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摘　　要

本論文主要研究二維與三維的花樣生成問題。不論在二維或三維狀況中，適當給定局部花樣的次序可定義出次序矩陣，此次序矩陣的特殊結構可使得較大有限花樣所對應的高次次序矩陣被有系統的生成出來。給定某局部花樣子集合，由次序矩陣可定義出轉化矩陣。利用低次與高次次序矩陣的特殊關係，可得到相對應轉換矩陣的遞迴公式。空間熵的正則性是判斷包含所有可允許的全局花樣集合複雜性的重要指標，而在此論文中，空間熵可藉由一組轉換矩陣的最大特徵值計算出，一般而言，因為轉換矩陣的大小呈指數增長，使得空間熵不易準確的計算出。因此定義所謂的連接算子，並利用其來估計空間熵的下界，進而來驗證空間熵的正則性。另外，在二維情況中，可定義跡矩陣，利用其估計更好的空間熵上界。在三維情況中，將以三維細胞類神經網路為例，呈現三維花樣生成問題的應用。此博士論文所建立的理論，在研究網格動態系統及類神經網路中全局解的複雜性上有極大的幫助。

Patterns Generation Problems in Two and Three-dimensional Lattice Models

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Abstract

This dissertation investigates two and three-dimensional patterns generation problems. Both in two and three-dimensional cases, an ordering matrix for the set of all local patterns is established to derive a recursive formula for the ordering matrix for a larger finite lattice. For a given admissible set of local patterns, the transition matrix is defined and the recursive formula of high order transition matrix is presented. Then, the spatial entropy is obtained by computing the maximum eigenvalues of a sequence of transition matrices. The connecting operators are used to verify the positivity of the spatial entropy, which is important in determining the complexity of the set of admissible global patterns. Moreover, trace operator can be also introduced to give a good estimate of the upper bound of spatial entropy. In three-dimensional case, applications to three-dimensional Cellular Neural Networks is presented. The results are useful in studying a set of global stationary solutions in various Lattice Dynamical Systems and Cellular Neural Networks.

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Contents

1 Introduction

Lattices are important in scientifically modelling underlying spatial structures. Investigations in this field have covered phase transition [14], [15], [39], [40], [41], [42], [43], [50], [51], [52], [53], chemical reaction [9], [10], [27], biology [11], [12], [24], [25], [26], [36], [37], [38] and image processing and pattern recognition [19], [20], [21], [22], [23], [28], [32], [33]. In the field of lattice dynamical systems (LDS) and cellular neural networks (CNN), the complexity of the set of all global patterns recently attracted substantial interest. In particular, its spatial entropy has received considerable attention [1],[2], [3], [4], [5], [6], [7], [8], [16], [17], [18], [31], [34],[35], [44], [45], [46], [47], [48], [49].

The one dimensional spatial entropy h can be found from an associated transition matrix $\mathbb{T}$. The spatial entropy h equals to $\log\rho(\mathbb{T})$, where $\rho(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

In two-dimensional situations, higher transition matrices have been discovered in [35] and developed systematically [4] by studying the patterns generation problem.

This study extends our previous work [4]. For simplicity, two symbols on 2×2 lattice $\mathbb{Z}_{2\times 2}$ are considered. A transition matrix in the horizontal (or vertical) direction

$$\mathbb{A}_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \tag{1.1}$$

which is linked to a set of admissible local patterns on $\mathbb{Z}_{2\times 2}$ is considered, where $a_{ij}\in\{0,1\}$ for $1\leq i,j\leq 4$. The associated vertical (or horizontal) transition matrix $\mathbb{B}_2$ is given by

$$\mathbb{B}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \tag{1.2}$$

$\mathbb{A}_2$ and $\mathbb{B}_2$ are connected to each other as follows.

$$\mathbb{A}_2 = \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{21} & b_{22} \\ b_{13} & b_{14} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{41} & b_{42} \\ b_{33} & b_{34} & b_{43} & b_{44} \end{array}\right] = \begin{bmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{bmatrix}, \tag{1.3}$$

and

$$\mathbb{B}_2 = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{21} & a_{22} \\ a_{13} & a_{14} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{41} & a_{42} \\ a_{33} & a_{34} & a_{43} & a_{44} \end{array}\right] = \begin{bmatrix} B_{2;1} & B_{2;2} \\ B_{2;3} & B_{2;4} \end{bmatrix}. \tag{1.4}$$

Notably if $\mathbb{A}_2$ represents the horizontal (or vertical) transition matrix then $\mathbb{B}_2$ represents the vertical (or horizontal) transition matrix. Results that hold for $\mathbb{A}_2$ are also valid for $\mathbb{B}_2$. Therefore, for simplicity, only $\mathbb{A}_2$ is presented herein.

