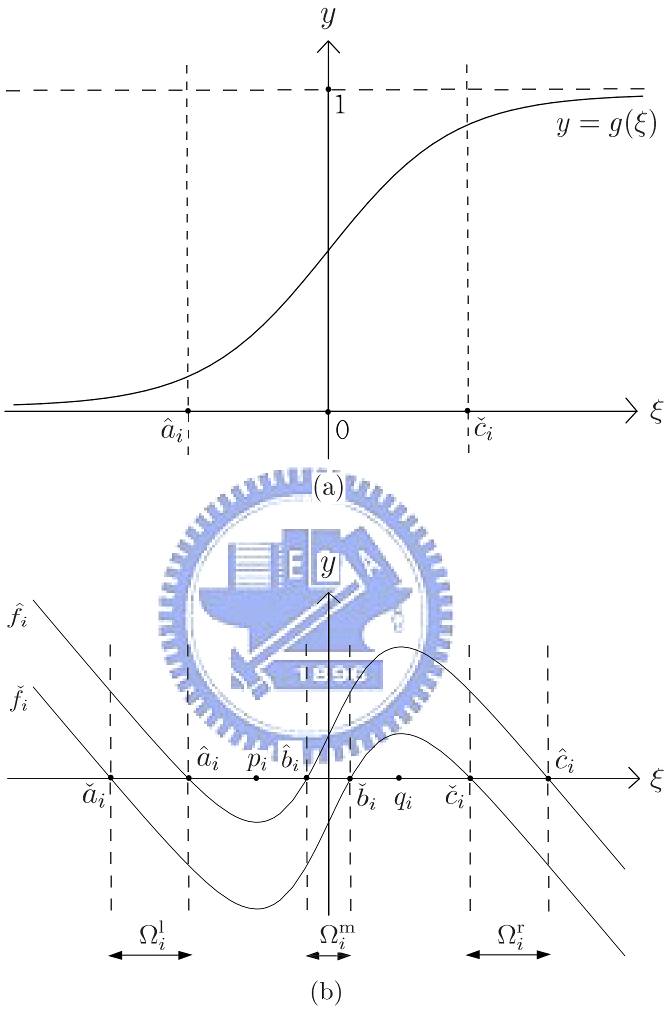

Figure 4.3: (a) The graph of activation function g_i in class $\mathcal{A}$, (b) Configurations of functions $\hat{f}_i$ and $\check{f}_i$.

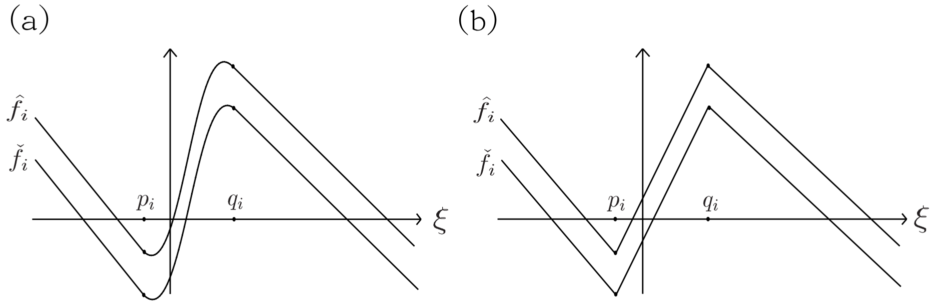

Figure 4.4: (a) The graphs of $\hat{f}_i$ and $\check{f}_i$ induced from the activation function of class $\mathcal{B}$. (b) The graphs of $\hat{f}_i$ and $\check{f}_i$ induced from the standard activation function $\overline{g}$.

4.4 Stability of Equilibria and Basins of Attraction

In this section, we first establish some positively invariant sets for system (1.1) and investigate stability of the equilibrium in each invariant set. As a result, we also obtain the basin of attraction for each of the asymptotically stable equilibrium.

We consider the following 2^n subsets of $\mathcal{C}([-\tau,0],\mathbb{R}^n)$. Let $\mathbf{w} = (w_1,\cdots,w_n)$ with $w_i =$ "l" or "r", and set

$$\tilde{\mathbf{\Lambda}}^{\mathbf{w}} = \{\varphi = (\varphi_1,\cdots,\varphi_n) \mid \varphi_i \in \tilde{\Lambda}_i^{\mathrm{l}} \text{ if } w_i = \text{"l"},\ \varphi_i \in \tilde{\Lambda}_i^{\mathrm{r}} \text{ if } w_i = \text{"r"}\}, \tag{4.14}$$

where

$$\begin{aligned}\tilde{\Lambda}_i^{\mathrm{l}} &= \{\varphi_i \in \mathcal{C}([-\tau,0],\mathbb{R}) \mid \varphi_i(\theta) < \hat{b}_i \text{ for all } \theta \in [-\tau,0]\},\\ \tilde{\Lambda}_i^{\mathrm{r}} &= \{\varphi_i \in \mathcal{C}([-\tau,0],\mathbb{R}) \mid \varphi_i(\theta) > \check{b}_i \text{ for all } \theta \in [-\tau,0]\}.\end{aligned}$$

Theorem 4.4.1. *Assume that* (H_1^*), (H_2), $* = \mathcal{A},\mathcal{B}$, *and* $\beta_{ii} > 0, i = 1,\cdots,n$, *then each* $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ *is positively invariant under the solution flow generated by system* (1.1) *with activation functions in class* $*$.

Proof. We only prove the $\mathcal{A}$ case. Let $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ be a subset defined in (4.14). Consider any initial condition $\phi = (\phi_1,\cdots,\phi_n) \in \tilde{\mathbf{\Lambda}}^{\mathbf{w}}$, there exists a sufficiently small constant $\varepsilon_0 > 0$ such that $\phi_i(\theta) \geq \check{b}_i + \varepsilon_0$ for all $\theta \in [-\tau,0]$, if $w_i =$"r", and $\phi_i(\theta) \leq \hat{b}_i - \varepsilon_0$ for all $\theta \in [-\tau,0]$, if $w_i =$"l". We claim that the solution $\mathbf{x}(t;\phi)$ remains in $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ for all

$t \geq 0$. If this is not true, there exists a component of $\mathbf{x}(t;\phi)$ which is the first one (or one of the first ones) decreasing across the value $\check{b}_i + \varepsilon_0$ or increasing across the value $\hat{b}_i - \varepsilon_0$; i.e., there exists some $i \in \{1, \cdots, n\}$ and $t_1 > 0$ such that either $x_i(t_1) = \check{b}_i + \varepsilon_0$, $(dx_i/dt)(t_1) \leq 0$, and $x_i(t) > \check{b}_i + \varepsilon_0$ for $-\tau \leq t < t_1$ or $x_i(t_1) = \hat{b}_i - \varepsilon_0$, $(dx_i/dt)(t_1) \geq 0$ and $x_i(t) < \hat{b}_i - \varepsilon_0$ for $-\tau \leq t < t_1$. For the first case, we derive from Eq (1.1) that

$$\begin{aligned}\frac{dx_i}{dt}(t_1) = & - \mu_i(\check{b}_i + \varepsilon_0) + \alpha_{ii} g_i(\check{b}_i + \varepsilon_0) + \beta_{ii} g_i(x_i(t_1 - \tau_{ii})) \\ & + \sum_{j=1, j\neq i}^{n} \alpha_{ij} g_j(x_j(t_1)) + \sum_{j=1, j\neq i}^{n} \beta_{ij} g_j(x_j(t_1 - \tau_{ij})) + I_i \leq 0. \end{aligned} \tag{4.15}$$

On the other hand,

$$\begin{aligned} & -\mu_i(\check{b}_i + \varepsilon_0) + \alpha_{ii} g_i(\check{b}_i + \varepsilon_0) + \beta_{ii} g_i(x_i(t_1 - \tau_{ii})) \\ + & \sum_{j=1, j\neq i}^{n} \alpha_{ij} g_j(x_j(t_1)) + \sum_{j=1, j\neq i}^{n} \beta_{ij} g_j(x_j(t_1 - \tau_{ij})) + I_i \\ \geq & -\mu_i(\check{b}_i + \varepsilon_0) + (\alpha_{ii} + \beta_{ii}) g_i(\check{b}_i + \varepsilon_0) - \sum_{j=1, j\neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i \\ = & \check{f}_i(\check{b}_i + \varepsilon_0) > 0, \end{aligned} \tag{4.16}$$

due to (H_2), $\beta_{ii} > 0$, $|g_j(\cdot)| \leq \rho_j$, and $g_i(x_i(t_1 - \tau_{ii})) \geq g_i(\check{b}_i + \varepsilon_0)$, from the monotonicity of g_i and the definition of t_1. This yields a contradiction to Eq (4.15). Hence, $x_i(t) \geq \check{b}_i + \varepsilon_0$ for all $t > 0$. Similar arguments can be employed to show that $x_i(t) \leq \hat{b}_i - \varepsilon_0$, for all $t > 0$ for the situation that $x_i(t_1) = \hat{b}_i - \varepsilon_0$ and $(dx_i/dt)(t_1) \geq 0$. Therefore, $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ is positively invariant under the flow generated by system (1.1). The proof is completed. $\square$

Next, we consider the following criterion concerning stability of the equilibria for classes $\mathcal{A}$ and $\mathcal{B}$. Let η_j be real numbers satisfying $\gamma_j \geq \eta_j \geq \max\{g_j'(\xi) \mid \xi = \check{c}_j,\ \hat{a}_j\}$ for $j = 1, \cdots, n$. Consider

$$(\mathrm{H}_3)\ :\ \mu_i > \sum_{j=1}^{n} \eta_j(|\alpha_{ij}| + |\beta_{ij}|), i = 1, \cdots, n.$$

For activation functions $g_j(\cdot)$ in classes $\mathcal{A}$, we define $\underline{d}_j$ and $\bar{d}_j$ as

$$\underline{d}_j = \min\{\xi | g_j'(\xi) = \eta_j\}, \bar{d}_j = \max\{\xi | g_j'(\xi) = \eta_j\}. \tag{4.17}$$

Then $\underline{d}_j \geq \hat{a}_j$, $\overline{d}_j \leq \check{c}_j$. For the activation functions g_j in class $\mathcal{B}'$, $\tilde{g}_i$ in (4.7), and $\overline{g}$ in Eq (4.8), we define, respectively,

$$\underline{d}_j = \tilde{p}_j, \quad \overline{d}_j = \tilde{q}_j; \quad \underline{d}_j = p_j, \quad \overline{d}_j = q_j; \quad \underline{d}_j = -1, \; \overline{d}_j = 1. \tag{4.18}$$

We consider the following 2^n subsets of $\mathcal{C}([-\tau,0],\mathbb{R}^n)$. Let $\mathbf{w} = (w_1, \cdots, w_n)$ with $w_i =$ "l" or "r", and set

$$\mathbf{\Lambda^w} = \{\varphi = (\varphi_1, \cdots, \varphi_n) \mid \varphi_i \in \Lambda_i^{\mathrm{l}} \text{ if } w_i = \text{"l"}, \; \varphi_i \in \Lambda_i^{\mathrm{r}} \text{ if } w_i = \text{"r"}\}, \tag{4.19}$$

where

$$\begin{aligned} \Lambda_i^{\mathrm{l}} &= \{\varphi_i \in \mathcal{C}([-\tau,0],\mathbb{R}) \mid \varphi_i(\theta) < \underline{d}_i, \; \forall \, \theta \in [-\tau,0]\}, \\ \Lambda_i^{\mathrm{r}} &= \{\varphi_i \in \mathcal{C}([-\tau,0],\mathbb{R}) \mid \varphi_i(\theta) > \overline{d}_i, \; \forall \, \theta \in [-\tau,0]\}. \end{aligned}$$

In the following, we will derive that each of these 2^n subsets $\mathbf{\Lambda^w}$ of $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ is a basin of attraction for the respective equilibrium and justify that these 2^n equilibria are exponentially stable.