The recursive formulae for n-th order transition matrices $\mathbb{A}_n$ defined on $\mathbb{Z}_{2\times n}$ were obtained [4] as follows

$$\mathbb{A}_{n+1} = \begin{bmatrix} b_{11}A_{n;1} & b_{12}A_{n;2} & b_{21}A_{n;1} & b_{22}A_{n;2} \\ b_{13}A_{n;3} & b_{14}A_{n;4} & b_{23}A_{n;3} & b_{24}A_{n;4} \\ b_{31}A_{n;1} & b_{32}A_{n;2} & b_{41}A_{n;1} & b_{42}A_{n;2} \\ b_{33}A_{n;3} & b_{34}A_{n;4} & b_{43}A_{n;3} & b_{44}A_{n;4} \end{bmatrix} \tag{1.5}$$

whenever

$$\mathbb{A}_n = \begin{bmatrix} A_{n;1} & A_{n;2} \\ A_{n;3} & A_{n;4} \end{bmatrix}, \tag{1.6}$$

for $n \geq 2$, or equivalently,

$$A_{n+1;\alpha} = \begin{bmatrix} b_{\alpha 1}A_{n;1} & b_{\alpha 2}A_{n;2} \\ b_{\alpha 3}A_{n;3} & b_{\alpha 4}A_{n;4} \end{bmatrix}, \tag{1.7}$$

for $\alpha \in \{1, 2, 3, 4\}$. The number of all admissible patterns defined on $\mathbb{Z}_{m\times n}$ which can be generated from $\mathbb{A}_2$ is now defined by

$$\begin{aligned} \Gamma_{m,n}(\mathbb{A}_2) &= |\mathbb{A}_n^{m-1}| \\ &= \text{the summation of all entries in } 2^n \times 2^n \text{ matrix } \mathbb{A}_n^{m-1}. \end{aligned} \tag{1.8}$$

The spatial entropy $h(\mathbb{A}_2)$ is defined as

$$h(\mathbb{A}_2) = \lim_{m,n\to\infty} \frac{1}{mn} \log \Gamma_{m,n}(\mathbb{A}_2) = \lim_{m,n\to\infty} \frac{1}{mn} \log |\mathbb{A}_n^{m-1}|. \tag{1.9}$$

The existence of the limit (1.9) has been shown in [4], [18], [35]. When $h(\mathbb{A}_2) > 0$, the number of admissible patterns grows exponentially with the lattice size $m \times n$. In this situation, spatial chaos arises. When $h(\mathbb{A}_2) = 0$, pattern formation occurs.

To compute the double limit in (1.9), $n \geq 2$ can be fixed initially and m allowed to tend to infinite [35] and [4]; then the Perron-Frobenius theorem is applied;

$$\lim_{m\to\infty} \frac{1}{m} \log |\mathbb{A}_n^{m-1}| = \log \rho(\mathbb{A}_n), \tag{1.10}$$

which implies

$$h(\mathbb{A}_2) = \lim_{n\to\infty} \frac{1}{n} \log \rho(\mathbb{A}_n), \tag{1.11}$$

where $\rho(M)$ is the maximum eigenvalue of matrix M. $\mathbb{A}_n$ is a $2^n \times 2^n$ matrix, so computing $\rho(\mathbb{A}_n)$ is usually quite difficult when n is larger. Moreover, (1.11) does not produce any error estimation in the estimated sequence $\frac{1}{n} \log \rho(\mathbb{A}_n)$ and its limit $h(\mathbb{A}_2)$. This causes a serious problem in computing the entropy. However, for a class of $\mathbb{A}_2$, the recursive formulae for $\rho(\mathbb{A}_n)$ can be discovered, along with a limiting equation to $\rho^* = \exp(h(\mathbb{A}_2))$, as in [4].

This study takes a different approach to resolve these difficulties. Previously, the double limit (1.9) was initially examined by taking the m-limit firstly as in (1.10). Now, for each fixed $m \geq 2$, the n-limit in (1.9) is studied. Therefore, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathbb{A}_n^{m-1}| \tag{1.12}$$

is considered. Write

$$\mathbb{A}_n^m = \begin{bmatrix} A_{m,n;1} & A_{m,n;2} \\ A_{m,n;3} & A_{m,n;4} \end{bmatrix}. \tag{1.13}$$

The investigation of (1.12) would be simpler if a recursive formula such as (1.7) could be found for $A_{m,n;\alpha}$. The first task in this study is to solve this problem. For matrix multiplication, the indices of $A_{n;\alpha},\ \alpha \in \{1,2,3,4\}$ are conveniently expressed as

$$\mathbb{A}_n = \begin{bmatrix} A_{n;11} & A_{n;12} \\ A_{n;21} & A_{n;22} \end{bmatrix}. \tag{1.14}$$

Then

$$A_{m,n;\alpha} = \sum_{k=1}^{2^{m-1}} A_{m,n;\alpha}^{(k)}, \tag{1.15}$$

where

$$A_{m,n;\alpha}^{(k)} = A_{n;j_1 j_2} A_{n;j_2 j_3} \cdots A_{n;j_m j_{m+1}}, \tag{1.16}$$

$$k = 1 + \sum_{i=2}^{m} 2^{m-i}(j_i - 1), \tag{1.17}$$

and

$$\alpha = 2(j_1 - 1) + j_{m+1}. \tag{1.18}$$

$A_{m,n;\alpha}^{(k)}$ in (1.16) is called an elementary pattern of order (m,n), and is a fundamental element in constructing $A_{m,n;\alpha}$ in (1.15). Notably the elementary patterns are in lexicographic order, according to (1.17). As in [4], the following m-th order ordering matrix.

$$\mathbb{X}_{m,n} = \begin{bmatrix} X_{m,n;1} & X_{m,n;2} \\ X_{m,n;3} & X_{m,n;4} \end{bmatrix}, \tag{1.19}$$

is represented to record systematically these elementary patterns, where

$$X_{m,n;\alpha} = (A_{m,n;\alpha}^{(k)})^t_{1 \le k \le 2^{m-1}} \tag{1.20}$$

is a 2^{m-1} column vector.

The first main result of this study is to introduce the connecting operator $\mathbb{C}_m$, and to use it to derive a recursive formula like (1.7) for $A_{m,n;\alpha}^{(k)}$. Indeed,

$$\mathbb{C}_m = \begin{bmatrix} C_{m;11} & C_{m;12} & C_{m;13} & C_{m;14} \\ C_{m;21} & C_{m;22} & C_{m;23} & C_{m;24} \\ C_{m;31} & C_{m;32} & C_{m;33} & C_{m;34} \\ C_{m;41} & C_{m;42} & C_{m;43} & C_{m;44} \end{bmatrix} \tag{1.21}$$

$$= \begin{bmatrix} S_{m;11} & S_{m;12} & S_{m;21} & S_{m;22} \\ S_{m;13} & S_{m;14} & S_{m;23} & S_{m;24} \\ S_{m;31} & S_{m;32} & S_{m;41} & S_{m;42} \\ S_{m;33} & S_{m;34} & S_{m;43} & S_{m;44} \end{bmatrix}, \tag{1.22}$$

where

$$\begin{aligned} C_{m;ij} \quad = \quad & \left(\left[\begin{array}{cc} a_{i1} & a_{i2} \\ a_{i3} & a_{i4} \end{array} \right] \circ \left(\hat{\otimes} \left[\begin{array}{cc} B_{2;1} & B_{2;2} \\ B_{2;3} & B_{2;4} \end{array} \right]^{m-2} \right)_{2\times 2} \right)_{2^{m-1}\times 2^{m-1}} \\ \circ \quad & \left(E_{2^{m-2}\times 2^{m-2}} \otimes \left[\begin{array}{cc} a_{1j} & a_{2j} \\ a_{3j} & a_{4j} \end{array} \right] \right)_{2^{m-1}\times 2^{m-1}} \end{aligned} \tag{1.23}$$

is a $2^{m-1} \times 2^{m-1}$ matrix where $E_{k\times k}$ is the $k \times k$ full matrix; $\otimes$ denotes the Kronecker product, $\circ$ denotes the Hadamard product and the product $\hat{\otimes}$ which involves both the Kronecker product and the Hadamard product, as stipulated by Definition 2.1.2.