Theorem 4.4.2. *Under conditions* $(\mathrm{H}_1^{\mathcal{A}})$, (H_2), (H_3), *and* $\beta_{ii} > 0, i = 1, \cdots, n$, *there exist* 2^n *exponentially stable equilibria for system* (1.1) *with activation functions of class* $\mathcal{A}$. *Same assertion holds for activation functions of class* $\mathcal{B}$, *under conditions* $(\mathrm{H}_1^{\mathcal{B}})$, (H_2).

Proof. We only prove the case of class $\mathcal{A}$. Let $\mathbf{\Lambda^w}$ be a subset defined in (4.19) and $\bar{\mathbf{x}}$ be an equilibrium lying in $\mathbf{\Lambda^w}$. For each $i = 1, \cdots, n$, we consider the single-variable function $G_i(\zeta) = \mu_i - \zeta - \sum_{j=1}^n \eta_j |\alpha_{ij}| - \sum_{j=1}^n \eta_j |\beta_{ij}| e^{\zeta \tau_{ij}}$. Then, (H_3) implies $G_i(0) > 0$, and there exists a constant $\lambda > 0$ such that $G_i(\lambda) > 0$, for all $i = 1, \cdots, n$, due to continuity of G_i. Let $\mathbf{x}(t) = \mathbf{x}(t;\phi)$ be the solution to system (1.1) with initial condition $\phi \in \mathbf{\Lambda^w}$. With translation $\mathbf{y}(t) = \mathbf{x}(t) - \bar{\mathbf{x}}$, system (1.1) becomes

$$\frac{dy_i(t)}{dt} = -\mu_i y_i(t) + \sum_{j=1}^n \alpha_{ij}[g_j(x_j(t)) - g_j(\overline{x}_j)] + \sum_{j=1}^n \beta_{ij}[g_j(x_j(t-\tau_{ij})) - g_j(\overline{x}_j)], \tag{4.20}$$

where $\mathbf{y} = (y_1, \cdots, y_n)$. Now, consider functions $z_i(\cdot)$ defined by $z_i(t) = e^{\lambda t}|y_i(t)|$, $i = 1, \cdots, n$. Let $\delta > 1$ and let $K = \max_{1\le i\le n}\{\sup_{\theta\in[-\tau,0]}|x_i(\theta) - \bar{x}_i|\} > 0$. It follows that $z_i(t) < K\delta$, for $t \in [-\tau, 0]$ and $i = 1, \cdots, n$. We shall justify that

$$z_i(t) < K\delta, \text{ for all } t > 0, \; i = 1, \cdots, n. \tag{4.21}$$

Suppose Eq (4.21) does not hold, then there is a $k \in \{1, \cdots, n\}$ and a $t_1 > 0$ for the first time such that $z_i(t) \le K\delta$, $t \in [-\tau, t_1]$, $i = 1, \cdots, n$, $i \ne k$, $z_k(t) \le K\delta$, $t \in [-\tau, t_1)$, and $z_k(t_1) = K\delta$, with $\dot{z}_k(t_1) \ge 0$. Note that $|y_k(t)|$ and $z_k(t)$ are differentiable at $t = t_1$, since $z_k(t_1) = K\delta > 0$ implies $y_k(t_1) \ne 0$. From Eq (4.20), we compute that

$$\frac{d}{dt}|y_k(t_1)| \le -\mu_k|y_k(t_1)| + \sum_{j=1}^{n}|\alpha_{kj}g_j'(\xi_j)y_j(t)| + \sum_{j=1}^{n}|\beta_{kj}g_j'(\varsigma_j)y_j(t_1 - \tau_{kj})|,$$

for some ξ_j between $x_j(t_1)$ and $\bar{x}_j$ as well as ς_j between $x_j(t_1 - \tau_{kj})$ and $\bar{x}_j$. Hence,

$$\begin{aligned}
&\frac{dz_k(t_1)}{dt} \\
\le\;& \lambda e^{\lambda t_1}|y_k(t_1)| + e^{\lambda t_1}[-\mu_k|y_k(t_1)| + \sum_{j=1}^{n}|\alpha_{kj}g_j'(\xi_j)y_j(t)| + \sum_{j=1}^{n}|\beta_{kj}g_j'(\varsigma_j)y_j(t_1 - \tau_{kj})|] \\
=\;& \lambda z_k(t_1) - \mu_k z_k(t_1) + \sum_{j=1}^{n}|\alpha_{kj}||g_j'(\xi_j)z_j(t_1) + \sum_{j=1}^{n}|\beta_{kj}||g_j'(\varsigma_j)e^{\lambda\tau_{kj}}z_j(t_1 - \tau_{kj}) \\
\le\;& -(\mu_k - \lambda)z_k(t_1) + \sum_{j=1}^{n}|\alpha_{kj}|\eta_j z_j(t_1) + \sum_{j=1}^{n}|\beta_{kj}|\eta_j e^{\lambda\tau_{kj}}[\sup_{\theta\in[t_1-\tau,t_1]} z_j(\theta)].
\end{aligned}$$

Herein, the positive invariance property of $\mathbf{\Lambda^w}$ can be verified using the same treatment as the proof of Theorem 4.4.1, under condition $\beta_{ii} > 0, i = 1, \cdots, n$, for activation functions in class $\mathcal{A}$ (and for $\mathcal{B}$). Due to $G_k(\lambda) > 0$, we obtain a contradiction that

$$0 \le \frac{dz_k(t_1)}{dt} \le -\{\mu_k - \lambda - \sum_{j=1}^{n}\eta_j|\alpha_{kj}| - \sum_{j=1}^{n}\eta_j|\beta_{kj}|e^{\lambda\tau_{kj}}\}K\delta < 0.$$

Hence assertion (4.21) holds and $z_i(t) \le K$ for all $t > 0$, $i = 1, \cdots, n$, by taking $\delta \to 1^+$. We thus obtain $|x_i(t) - \bar{x}_i| \le e^{-\lambda t}\max_{1\le j\le n}\{\sup_{\theta\in[-\tau,0]}|x_j(\theta) - \bar{x}_j|\}$, for $t > 0$ and $i = 1, \cdots, n$. Therefore, $\mathbf{x}(t)$ converges to $\bar{\mathbf{x}}$ exponentially. This completes the proof. □

In the above theorem, we have imposed a restriction: $\beta_{ii} > 0, i = 1, \cdots, n$ (positive self-feedback delays) for the cases of activation functions $\mathcal{A}, \mathcal{B}$. The situation is different for the activation functions in class $\mathcal{B}'$. In fact, since the slopes $\nu_i = (v_i - u_i)/(q_i - p_i)$ in the middle parts of the activation functions in $\mathcal{B}'$ are fixed, there can not exist parameters $\mu_i, \alpha_{ij}, \beta_{ij}$, and η_i satisfying both (H_3) and ($H_1^{\mathcal{B}'}$). Indeed, a contradiction arises in $\mu_i > \nu_i(\sum_{j=1}^n |\alpha_{ij}| + |\beta_{ij}|)$ versus $\nu_i(\alpha_{ii} + \beta_{ii}) > \mu_i$. Thus, the definition of $\mathbf{\Lambda^w}$ for the activation functions in $\mathcal{B}'$ and the standard activation function $\overline{g}$ are as indicated in (4.18) and every $\mathbf{\Lambda^w}$ lies in the saturated parts corresponding to the activation functions.

Corollary 4.4.3. *Each of these 2^n subsets $\mathbf{\Lambda^w}$ of $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$, defined in (4.19), is a basin of attraction for the unique equilibrium lying in $\mathbf{\Lambda^w}$, under the assumptions of Theorem 3.*

Corollary 4.4.4. *Under condition $\alpha_{ii} + \beta_{ii} - \sum_{j=1, j\neq i}^n (|\alpha_{ij}| + |\beta_{ij}|) - |I_i| > \mu_i,\ i = 1, \cdots, n$, there exist 2^n exponentially stable equilibria for Eq (1.1) with activation function $\overline{g}$ in (4.8).*

Proof. The condition yields 4.10, and condition (H_2) with $\tilde{p}_i = -1$ and $\tilde{q}_i = 1$ for all $i = 1, \cdots, n$. Hence, the assertion follows from Theorem 4.4.2. □

Remark. (i) Theorems 4.4.2 indicates that there exists an unique equilibrium in each of the 2^n regions $\mathbf{\Lambda^w}$, $\mathbf{w} = (w_1, w_2, \cdots, w_n)$, w_i = "l" or "r", under respective conditions.

(ii) There exists a globally attracting set for system (1.1), according to [35]. Therefore, it can be concluded that every solution of (1.1) is bounded in forward time.