In Theorem 2.1.4, $C_{m;ij}$ is shown to be $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_mi_{m+1}}$, with $i_1 = i$ and $i_{m+1} = j$. Therefore, all admissible paths of $\mathbb{A}_2$ from i to j with length m are arranged systematically in matrix $C_{m;ij}$. Now, the recursive formula is

$$A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} \sum\limits_{l=1}^{2^{m-1}} (S_{m;\alpha 1})_{kl} A^{(l)}_{m,n;1} & \sum\limits_{l=1}^{2^{m-1}} (S_{m;\alpha 2})_{kl} A^{(l)}_{m,n;2} \\ \sum\limits_{l=1}^{2^{m-1}} (S_{m;\alpha 3})_{kl} A^{(l)}_{m,n;3} & \sum\limits_{l=1}^{2^{m-1}} (S_{m;\alpha 4})_{kl} A^{(l)}_{m,n;4} \end{bmatrix}, \tag{1.24}$$

for $m \geq 2,\ n \geq 2,\ 1 \leq k \leq 2^{m-1}$ and $1 \leq \alpha \leq 4$. (1.24) is the generalization of (1.7).

The recursive formula (1.24) immediately yields a lower bound on entropy. Indeed, for any positive integer K and diagonal periodic cycle $\beta_1\beta_2\cdots\beta_K\beta_{K+1}$, where $\beta_j \in \{1, 4\}$ and $\beta_{K+1} = \beta_1$,

$$h(\mathbb{A}_2) \geq \frac{1}{mK} \log \rho(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_{K+1}}). \tag{1.25}$$

Equation (1.25) implies $h(\mathbb{A}_2) > 0$, if a diagonal periodic cycle of $\beta_1\beta_2\cdots\beta_K\beta_1$ applies, with a maximum eigenvalue of $S_{m;\beta_1\beta_2}\cdots S_{m;\beta_K\beta_1}$ that greater than one. This method powerfully yields the positivity of spatial entropy, which is hard in examining the complexity of patterns generation problems.

However, the subadditivity of $\Gamma_{m,n}(\mathbb{A}_2)$ is known to imply

$$h(\mathbb{A}_2) \leq \frac{1}{mn} \log \Gamma_{m,n}(\mathbb{A}_2) \tag{1.26}$$

as in [18]. Consequently, (1.8), (1.10) and (1.26) indicate an upper bound of entropy as

$$h(\mathbb{A}_2) \leq \frac{1}{n} \log \rho(\mathbb{A}_n), \tag{1.27}$$

for any $n \geq 2$.

However, the Perron-Frobenius theorem also implies

$$\limsup_{m\to\infty} \frac{1}{m} \log tr(\mathbb{A}_n^{m-1}) = \log \rho(\mathbb{A}_n), \tag{1.28}$$

where $tr(M)$ denotes the trace of matrix M [29], [30]. Therefore, (1.28) implies

$$h(\mathbb{A}_2) = \limsup_{m,n\to\infty} \frac{1}{mn} \log tr(\mathbb{A}_n^{m-1}). \tag{1.29}$$

In studying the double-limit of (1.29), for each fixed $m \geq 2$, the n-limit in (1.29)

$$\limsup_{n\to\infty} \frac{1}{n} \log tr(\mathbb{A}_n^{m-1}) \tag{1.30}$$

is first considered. (1.30) can be studied by introducing the following trace operator

$$\mathbb{T}_m = \begin{bmatrix} C_{m;11} & C_{m;22} \\ C_{m;33} & C_{m;44} \end{bmatrix}. \tag{1.31}$$

Then, a recursive formula for $tr(\mathbb{A}_n^m)$ can be verified

$$tr(\mathbb{A}_n^m) = \left| \mathbb{T}_m^{n-2} \begin{pmatrix} trX_{m,2;1} \\ \\ trX_{m,2;4} \end{pmatrix} \right|, \tag{1.32}$$

for $n \geq 2$, where $tr(X_{m,n;\alpha}) = (trA_{m,n;\alpha}^{(k)})^t_{1\leq k\leq 2^{m-1}}$ and $|v| = \sum_{j=1}^{l} v_j$ for vector $v = (v_1, \cdots, v_l)^t$. Consequently, (1.29) and (1.32) yield

$$h(\mathbb{A}_2) = \limsup_{m\to\infty} \frac{1}{m} \log \rho(\mathbb{T}_m). \tag{1.33}$$

Notably, for a large class of $\mathbb{A}_2$, the limit sup in (1.28), (1.29), (1.30) and (1.33) can be replaced by limit. See section 2.2 for details.

Now, (1.33) can be applied to find the upper bounds of entropy. For example, when $\mathbb{A}_2$ is symmetric,

$$h(\mathbb{A}_2) \leq \frac{1}{2m} \log \rho(\mathbb{T}_{2m}), \tag{1.34}$$

for any $m \geq 1$. Since

$$\mathbb{T}_n \leq \mathbb{B}_n \tag{1.35}$$

can be shown for any $n \geq 2$. Generally, (1.33) and (1.34) yield better approximation than (1.11) and (1.27).

Moreover, this dissertation develops a general method to investigate three-dimensional pattern generation problems, extending other studies [4] and [5] to the three-dimensional case. It focuses on ordering matrices of patterns and on the connecting operator in the three-dimensional case. The trace operator has been described elsewhere [8]. This work is motivated by 3DCNN, so it is a major tool to study global patterns in 3DCNN.

Three-dimensional pattern generation problems are considered initially. Let $\mathcal{S}$ be a finite set of $p \geq 2$ colors, where $\mathbf{Z}^3$ denotes the integer lattice of $\mathbb{R}^3$. Denote, $U : \mathbf{Z}^3 \to \mathcal{S}$, a global pattern by $U(\alpha_1, \alpha_2, \alpha_3) = u_{\alpha_1\alpha_2\alpha_3}$. The set of all patterns with p colors in a three-dimensional lattice is expressed as $\Sigma_p^3 \equiv \mathcal{S}^{\mathbf{Z}^3} = \{U | U : \mathbf{Z}^3 \to \mathcal{S}\}$. The set of all local patterns on $\mathbf{Z}_{m_1\times m_2\times m_3}$ is denoted by

$$\Sigma_{m_1\times m_2\times m_3} \equiv \{U|_{\mathbf{Z}_{m_1\times m_2\times m_3}} | U \in \Sigma_p^3\}$$

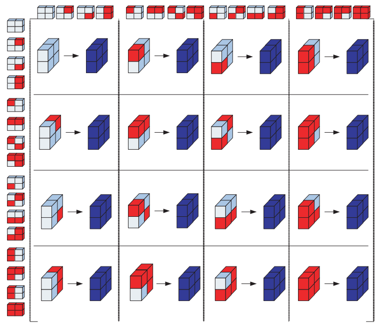
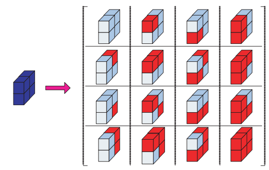
where $\mathbf{Z}_{m_1\times m_2\times m_3} = \{(\alpha_1, \alpha_2, \alpha_3)|\ 1 \leq \alpha_i \leq m_i,\ 1 \leq i \leq 3\}$ is an $m_1 \times m_2 \times m_3$ finite rectangular lattice. For simplicity, two colors on the $2\times2\times2$ lattice $\mathbf{Z}_{2\times2\times2}$ are considered here. Given a basic set $\mathcal{B} \subset \Sigma_{2\times2\times2}$, the spatial entropy can be defined as

$$h(\mathcal{B}) = \lim_{m_1,m_2,m_3\to\infty} \frac{\log \Gamma_{m_1\times m_2\times m_3}(\mathcal{B})}{m_1 m_2 m_3}, \tag{1.36}$$

where $\Gamma_{m_1\times m_2\times m_3}(\mathcal{B})$ is the number of distinct patterns in $\Sigma_{m_1\times m_2\times m_3}(\mathcal{B})$ and $\Sigma_{m_1\times m_2\times m_3}(\mathcal{B})$ is the set of all local patterns on $\mathbf{Z}_{m_1\times m_2\times m_3}$, which can be generated from $\mathcal{B}$, as described elsewhere [18]. Six different orderings

$$\begin{array}{lllllll}
[x] & : & [1] & \succ & [2] & \succ & [3] \\
[y] & : & [2] & \succ & [1] & \succ & [3] \\
[z] & : & [3] & \succ & [1] & \succ & [2] \\
[\hat{x}] & : & [1] & \succ & [3] & \succ & [2] \\
[\hat{y}] & : & [2] & \succ & [3] & \succ & [1] \\
[\hat{z}] & : & [3] & \succ & [2] & \succ & [1]
\end{array}$$

are obtained and the ordering matrix $\mathbb{W}_{2\times2\times2}$ for $\Sigma_{2\times2\times2}$ can be introduced according to the different ordering $[\omega]$. Without loss of generality, $\mathbb{X}_{2\times2\times2}$ is considered

 where  (1.37)

and the other cases are similar.