(iii) In [56], (1.1) with $\mu_i = 1,\ i = 1, \cdots, n$, and standard activation function (4.8) is investigated. It was proved therein that, under condition

$$\alpha_{ii} - \sum_{j=1, j\neq i}^{n} |\alpha_{ij}| - \sum_{j=1}^{n} |\beta_{ij}| - |I_i| > 1,\ i = 1, \cdots, n, \tag{4.22}$$

there exist exactly 2^n isolated locally exponential stable equilibria. It is obvious that our condition in Corollary 4.4.4. is weaker than (4.22). In addition, it was shown that the set $\{\mathbf{x} \mid \mathbf{x} = (x_1, \cdots, x_n),\ x_i < -1 \text{ or } x_i > 1\}$ is positively invariant under the flow induced by (1.1). Our Theorem 4.4.1. has exploited larger positively invariant set $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ under the flow induced by (1.1). The computations in deriving the results in [56] heavily depends on the saturation properties of the output functions. Restated, as $x_j(t-\tau_{ij})$ lies in $\{\xi < -1\}$ or $\{\xi > 1\}$, the output $\overline{g}(x_j(t-\tau_{ij}))$ is either -1 or 1, and thus the delay in the equation (1.1) does not have any actual effect in these regions. The numerical simulations therein are thus dealing with ordinary differential equations basically. As mentioned in Section 4.2, the transition from the linear regime to the saturated regime in the standard output function is smooth in practical situation. Our theory is based on a geometrical observation and has been established to take into account these practical considerations.

(iv) It will be justified in Section 4.5 that the basins of attractions for the equilibrium can be extended from $\mathbf{\Lambda}^{\mathbf{w}}$ to $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$. Moreover, the solution lying entirely in $\mathbf{\Lambda}^{\mathbf{w}}$ converges exponentially to the equilibrium in $\mathbf{\Lambda}^{\mathbf{w}}$, whereas the convergence for the solutions lying entirely in $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ may not have exponential rates.

(v) Consider (1.1) with periodic input, i.e. $I_i = I_i(t) = I_i(t+T)$ for all $t \geq 0$, for some $T > 0$. It could be established that there exist 2^n exponentially stable T-period solutions for the system with activation functions of class $*$, under conditions (H_1^*), (H_2), (H_3), $* = \mathcal{A}, \mathcal{B}$ and $\beta_{ii} > 0, i = 1, \cdots, n$, respectively.

The result in Theorem 4.4.2 confirms the exponential stability of 2^n equilibria. Other criterion for concluding exponential stability of the equilibria can be derived through different treatments. The further result concerns different parameters where the previous criterion could not apply. Herein we also consider the following criterion concerning exponential stability of the equilibria.

$$(\mathrm{H}_3') \ :\ \exists\ \eta_i > \max\{g_i'(\xi) \mid \xi = \check{c}_i,\ \hat{a}_i\} \text{ such that } \mu_i > \eta_i \sum_{j=1}^{n} (|\alpha_{ji}| + |\beta_{ji}|),$$

for all $i = 1, 2, \cdots, n$.

Let D_r denote the right-hand derivative of a function.

Lemma 4.4.5. *Let* $\mathbf{u}(t) = (u_1(t), u_2(t), \cdots, u_n(t))$ *and* $\tilde{\mathbf{u}}(t) = (\tilde{u}_1(t), \tilde{u}_2(t), \cdots, \tilde{u}_n(t))$

be continuously differentiable functions on $c \le t \le d$ *and both satisfy*

$$\dot{y}_i(t) = -\mu_i y_i(t) + F_i(\mathbf{y}_t),\ \ i = 1, 2, \cdots, n,$$

where $F_i : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ *is continuously differentiable and* $\mathbf{y} = (y_1, \cdots, y_n)$*. Then* $D_r|\mathbf{u}(t) - \tilde{\mathbf{u}}(t)|$ *exists on* $c \le t < d$ *and*

$$D_r|u_i(t) - \tilde{u}_i(t)| \le -\mu_i|u_i(t) - \tilde{u}_i(t)| + |F_i(\mathbf{u}_t) - F_i(\tilde{\mathbf{u}}_t)|.$$

Proof. Via a similar argument as Lemma I.6.1 in [23], $D_r|u_i(t) - \tilde{u}_i(t)|$ exists and

$$\begin{aligned}
& D_r|u_i(t) - \tilde{u}_i(t)| \\
= & \lim_{h\to 0^+} \frac{|u_i(t) - \tilde{u}_i(t) + h[-\mu_i u_i(t) + F_i(\mathbf{u}(t)) + \mu_i\tilde{u}_i(t) - F_i(\tilde{\mathbf{u}}(t))]| - |u_i(t) - \tilde{u}_i(t)|}{h}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& D_r|u_i(t) - \tilde{u}_i(t)| \\
\le & \lim_{h\to 0^+} \frac{|(1 - h\mu_i)[u_i(t) - \tilde{u}_i(t)]| + |h[F_i(\mathbf{u}_t) - F_i(\tilde{\mathbf{u}}_t)]| - |u_i(t) - \tilde{u}_i(t)|}{h} \\
= & \lim_{h\to 0^+} \frac{(1 - h\mu_i)|u_i(t) - \tilde{u}_i(t)| + |h[F_i(\mathbf{u}_t) - F_i(\tilde{\mathbf{u}}_t)]| - |u_i(t) - \tilde{u}_i(t)|}{h} \\
= & -\mu_i|u_i(t) - \tilde{u}_i(t)| + |F_i(\mathbf{u}_t) - F_i(\tilde{\mathbf{u}}_t)|.
\end{aligned}$$

The assertion is justified. □

Theorem 4.4.6. *Assume that conditions* $(\mathrm{H}_1^{\mathcal{A}})$, (H_2), *and* (H_3') *hold and* $\beta_{ii} > 0$, *then there exist* 2^n *exponentially stable equilibria for DRNN* (1.1) *with activation functions of class* $\mathcal{A}$*.*

Proof. Consider an equilibrium $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \mathbf{\Lambda}^{\mathbf{w}}$, for some $\mathbf{w} = (w_1, w_2, \cdots, w_n)$, with w_i = "l" or "r". For a fixed i, we consider the single-variable continuous function $L_i(\cdot)$, defined by

$$L_i(\zeta) = \mu_i - \zeta - \eta_i\sum_{j=1}^{n}|\alpha_{ji}| - \eta_i\sum_{j=1}^{n}|\beta_{ji}|e^{\zeta\tau_{ji}}.$$

Then, $L_i(0) > 0$ from (H_3'). Moreover, there exists a constant $\lambda > 0$ such that $L_i(\lambda) > 0$, for all $i = 1, 2, \cdots, n$, due to continuity of L_i. Let $\mathbf{x}(t) = \mathbf{x}(t; \phi)$ be the solution to (1.1) with initial condition $\phi \in \mathbf{\Lambda}^{\mathbf{w}}$. From (1.1) and Lemma 4.4.5, we obtain

$$\begin{aligned} D_r|x_i(t) - \bar{x}_i| \le -\mu_i|x_i(t) - \bar{x}_i| \ &+ \ \sum_{j=1}^{n} \eta_j|\alpha_{ij}||x_j(t) - \bar{x}_j| \\ &+ \ \sum_{j=1}^{n} \eta_j|\beta_{ij}||x_j(t - \tau_{ij}) - \bar{x}_j|, \end{aligned} \tag{4.23}$$

for all $t > 0$. Define functions

$$z_i(t) = e^{\lambda t}|x_i(t) - \bar{x}_i|, \ t \in [-\tau, \infty), \ i = 1, 2, \cdots, n. \tag{4.24}$$

Then, by (4.23) and (4.24) we have

$$D_r z_i(t) \le -(\mu_i - \lambda)z_i(t) + \sum_{j=1}^{n} \eta_j|\alpha_{ij}|z_j(t) + \sum_{j=1}^{n} \eta_j|\beta_{ij}|e^{\lambda\tau_{ij}} z_j(t - \tau_{ij}), \tag{4.25}$$

for all $t > 0$, $i = 1, 2, \cdots, n$. Next, we define a Lyapunov functional V as follows:

$$V(\mathbf{z}_t) = \sum_{i=1}^{n} \left(z_i(t) + \sum_{j=1}^{n} \eta_j|\beta_{ij}|e^{\lambda\tau_{ij}} \int_{t-\tau_{ij}}^{t} z_j(s)ds \right).$$

Then, by (4.25) and $L_i(\lambda) > 0$, we derive

$$\begin{aligned} D_r V(t) \ &\le \ \sum_{i=1}^{n}[-(\mu_i - \lambda)z_i(t) + \sum_{j=1}^{n} \eta_j|\alpha_{ij}|z_j(t) + \sum_{j=1}^{n} \eta_j|\beta_{ij}|e^{\lambda\tau_{ij}} z_j(t - \tau_{ij}) \\ &\quad + \sum_{j=1}^{n} \eta_j|\beta_{ij}|e^{\lambda\tau_{ij}} z_j(t) - \sum_{j=1}^{n} \eta_j|\beta_{ij}|e^{\lambda\tau_{ij}} z_j(t - \tau_{ij})] \\ &= \ \sum_{i=1}^{n}[-(\mu_i - \lambda)z_i(t) + \sum_{j=1}^{n} \eta_j|\alpha_{ij}|z_j(t) + \sum_{j=1}^{n} \eta_j|\beta_{ij}|e^{\lambda\tau_{ij}} z_j(t)] \\ &= \ -\sum_{i=1}^{n}[\mu_i - \lambda - \eta_i \sum_{j=1}^{n} |\alpha_{ji}| - \eta_i \sum_{j=1}^{n} |\beta_{ji}|e^{\lambda\tau_{ji}}]z_i(t) \\ &< \ 0, \end{aligned} \tag{4.26}$$

for all $t > 0$. Since for given initial condition $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, $V(t)$ is continuous in t, (4.26) implies $V(t) \leq V(0)$ for all $t > 0$. Consequently, we obtain

$$\begin{aligned}\sum_{i=1}^{n} z_i(t) &\leq V(t) \leq V(0) = \sum_{i=1}^{n}[(z_i(0) + \sum_{j=1}^{n} \eta_j |\beta_{ij}| e^{\lambda\tau_{ij}} \int_{-\tau_{ij}}^{0} z_j(s)ds] \\ &= \sum_{i=1}^{n}[z_i(0) + \eta_i \sum_{j=1}^{n} |\beta_{ji}| e^{\lambda\tau_{ji}} \int_{-\tau_{ji}}^{0} z_i(s)ds],\end{aligned}$$

for all $t > 0$. From (4.24), we derive

$$\sum_{i=1}^{n} |x_i(t) - \bar{x}_i| \leq e^{-\lambda t} \sum_{i=1}^{n}(1 + \eta_i \sum_{j=1}^{n} |\beta_{ji}| e^{\lambda\tau_{ji}} \tau_{ji})(\sup_{s\in[-\tau,0]} |x_i(s) - \bar{x}_i|).$$

Therefore, $\mathbf{x}(t)$ converges to $\bar{\mathbf{x}}$ exponentially. This completes the proof. □

Remark. (i) Theorem 4.4.6 indicates that there exists an unique equilibrium in each of the 2^n regions $\mathbf{\Lambda}^{\mathbf{w}}$, $\mathbf{w} = (w_1, w_2, \cdots, w_n)$, w_i = "l" or "r", under respective conditions.

(ii) The basin of the equilibrium in each $\mathbf{\Lambda}^{\mathbf{w}}$ could be proved to be as large as the positively invariant region by Theorem 4.4.6, although we are uncertain of the exponential stability.

4.5 Numerical Illustrations

In this section, two two-dimensional examples are presented to illustrate our theory. In particular, Example 4.5.2 demonstrates the multistability of system (1.1) with the standard activation function (4.8). This example adopts parameters satisfying the criteria in our theory but not the one in [56].

Example 4.5.1. Consider the following system with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$, which belongs to class $\mathcal{A}$:

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -x_1(t) + 4g_1(x_1(t)) + g_2(x_2(t)) + 3g_1(x_1(t-10)) + g_2(x_2(t-10)) \\ \frac{dx_2(t)}{dt} &= -3x_2(t) + 2g_1(x_1(t)) + 7g_2(x_2(t)) + g_1(x_1(t-10)) + 5g_2(x_2(t-10)).\end{aligned}$$

Direct computation gives $\hat{f}_1(x_1) = -x_1+7g(x_1)+2$, $\check{f}_1(x_1) = -x_1+7g(x_1)-2$, $\hat{f}_2(x_2) = -3x_2 + 12g(x_2) + 3$, $\check{f}_2(x_2) = -3x_2 + 12g(x_2) - 3$. Herein, the parameters satisfy our conditions in Theorem 4.4.2:

$$
\begin{aligned}
\text{Condition } (\mathrm{H}_1^A): &\quad 0 < \mu_1/(\alpha_{11} + \beta_{11}) = 1/7 < 1, 0 < \mu_2/(\alpha_{22} + \beta_{22}) = 3/12 < 1. \\
\text{Condition } (\mathrm{H}_2): &\quad \hat{f}_1(p_1) = -2.8524 < 0, \check{f}_1(q_1) = 2.8524 > 0, \\
&\quad \hat{f}_2(p_2) = -3.4414 < 0, \check{f}_2(q_2) = 3.4414 > 0. \\
\text{Condition } (\mathrm{H}_3): &\quad \mu_1 = 1 > 0.98 = (|\alpha_{11}| + |\beta_{11}|)\eta_1 + (|\alpha_{12}| + |\beta_{12}|)\eta_2, \\
&\quad \mu_2 = 3 > 1.98 = (|\alpha_{21}| + |\beta_{21}|)\eta_1 + (|\alpha_{22}| + |\beta_{22}|)\eta_2,
\end{aligned}
$$

where $\eta_1 = 0.1$ and $\eta_2 = 0.14$ are chosen in (H_3) and the other related numbers are listed in Table 4.1.

$\hat{a}_1 = -4.9994$	$\underline{d}_1 = -1.8184$	$\tilde{p}_1 = -1.6283$	$\hat{b}_1 = -0.3491$	$\tilde{q}_1$=1.6283	$\overline{d}_1 = 1.8184$	$\hat{c}_1$=9.0000
$\check{a}_1 = -9.0000$		$\check{b}_1$=0.3491		$\check{c}_1$=4.9993		
$\hat{a}_2 = -2.9793$	$\underline{d}_2 = -1.6392$	$\tilde{p}_2 = -1.3170$	$\hat{b}_2 = -0.3518$	$\tilde{q}_2$=1.3170	$\overline{d}_2 = 1.6392$	$\hat{c}_2$=4.9996
$\check{a}_2 = -4.9996$		$\check{b}_2$=0.3518		$\check{c}_2$=2.9793		

Table 4.1: Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$.

The dynamics of this system are illustrated in Figure 4.5, where evolutions of 72 initial conditions have been tracked. The constant initial conditions are plotted in red color, and the time-dependent initial conditions are plotted in purple. There are four exponentially stable equilibria in the system, as confirmed by our theory. The simulation demonstrates convergence to these four equilibria from initial functions ϕ lying in the respective basin for the equilibrium.

Example 4.5.2. Consider the following system with the standard activation function (4.8):

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= -x_1(t) + 2g_1(x_1(t)) + g_2(x_2(t)) + 3g_1(x_1(t-5)) + g_2(x_2(t-5)) \\
\frac{dx_2(t)}{dt} &= -x_2(t) - g_1(x_1(t)) + 4g_2(x_2(t)) + 2g_1(x_1(t-5)) + 5g_2(x_2(t-5)) + 1,
\end{aligned}
$$

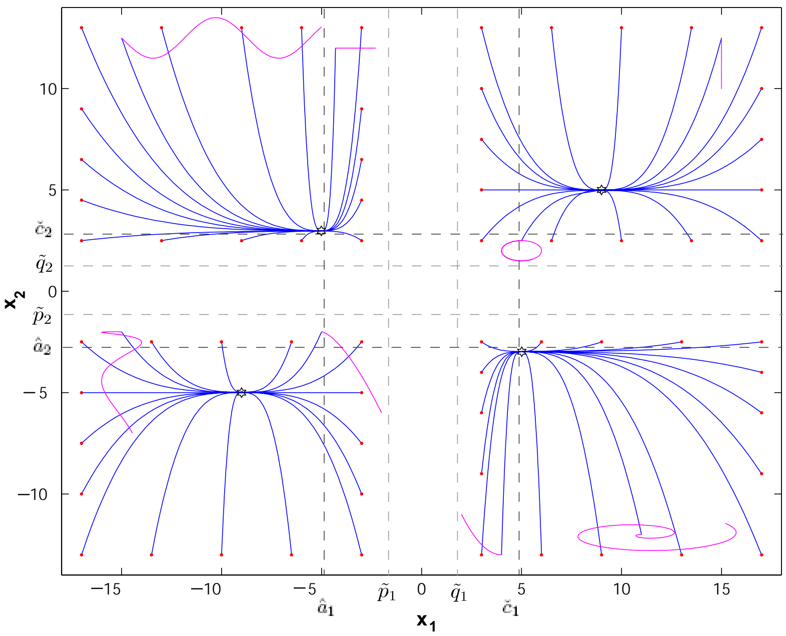

Figure 4.5: Illustration for the dynamics in Example 4.5.1.

where $g_1(\xi) = g_2(\xi) = \overline{g}(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|)$. The parameters satisfy the criterion in Corollary 4.4.4:

$$\alpha_{11} + \beta_{11} - (|\alpha_{12}| + |\beta_{12}|) - |I_1| = 3 > 1 = \mu_1,$$

$$\alpha_{22} + \beta_{22} - (|\alpha_{21}| + |\beta_{21}|) - |I_2| = 5 > 1 = \mu_2.$$

Therefore, there exist 2^n exponentially stable equilibria. The parameters herein do not satisfy the criterion (4.22) for the theory in [56]: $\alpha_{11} - |\alpha_{12}| - (|\beta_{11}| + |\beta_{12}|) - |I_1| = -3 < 1 = \mu_1$. The dynamic of the system is illustrated in Figure 4.6.

4.6 Extending Basins of Attraction

In the previous section, the basins of attraction of stationary solutions for DRNN (1.1) were derived from a criteria related to the slope of the activation functions. The ranges of the basins depend on the parameters therein. As mentioned in the previous section, the basin of attraction of each equilibrium confirmed in Theorem 4.4.2 and 4.4.6 is not

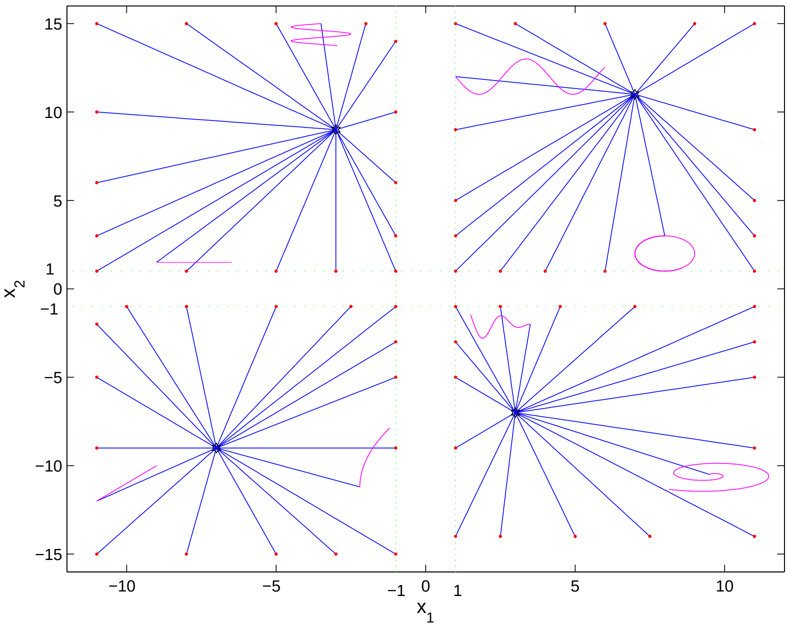

Figure 4.6: Illustration for the dynamics in Example 4.5.2 ($\tilde{p}_i = -1,\ \tilde{q}_i = 1$).

as large as the positive invariant regions. This section is dedicated to extending basins of attraction of 2^n stable stationary solutions to their confirmed positively invariant regions.

Theorem 4.6.1. *Suppose that* $(\mathrm{H}_1^{\mathcal{A}})$, (H_2), (H_3) *hold and* $\beta_{ii} > 0$ *for all* i*, then each* $\tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ *is the basin of the equilibrium therein of system* (1.1) *with activation functions of classes* $\mathcal{A}$*. Same assertion holds for activation functions of class* $\mathcal{B}$*, under conditions* $(\mathrm{H}_1^{\mathcal{B}})$, (H_2)

Proof. We only prove the case of class $\mathcal{A}$. For a fixed $\mathbf{w}$, let $\phi \in \tilde{\mathbf{\Lambda}}^{\mathbf{w}}$ be an initial condition. Consider any neuron, say, the i-th one, in the case $w_i =$ "r" (the argument also works in the case $w_i =$ "l"). By the result in Theorem 4.4.2, it is sufficient to prove it for $\hat{b}_i < \min_{\theta\in[-\tau,0]} \phi_i(\theta) \leq \overline{d}_i$. Since $\phi_i(\cdot)$ is a continuous function, there exists a positive constant η_1 such that $\phi_i(\theta) \geq \check{b}_i + \eta_1$ for all $\theta \in [-\tau, 0]$. Define

$$t_1 := \max\{t | \phi_i(t) = \check{b}_i + \eta_1,\ t \in [-\tau, 0]\}.$$

We claim that the state x_i with initial condition ϕ will run into Λ_i^{r} in finite time and justify it in four steps.

Claim 1: If $t_1 = 0$, then $\frac{dx_i}{dt}(t_1) > 0$.
Since $\beta_{ii} > 0$ and the activation function g_i is increasing, $\beta_{ii} g_i(\check{b}_i + \eta_1) \le \beta_{ii} g_i(x_i(t_1 - \tau_{ii}))$. Hence,

$$\begin{aligned}
\frac{dx_i}{dt}(t_1) &= -\mu_i x_i(t_1) + \alpha_{ii} g_i(x_i(t_1)) + \beta_{ii} g_i(x_i(t_1 - \tau_{ii})) \\
&\quad + \sum_{j=1, j\neq i}^{n} \alpha_{ij} g_j(x_j(t_1)) + \sum_{j=1, j\neq i}^{n} \beta_{ij} g_j(x_j(t_1 - \tau_{ij})) + I_i \\
&\ge -\mu_i(\check{b}_i + \eta_1) + \alpha_{ii} g_i(\check{b}_i + \eta_1) + \beta_{ii} g_i(\check{b}_i + \eta_1) - \sum_{j=1, j\neq i}^{n} \rho_j(|\alpha_{ij} + \beta_{ij}|) + I_i \\
&> 0.
\end{aligned}$$

Claim 2: There does not exist any $t > t_1$ such that $x_i(t) = \check{b}_i + \eta_1$.
If there exists the first time $t_2 > t_1$ such that $x_i(t_2) = \check{b}_i + \eta_1$, we have $\frac{dx_i}{dt}(t_2) \le 0$. This contradicts the fact that

$$\frac{dx_i}{dt}(t_2) \ge \check{f}_i(\check{b}_i + \eta_1) > 0.$$

The first inequality could be proved as previous step. Hence, $x_i(t) > \check{b}_i + \eta_1$, for all $t > t_1$.

Claim 3: There exists a positive constant η_2, depends on ϕ, such that $x_i(t) \ge \check{b}_i + \eta_1 + \eta_2$, for all $t \ge \tau$.
Since the state $x_i(t)$ is a continuous function, the minimum on the compact set $[\tau, 2\tau]$ exists. By Claim 2, the minimum is greater than $\check{b}_i + \eta_1$. We denote

$$\eta_2 := \left(\min_{t \in [\tau, 2\tau]} x_i(t) \right) - (\check{b}_i + \eta_1).$$

Then η_2 is positive. As in Claim 2, we could show that $x_i(t) \ge \check{b}_i + \eta_1 + \eta_2$ for all $t \ge \tau$.

Claim 4: There exists a finite $T_i > 0$ such that $x_i(t) > \overline{d}_i$ for all $t \ge T_i$.
Suppose there exists a finite $T_i > 0$ such that $x_i(t) > \overline{d}_i$ for $T_i - \tau \le t \le T_i$. Because $\check{f}_i(\overline{d}_i) > 0$, we could obtain $x_i(t) > \overline{d}_i$ for all $t \ge T_i$ by the argument in Claim 1 and 2. On the other hand, suppose there does not exist any finite $T_i > 0$ such that $x_i(t) > \overline{d}_i$

for $T_i - \tau \le t \le T_i$. Since $x_i(t) \ge \check{b}_i + \eta_1 + \eta_2$ for $\tau \le t \le 2\tau$, there exists a finite constant $\eta_3 > 0$, justified as in Claim 3, such that $x_i(t) \ge \check{b}_i + \eta_1 + \eta_2 + \eta_3$ for all $t \ge 3\tau$. Under the hypotheses of nonexistence of finite $T_i > 0$ with $x_i(t) > \overline{d}_i$ for all $T_i - \tau \le t \le T_i$, there exists a sequence of positive constants $\{\eta_k\}_{k=1}^{\infty}$, such that

$$\begin{aligned} x_i(t) &\ge \check{b}_i + \sum_{k=1}^{N+1} \eta_k, \text{ for all } t \ge (2N-1)\tau, \\ x_i(t) &\ge \check{b}_i + \sum_{k=1}^{N} \eta_k, \text{ for all } t \ge (2N-2)\tau. \end{aligned} \tag{4.27}$$

Suppose that

$$M_i := \check{b}_i + \sum_{k=1}^{\infty} \eta_k \le \overline{d}_i. \tag{4.28}$$

Then since $\check{b}_i < M_i \le \overline{d}_i$ we have

$$\check{f}_i(M_i) = -\mu_i M_i + (\alpha_{ii} + \beta_{ii}) g_i(M_i) - \sum_{j=1, j\neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i > 0.$$

By the continuity of activation functions g_i, there exists positive constant ρ_0 such that

$$-\mu_i \xi_1 + \alpha_{ii} g_i(\xi_1) + \beta_{ii} g_i(\xi_2) - \sum_{j=1, j\neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i > 0, \tag{4.29}$$

whenever $\xi_1, \xi_2 \in [M_i - \rho_0, M_i + \rho_0]$. For this ρ_0, there exists integer N_0 such that

$$\check{b}_i + \sum_{k=1}^{N_0} \eta_k > M_i - \rho_0.$$

Then, for $t \ge (2N_0 - 3)\tau$,

$$x_i(t) \ge \check{b}_i + \sum_{k=1}^{N_0} \eta_k > M_i - \rho_0.$$

When $t \ge (2N_0 - 2)\tau$ and $x_i(t) \in [M_i - \rho_0, M_i]$,

$$\begin{aligned} \frac{dx_i}{dt}(t) &\ge -\mu_i x_i(t) + \alpha_{ii} g_i(x_i(t)) + \beta_{ii} g_i(x_i(t - \tau_{ii})) - \sum_{j=1, j\neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i \\ &\ge -\mu_i x_i(t) + \alpha_{ii} g_i(x_i(t)) + \beta_{ii} g_i(M_i - \rho_0) - \sum_{j=1, j\neq i}^{n} \rho_j(|\alpha_{ij}| + |\beta_{ij}|) + I_i \\ &=: K_0 > 0. \end{aligned}$$

The last inequality is due to (4.29). Hence, there exists a finite time T_0 such that $x_i(t) > M_i$ for all $t > T_0$, contradicting with (4.28). Thus, we have $\check{b}_i + \sum_{k=1}^{\infty} \eta_k > \overline{d}_i$, then there exists integer N_i such that $\check{b}_i + \sum_{k=1}^{N_i} \eta_k \geq \overline{d}_i$. And then we conclude that for all $t > (2N_i - 3)\tau =: T_i$,

$$x_i(t) \geq \check{b}_i + \sum_{k=1}^{N_i} \eta_k \geq \overline{d}_i.$$

It means that the state x_i with initial condition ϕ_i runs into Λ_i^{r} in finite time T_i.

Next, taking $T := \max_{1 \leq i \leq n} T_i$, we derive that the solution $\mathbf{x}(0, \phi)$ runs into the region $\mathbf{\Lambda}^{\mathbf{w}}$ in finite time T. Therefore, by Theorem 4.4.2, the solution $\mathbf{x}(0, \phi)$ approaches the equilibrium therein. The proof is completed. □

The estimates in confirming that the size of basins of attraction for the equilibria are at least as large as the established positively invariant sets are independent to the ones for deriving exponential stability for the equilibria. Although the confirmed attracting domains are extended, the rates of convergence to the equilibrium for the solutions lying in the larger and smaller regions may be different. In fact, our derivation only indicates convergence to the equilibrium for the solutions starting from the larger regions, while convergence to the equilibrium for the solutions starting from the smaller regions is of exponential rates.

4.7 Numerical Illustrations

Herein, we present a two-dimensional systems to illustrate our theory for system (1.1). Example 4.7.1 demonstrates the coincidence of positively invariant regions and basins of traction which are confirmed in this work.

Example 4.7.1. Consider the following system with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$, which belongs to class $\mathcal{A}$:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -2x_1(t) + 3g_1(x_1(t)) + g_2(x_2(t)) + 3g_1(x_1(t-10)) + g_2(x_2(t-10)) \\ \frac{dx_2(t)}{dt} &= -3x_2(t) + 2g_1(x_1(t)) + 6g_2(x_2(t)) + g_1(x_1(t-10)) + 6g_2(x_2(t-10)). \end{aligned}$$

A computation gives

$$\hat{f}_1(x_1) = -2x_1 + 6g(x_1) + 2,\ \check{f}_1(x_1) = -2x_1 + 6g(x_1) - 2,$$

$$\hat{f}_2(x_2) = -3x_2 + 12g(x_2) + 3,\ \check{f}_2(x_2) = -3x_2 + 12g(x_2) - 3.$$

Herein, the parameters satisfy our conditions in Theorem 3:

$$\begin{aligned}
\text{Condition } (\mathrm{H}_1): &\quad 0 < \mu_1/(\alpha_{11} + \beta_{11}) = 1/3 < 1, 0 < \mu_2/(\alpha_{22} + \beta_{22}) = 1/4 < 1. \\
\text{Condition } (\mathrm{H}_2): &\quad \hat{f}_1(\tilde{p}_1) = -0.6065 < 0, \check{f}_1(\tilde{q}_1) = 0.6065 > 0, \\
&\quad \hat{f}_2(\tilde{p}_2) = -3.4414 < 0, \check{f}_2(q_2) = 3.4414 > 0. \\
\text{Condition } (\mathrm{H}_3'): &\quad \mu_1 = 2 > 1.6 = (|\alpha_{11}| + |\beta_{11}|)\eta_1 + (|\alpha_{12}| + |\beta_{12}|)\eta_2, \\
&\quad \mu_2 = 3 > 2.7 = (|\alpha_{21}| + |\beta_{21}|)\eta_1 + (|\alpha_{22}| + |\beta_{22}|)\eta_2,
\end{aligned}$$

where $\eta_1 = 0.2$ and $\eta_2 = 0.18$ are chosen and the other related numbers are listed in Table 1. The dynamics of this system and the evolutions of state variables x_1,

$\hat{a}_1 = -1.8573$	$\tilde{p}_1 = -1.1462$	$\hat{b}_1 = -0.5903$	$\tilde{q}_1$=1.1462	$\hat{c}_1$=3.9980
$\check{a}_1 = -3.9980$		$\check{b}_1$=0.5903		$\check{c}_1$=1.8573
$\hat{a}_2 = -2.9794$	$\tilde{p}_2 = -1.31705$	$\hat{b}_2 = -0.3518$	$\tilde{q}_2$=1.3170	$\hat{c}_2$=4.9996
$\check{a}_2 = -4.9996$		$\check{b}_2$=0.3518		$\check{c}_2$=2.9794

Table 4.2: Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$.

x_2 are illustrated in Figure 4.7-4.9. The constant initial conditions are plotted in res color, and the time-dependent initial conditions are plotted in purple. The simulation demonstrates that the basins of attraction of each equilibrium is at least as large as the positively invariant regions.

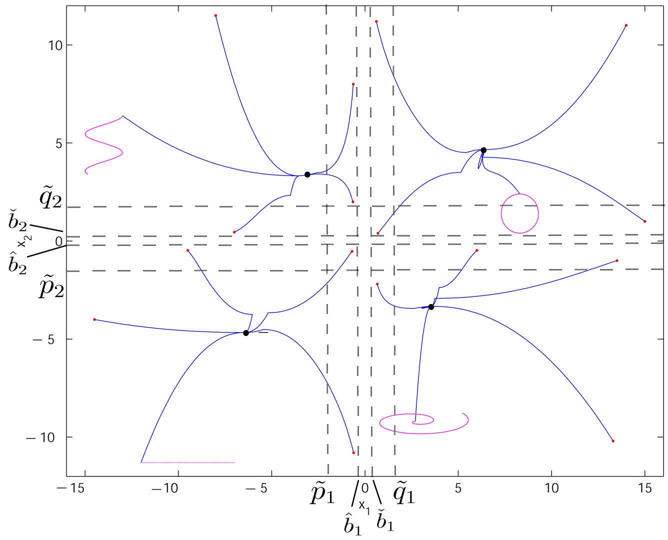

Figure 4.7: Illustrations for the dynamics in Example 4.7.1.

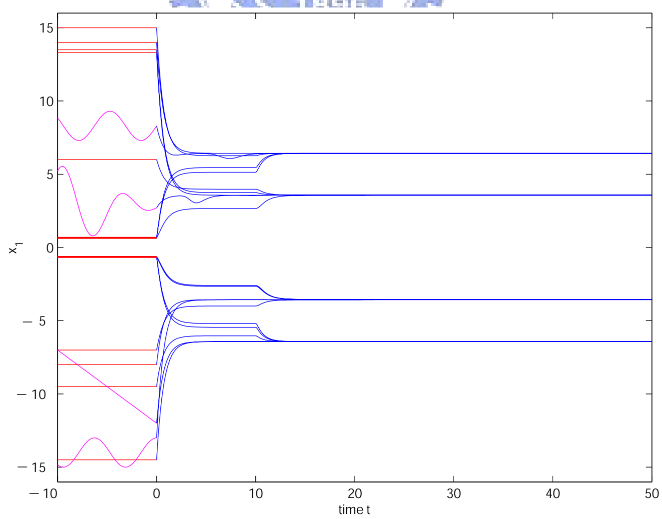

Figure 4.8: Evolution of state variable $x_1(t)$ in Example 4.7.1.

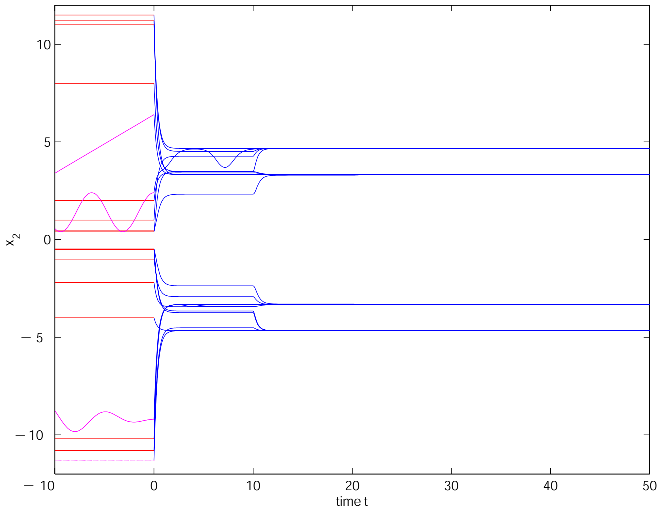

Figure 4.9: Evolution of state variable $x_2(t)$ in Example 4.7.1.

Chapter 5

Monotonicity, Convergence and Quasiconvergence in Delayed Neural Networks

Global convergence had been investigated in delayed Hopfield neural network models by employing the theory of monotone dynamical systems [52]. It was proved that under some additional conditions all solutions converge to the unique equilibrium provided that the negative delay feedback time is sufficiently small. This result was based on the globally convergent criterion in Theorem 3.4.1. Contrasting with single equilibrium, the existence of multiple equilibria has been declared in the previous chapter. The structure of global dynamics is therefore the next issue. In the remainder of this chapter, we will discuss the monotonicity and quasiconvergence in general delayed neural networks with multiple equilibria.

5.1 Quasiconvergence

Generic convergence was proposed in a class of networks with interconnection matrix satisfying "sign symmetry" and "irreducibility" properties and without delays in [26, 28]. Therein, the convergence is guaranteed for almost every trajectory in the term of Lebesgue measure zero. Relatively, in this section, we discuss the monotone dynamics for delayed system (1.1) by the theory of Smith and Thieme [48] and confirm that quasiconvergence is generic for the networks through justifying the strongly order preserving property. Explicitly, the system (1.1) possesses multiple equilibria and the set of quasiconvergent points is more than dense in the phase space $\mathcal{C}$. Let us first recall the following definition.

Definition 5.1.1. *Let $\mathcal{E}$ be the set of all equilibrium points. We say that $\phi \in \mathcal{C}$ is a quasiconvergent point, if its ω-limit set $\omega(\phi) \subset \mathcal{E}$. The set of such points is denoted by Q. A point $\phi \in \mathcal{C}$ is called a convergent point, if $\omega(\phi)$ consists of a single point of $\mathcal{E}$.*

Note that if all equilibria are isolated, then quasiconvergence yields convergence for continuous-time dynamical systems. In order to apply the theory of monotone dynamical systems, we need the following notations and definitions. Consider the standard componentwise partial order " $\leq$ " and inequality "<" on $\mathbb{R}^n$:

$$\begin{aligned} \mathbf{x} \leq \mathbf{y} \quad &\Leftrightarrow \quad x_i \leq y_i, \text{ for all } i, \\ \mathbf{x} < (\ll)\ \mathbf{y} \quad &\Leftrightarrow \quad \mathbf{x} \leq \mathbf{y} \text{ and } x_i < y_i \text{ for some (all) } i. \end{aligned}$$

Then the partial order " $\leq$ ", called the standard order, and the inequality " $<$ " on $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ are defined by

$$\begin{aligned} \phi \leq \psi \quad &\Leftrightarrow \quad \phi(\theta) \leq \psi(\theta) \text{ for } \theta \in [-\tau, 0], \\ \phi < \psi \quad &\Leftrightarrow \quad \phi \leq \psi \text{ and } \phi \neq \psi, \\ \phi \ll \psi \quad &\Leftrightarrow \quad \phi(\theta) \ll \psi(\theta) \text{ for all } \theta \in [-\tau, 0]. \end{aligned}$$

Definition 5.1.2. *(i) A semiflow Φ is said to be monotone provided $\Phi_t(\phi) \leq \Phi_t(\psi)$ whenever $\phi \leq \psi$ and $t \geq 0$. (ii) Φ is called strongly order preserving (SOP), if it is monotone and whenever $\phi < \psi$, there exist open subsets U, V of $\mathcal{C}$ with $\phi \in U$ and $\psi \in V$ and $t_0 > 0$ such that $\Phi_{t_0}(U) \leq \Phi_{t_0}(V)$.*

It has been shown in [47] that if the phase space can be approximated from below or above, then Int$Q$ is dense in $\mathcal{C}$ for a SOP system, under a compactness assumption. The conditions in this theorem can all be justified in our situations herein.

Trivially, the one-dimensional delayed equation

$$\frac{dx}{dt} = -ax(t) + bg(x(t-\tau)),\ a > 0,\ b < 0,$$

fails to be monotone under the standard ordering in $\mathcal{C}$ [47], so do the higher dimensional cases. We shall adopt a special order introduced in [48] to conclude the monotone behavior for system (1.1). Let M be an $n \times n$ essentially nonnegative matrix, which means that $M + \lambda I$ is entrywise nonnegative for all sufficiently large λ. Define

$$K_M := \{\psi \in \mathcal{C} | \psi \geq 0 \text{ and } e^{-tM}\psi(t) \geq e^{-sM}\psi(s), \text{ for } -\tau \leq s \leq t \leq 0\}. \tag{5.1}$$

Then K_M is a cone in the space $\mathcal{C}$, that is, under addition and scalar multiplication by nonnegative scalars, K_M is closed in $\mathcal{C}$ and $K_M \cap (-K_M) = \emptyset$. Moreover, K_M is a normal cone, which means that every order interval is a bounded set in $\mathcal{C}$ [1]. According to [48], K_M induces a partial order on $\mathcal{C}$.

Definition 5.1.3. *If $\phi, \psi \in \mathcal{C}$, we say $\phi \leq_M \psi$ whenever $\psi - \phi \in K_M$. We write $\phi <_M \psi$ to indicate that $\phi \leq_M \psi$ and $\phi \neq \psi$.*

Theorem 5.1.4. [48] *Consider the delayed differential equation*

$$\frac{d\mathbf{x}(t)}{dt} = F(\mathbf{x}_t), \tag{5.2}$$

where $F \in C^1(\mathcal{C}, \mathbb{R}^n)$. Then the semiflow Φ generated by (5.2) is SOP on $\mathcal{C}$ under order "$\leq_M$", if the following conditions hold :
(i) $dF(\phi)\psi - M\psi(0) \gg 0$ for every $\phi \in \mathcal{C}$ and every $\psi \in K_M$ with $\psi \gg 0$,
(ii) If $\phi \in \mathcal{C}$, $\psi \in K_M$ and J is a (nonempty) proper subset of $\{1, \cdots, n\}$ such that $\psi_j \gg 0$ for $j \in J$ and $\psi_k(0) = 0$ for $k \notin J$, then $(dF(\phi)\psi)_i > 0$, for some $i \notin J$.

Definition 5.1.5. *An $n \times n$ matrix $A = [A_{ij}]$ is called irreducible if whenever the set $\{1, 2, \cdots, n\}$ is expressed as the union of two disjoint proper subsets S, S', then for every $i \in S$ there exists j, $k \in S'$ such that $A_{ij} \neq 0$, $A_{ki} \neq 0$.*

Remark. This means that the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ does not map into itself any nonzero proper linear subspace spanned by a subset of the standard basis. Equivalently, the directed graph with vertices $1, 2, \cdots, n$ and directed edges (i, j) for $A_{ij} \neq 0$, is connected by directed paths.

Herein, we set the $n \times n$ matrix $M = \text{diag}(-\mu_1 - \nu_1, \cdots, -\mu_n - \nu_n)$, where $\nu_i > 0$ will be chosen later. Indeed, the matrix M is essentially nonnegative. Let $\gamma_i := \max_{\xi \in \mathbb{R}} g_i'(\xi)$.

Proposition 5.1.6. *Assume that one of the matrices A and B is irreducible, where $A = [\alpha_{ij}]$, $B = [\beta_{ij}]$, $\alpha_{ij} \geq 0$, $\beta_{ij} \geq 0$ for all $i \neq j$, $\alpha_{ii} + \beta_{ii} > 0$ for all i, and the time lags $\{\tau_{ij}\}$ satisfy*

$$\tau_{ii} \leq 1/(\mu_i + e|\beta_{ii}|\gamma_i), \tag{5.3}$$

for all i with $\beta_{ii} < 0$. Then the semiflow Φ generated by the solutions of (1.1) *is SOP in the order $\leq_M$.*

Proof. Recall the previous definition of F defined from (1.1):

$$F_i(\phi) = -\mu_i \phi_i(0) + \sum_{j=1}^{n} \alpha_{ij} g_j(\phi_j(0)) + \sum_{j=1}^{n} \beta_{ij} g_j(\phi_j(-\tau_{ij})) + I_i, \; i = 1, \cdots, n.$$

For any $\phi \in \mathcal{C}$ and $\psi \in K_M$, we have

$$\begin{aligned} & (dF(\phi)\psi)_i - (M\psi(0))_i \\ = \;& \nu_i \psi_i(0) + \sum_{j=1}^{n} \alpha_{ij} g_j'(\phi_j(0))\psi_j(0) + \sum_{j=1}^{n} \beta_{ij} g_j'(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}) \quad (5.4) \\ \geq \;& [(\nu_i e^{-\tau_{ii}(\mu_i+\nu_i)} + \beta_{ii} g_i'(\phi_i(-\tau_{ii}))]\psi_i(-\tau_{ii}) + \alpha_{ii} g_i'(\phi_i(0)))\psi_i(0) \\ & + \sum_{j=1, j\neq i}^{n} \alpha_{ij} g_j'(\phi_j(0))\psi_j(0) + \sum_{j=1, j\neq i}^{n} \beta_{ij} g_j'(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}), \quad (5.5) \end{aligned}$$

since $\psi_i(0) \geq e^{-\tau_{ii}(\mu_i+\nu_i)}\psi(-\tau_{ii})$, from $\psi \in K_M$, and $\psi(0) \geq e^{-sM}\psi(s)$, for all $s \in [-\tau, 0]$. Here, we take $\nu_i > 0$ satisfying $\nu_i = e|\beta_{ii}|\gamma_i$. If $\beta_{ii} < 0$, then $\alpha_{ii} > 0$, and the assumption $\tau_{ii} \leq 1/(\mu_i + e|\beta_{ii}|\gamma_i)$ yields $\nu_i \exp[-\tau_{ii}(\mu_i + \nu_i)] + \beta_{ii} g_i'(\phi_i(-\tau_{ii})) > 0$. Thus $(dF(\phi)\psi)_i - (M\psi(0))_i > 0$, from (5.5). When $\beta_{ii} \geq 0$, $(dF(\phi)\psi)_i - (M\psi(0))_i > 0$ follows from $\nu_i + \alpha_{ii}\gamma_i > 0$ and (5.4). Next, we will prove that condition (ii) in Theorem 5.1.4 holds. For any $\phi \in \mathcal{C}$ and $\psi \in K_M$, let J be a (nonempty) proper subset of $\{1, \cdots, n\}$ such that $\psi_j \gg 0$ for $j \in J$ and $\psi_k(0) = 0$ for $k \notin J$. Then $\psi_i(-\tau_{ii}) = 0$ for each $i \notin J$, due to $\psi_i(-\tau_{ii}) \le \exp[d_i\tau_{ii}]\psi_i(0)$. Since one of the matrices A and B is irreducible, there is some $i \notin J$ such that

$$\begin{aligned}
(\mathrm{d}F(\phi)\psi)_i &= -\mu_i\psi_i(0) + \sum_{j=1}^{n}\alpha_{ij}g_j^{'}(\phi_j(0))\psi_j(0) + \sum_{j=1}^{n}\beta_{ij}g_j^{'}(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}) \\
&= \sum_{j=1,j\neq i}^{n}\alpha_{ij}g_j^{'}(\phi_j(0))\psi_j(0) + \sum_{j=1,j\neq i}^{n}\beta_{ij}g_j^{'}(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}) \\
&= \sum_{j\in J}^{n}\alpha_{ij}g_j^{'}(\phi_j(0))\psi_j(0) + \sum_{j\in J}^{n}\beta_{ij}g_j^{'}(\phi_j(-\tau_{ij}))\psi_j(-\tau_{ij}) > 0.
\end{aligned}$$

Hence, it follows from Theorem 5.1.4 that the semiflow Φ generated by the solutions of (1.1) is SOP under order "$\le_M$". $\Box$

Notably, condition (H_1) yields $\alpha_{ii} + \beta_{ii} > 0$ for all i. Thus, under conditions (H_1) and (H_2), and the assumptions in Proposition 1, there are 3^n equilibria for (1.1) and $\mathrm{int}Q$ is dense in $\mathcal{C}$. In fact, the assumptions of irreducibility of A, B and *non-inhibitory interactions*, $\alpha_{ij}, \beta_{ij} \ge 0$ for all $i \neq j$, are not necessary. We will remove these assumptions by using a *decomposition approach* in competitive-cooperative systems [53, 12].

Theorem 5.1.7. *Assume that* (H_1) *and* (H_2) *hold and the delay time* $\{\tau_{ij}\}$ *satisfy* (5.3). *Then system* (1.1) *has* 3^n *equilibria and* $\mathrm{int}Q$ *is dense in* $\mathcal{C}$.

Proof. Define matrices $A^+ = [a_{ij}^+]$, $A^- = [a_{ij}^-]$, $B^+ = [b_{ij}^+]$ and $B^- = [b_{ij}^-]$ by

$$\begin{aligned}
a_{ij}^+ &= \begin{cases} \alpha_{ii}, & \text{for } j = i \\ \alpha_{ij}^+ + s, & \text{for } j \neq i, \end{cases} \quad a_{ij}^- = \begin{cases} 0, & \text{for } j = i \\ \alpha_{ij}^- + s, & \text{for } j \neq i, \end{cases} \\
b_{ij}^+ &= \begin{cases} \beta_{ii}, & \text{for } j = i \\ \beta_{ij}^+ + s, & \text{for } j \neq i, \end{cases} \quad b_{ij}^- = \begin{cases} 0, & \text{for } j = i \\ \beta_{ij}^- + s, & \text{for } j \neq i, \end{cases}
\end{aligned}$$

where $\alpha_{ij}^+ = \max\{\alpha_{ij}, 0\}, \alpha_{ij}^- = \max\{-\alpha_{ij}, 0\}$, similarly for $\beta_{ij}^+, \beta_{ij}^-$; $s > 0$ will be chosen latter. Since $\alpha_{ij} = a^+_{ij} - a^-_{ij}$ and $\beta_{ij} = b^+_{ij} - b^-_{ij}$, (1.1) becomes

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -\mu_i x_i(t) + \sum_{j=1}^{n} a^+_{ij} g_j(x_j(t)) - \sum_{j=1}^{n} a^-_{ij} g_j(x_j(t)) \\ &\quad + \sum_{j=1}^{n} b^+_{ij} g_j(x_j(t-\tau_{ij})) - \sum_{j=1}^{n} b^-_{ij} g_j(x_j(t-\tau_{ij})) + I_i, \end{aligned} \tag{5.6}$$

$i = 1, \cdots, n$. Define $y_i = -x_i$, and set $\tilde{g}_i(\xi) := -g_i(-\xi)$, $i = 1, \cdots, n$. Then (5.6) is embedded into the following system :

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -\mu_i x_i(t) + \sum_{j=1}^{n} a^+_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} a^-_{ij} \tilde{g}_j(y_j(t)) \\ &\quad + \sum_{j=1}^{n} b^+_{ij} g_j(x_j(t-\tau_{ij})) + \sum_{j=1}^{n} b^-_{ij} \tilde{g}_j(y_j(t-\tau_{ij})) + I_i \\ \frac{dy_i(t)}{dt} &= -\mu_i y_i(t) + \sum_{j=1}^{n} a^-_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} a^+_{ij} \tilde{g}_j(y_j(t)) \\ &\quad + \sum_{j=1}^{n} b^-_{ij} g_j(x_j(t-\tau_{ij})) + \sum_{j=1}^{n} b^+_{ij} \tilde{g}_j(y_j(t-\tau_{ij})) - I_i, \end{aligned} \tag{5.7}$$

$i = 1, \cdots, n$. Note that each $\tilde{g}_i$ also admits the characteristics of g_i. We define $z_k(t)$ and $h_k(\xi)$ by $z_i(t) = x_i(t)$, $z_{n+i}(t) = y_i(t)$, and $h_i(\xi) = g_i(\xi)$, $h_{n+i}(\xi) = \tilde{g}_i(\xi)$, for $i = 1, \cdots, n$. Then (5.7) can be written as

$$\frac{dz_i(t)}{dt} = -\tilde{\mu}_i z_i(t) + \sum_{j=1}^{2n} \tilde{a}_{ij} h_j(z_j(t)) + \sum_{j=1}^{2n} \tilde{b}_{ij} h_j(z_j(t-\tilde{\tau}_{ij})) + \tilde{I}_i, \tag{5.8}$$

$i = 1, \cdots, 2n$, where the $2n \times 2n$ matrices $\tilde{A}$ and $\tilde{B}$ are defined by

$$\tilde{A} = [\tilde{a}_{ij}] := \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}, \ \tilde{B} = [\tilde{b}_{ij}] := \begin{bmatrix} B^+ & B^- \\ B^- & B^+ \end{bmatrix},$$

and $\tilde{\mu}_i$, $\tilde{I}_i$, $\tilde{\tau}_{ij}$ are given by $\tilde{\mu}_i = \mu_i$, $\tilde{\mu}_{n+i} = \mu_i$; $\tilde{I}_i = I_i$, $\tilde{I}_{n+i} = -I_i$, $i = 1, \cdots, n$; $\tilde{\tau}_{ij} = \tilde{\tau}_{n+i,j} = \tilde{\tau}_{i,n+j} = \tilde{\tau}_{n+i,n+j} = \tau_{ij}$, $i, j = 1, \cdots, 2n$.

Note that $\tilde{A}, \tilde{B}$ are both irreducible and $\tilde{a}_{ij} > 0, \tilde{b}_{ij} > 0$, for all $i \neq j$. System (5.8) thus satisfies the assumptions other than $\tilde{a}_{ii} + \tilde{b}_{ii} > 0$, for all i, in Proposition

1. It can be justified that under (H_1) and (H_2) for system (1.1), conditions analogous to (H_1) and (H_2) hold for (5.8) so that there exist 3^{2n} equilibria for (5.8). One also observes that if $x_i(0) + y_i(0) = 0$, then $x_i(t) + y_i(t) = 0$ for all t for solutions $(x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_n(t))$ of (5.8). Restated, the dynamics of (5.8) on the invariant regions $\{x_1 = y_1, \cdots, x_n = y_n\}$ are exactly the dynamics for (1.1). Thereafter, under the assumption (H_1) and (H_2), there exist 3^n equilibria for (1.1) and we could choose $s > 0$ sufficiently small as in Proposition 5.1.6 so that, the semiflow Φ generated by the solutions of (5.8) is SOP. Therefore, Int$Q$ is dense in $\mathcal{C}([-\tau, 0], \mathbb{R}^{2n})$ for system (5.8), hence Int$Q$ is dense in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ for system (1.1), if $0 \le \tau_{ii} \le 1/(b_i + e|\beta_{ii}|\gamma_i)$, for $i = 1, \cdots, n$. $\square$

5.2 Numerical Illustrations

The parameters in Example 5.2.1 satisfy conditions (H_1^*), $* = \mathcal{A}, \mathcal{B}'$, and ($H_2$), but not ($H_3$).

Example 5.2.1. Consider the following system with activation functions $g_1(\xi) = g_2(\xi) = \tanh(\xi)$, which belongs to class $\mathcal{A}$,

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -x_1(t) + 7g_1(x_1(t)) + 0.5g_2(x_2(t)) - 4g_1(x_1(t-\tau_{11})) + 0.5g_2(x_2(t-\tau_{12}))\\ \frac{dx_2(t)}{dt} &= -x_2(t) + 0.5g_1(x_1(t)) + 7g_2(x_2(t)) + 0.5g_1(x_1(t-\tau_{21})) - 4g_2(x_2(t-\tau_{22})).\end{aligned}$$

Direct computation gives

$$\begin{aligned}\hat{f}_1(x_1) &= -x_1 + 7g(x_1) + 2,\\ \check{f}_1(x_1) &= -x_1 + 7g(x_1) - 2,\\ \hat{f}_2(x_2) &= -3x_2 + 12g(x_2) + 3,\\ \check{f}_2(x_2) &= -3x_2 + 12g(x_2) - 3.\end{aligned}$$

Herein, the parameters satisfy condition ($H_1^{\mathcal{A}}$):

$$0 < \mu_1/(\alpha_{11} + \beta_{11}) = \mu_2/(\alpha_{22} + \beta_{22}) = 1/3 < 1,$$

and condition (H_2):

$$\hat{f}_1(p_1) = -2.8524 < 0, \qquad \check{f}_1(q_1) = 2.8524 > 0,$$
$$\hat{f}_2(p_2) = -3.4414 < 0, \qquad \check{f}_2(q_2) = 3.4414 > 0.$$

The other related numbers are listed in Table 5.1.

$\hat{a}_1 = -1.8572$	$\tilde{p}_1 = -1.1462$	$\hat{b}_1 = -0.5903$	$\tilde{q}_1$=1.1462	$\hat{c}_1$=3.9980
$\check{a}_1 = -3.9980$		$\check{b}_1$=0.5903		$\check{c}_1$=1.8573
$\hat{a}_2 = -1.8572$	$p_2 = -1.1462$	$\hat{b}_2 = -0.5902$	q_2=1.1462	$\hat{c}_2$=3.9980
$\check{a}_2 = -3.9980$		$\check{b}_2$=0.5902		$\check{c}_2$=1.8572

Table 5.1: Local extreme points and zeros of $\hat{f}_1, \check{f}_1, \hat{f}_2, \check{f}_2$.

Note that $g'(\xi)$ is decreasing for $\xi > 0$ and increasing for $\xi < 0$. Condition (H_3) does not hold since $\mu_1 = 1 < (|\alpha_{11}| + |\beta_{11}|)g'(\hat{a}_1) + (|\alpha_{12}| + |\beta_{12}|)g'(\hat{a}_1) \simeq 11 \times 0.0929 + 1 \times 0.0929 = 1.1148$. We choose $\tau_{11} = 0.08, \tau_{12} = 10, \tau_{21} = 10, \tau_{22} = 0.08$ to satisfy Eq (5.3): $\tau_{11} = \tau_{22} = 0.08 < 1/(1 + 4e) \simeq 0.08475$. The dynamics of this system are illustrated in Figure 5.1.

Remark. Figure 5.2 depicts the dynamics for the system with the same parameters but with time lags $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 10$, which do not satisfy criterion (5.3). It appears that two of the four equilibria become unstable. The dynamics are apparently different if we replace the activation function $\tanh(\xi)$ by the standard activation function $\overline{g}(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|)$. There still exist four stable equilibria, as illustrated in Figure 5.3.

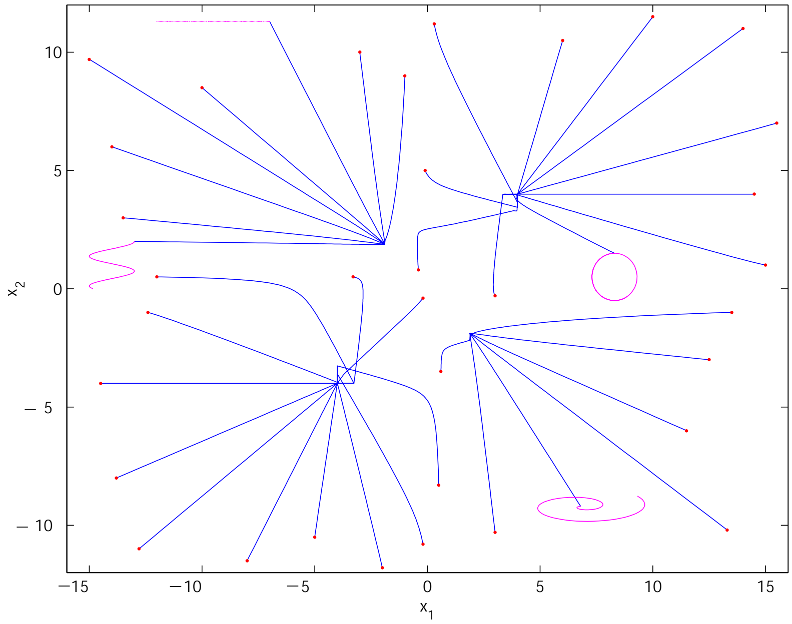

Figure 5.1: Illustration for the dynamics in Example 5.2.1 with activation function $g_i(\xi) = tanh(\xi)$ and $\tau_{11} = 0.08, \tau_{12} = 10, \tau_{21} = 10, \tau_{22} = 0.08$.

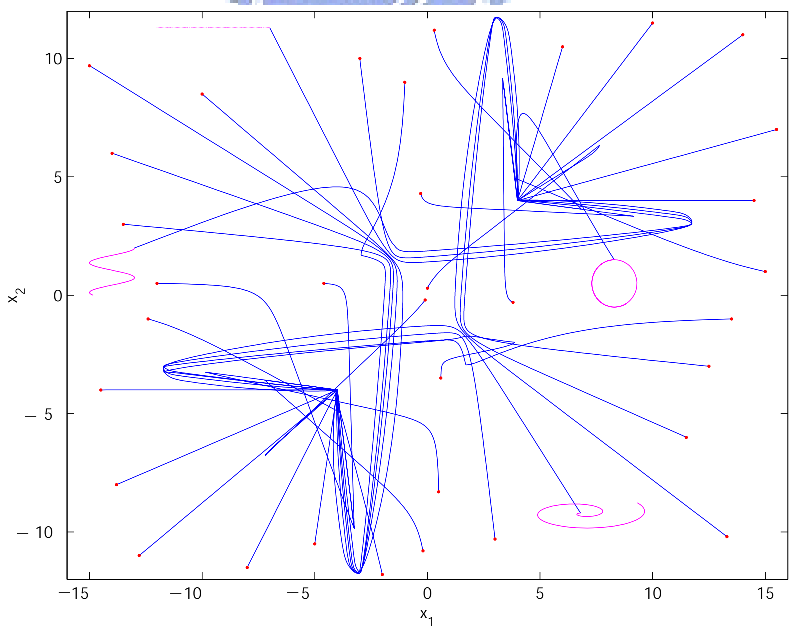

Figure 5.2: Illustration for the dynamics in Example 5.2.1 with activation function $g_i(\xi) = tanh(\xi)$ and $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 10$.

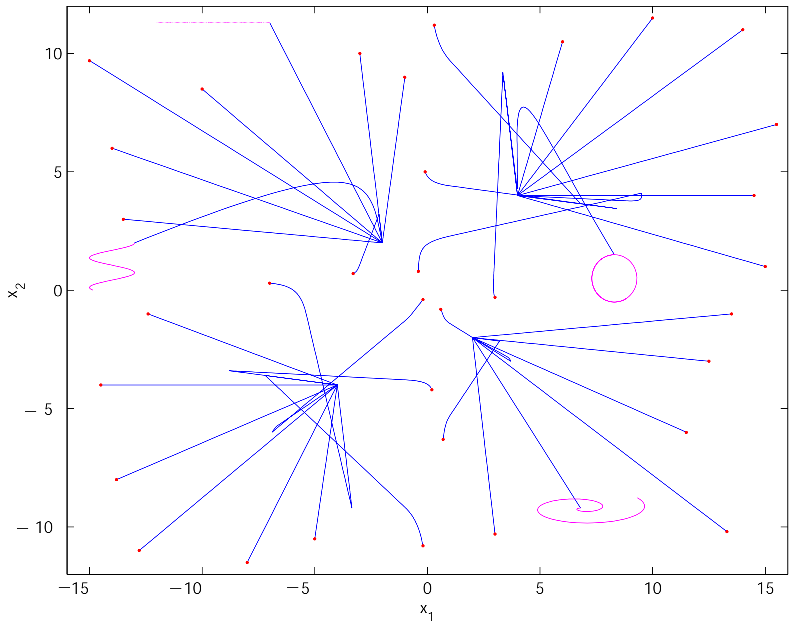

Figure 5.3: Illustration for the dynamics in Example 5.2.1 with the standard activation function $g_i(\xi) = \overline{g}(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|)$ and $\tau_{11} = \tau_{12} = \tau_{21} = \tau_{22} = 10$.

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