One of the main results is the construction of $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ from $\mathbb{X}_{2\times2\times2}$, where $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ represents the ordering matrix of $\Sigma_{2\times m_2\times m_3}$ according to $[\hat{x}]$-ordering. It can be addressed in the following three steps.

Step I : Apply $[x]$-ordering to $\mathbf{Z}_{1\times m_2\times 2}$

2	4	...	2k	...	$2m_2$-2	$2m_2$
1	3	...	2k-1	...	$2m_2$-3	$2m_2$-1

y →

and introduce ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$ as in Theorem 3.1.1. By Theorem 3.1.8, the transition matrix $\mathbb{A}_{x;2\times m_2\times 2}$ can be obtained from

$$\mathbb{A}_{x;2\times m_2\times 2} = (\mathbb{A}_{x;2\times(m_2-1)\times 2})_{2^{2(m_2-1)}\times 2^{2(m_2-1)}} \circ (E_{2^{2(m_2-2)}} \otimes \mathbb{A}_{x;2\times2\times2}),$$

$\otimes$ is the tensor product and $\circ$ is the Hadamard product, where E_{2^k} is the $2^k \times 2^k$ matrix with 1 as its entries, as in Eq. (3.29).

Step II : Convert $[x]$-ordering into $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times 2}$ using

z ↑

m_2+1	m_2+2	...	m_2+k	...	$2m_2$
1	2	...	k	...	m_2

and introduce the ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$ as in Theorem 3.1.4. The associated transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ is given by

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{A}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2},$$

where $\mathbb{P}_{x;2\times m_2\times 2}$ is the permutation matrix as in Theorem 3.1.10.

Step III : Define $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times m_3}$ as

z ↑

$(m_3$-1$)m_2$+1	$(m_3$-1$)m_2$+2	...	m_3m_2-1	m_3m_2
:	:	:	:	:
m_2+1	m_2+2	...	$2m_2$-1	$2m_2$
1	2	...	m_2-1	m_2

and introduce ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$ as in Theorem 3.1.5. The recursive formula for the transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be obtained by

$$\begin{aligned}\mathbb{A}_{\hat{x};2\times m_2\times m_3} &= (\mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)})_{2^{m_2(m_3-1)}\times 2^{m_2(m_3-1)}} \\ &\circ(E_{2^{m_2(m_3-2)}}\otimes \mathbb{A}_{\hat{x};2\times m_2\times 2})\end{aligned}$$

as in Theorem 3.1.11

Theorem 3.1.13 enables the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$ of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ to be computed, to yield the spatial entropy,

$$h(\mathcal{B}) = \lim_{m_2,m_3\to\infty}\frac{\log\lambda_{\hat{x};2,m_2,m_3}}{m_2m_3}.$$

However, some estimates of lower bound of spatial entropy $h(\mathcal{B})$ can be made using the connecting operator. Then, for fixed m_1, $m_2\geq 2$, the m_3-limit in Eq. (1.36) is studied:

$$\lim_{m_3\to\infty}\frac{1}{m_3}\log|\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}|. \tag{1.38}$$

The recursive formula of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ in m_3 is considered. Accordingly, the next task is to investigate Eq. (1.38). According to Eqs. (3.46) and (3.47),

$$\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3} = [A_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}},$$

$$A_{\hat{x};m_1,m_2,m_3;\alpha} = \sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$$

where $A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$ is called an elementary pattern of order (m_1,m_2,m_3) and is a fundamental element in constructing $A_{\hat{x};m_1,m_2,m_3;\alpha}$. $\mathbb{V}_{\hat{x};m_1,m_2,m_3}$ is defined as

$$\mathbb{V}_{\hat{x};m_1,m_2,m_3} = [V_{\hat{x};m_1,m_2,m_3;\alpha}],$$

$$V_{\hat{x};m_1,m_2,m_3;\alpha} = (A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha})^t$$

as in Eqs. (3.48) and (3.49), which specifies systematically these elementary patterns. The connecting operator $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ is introduced as in Definition 3.2.2, and used to derive a recursive formula for $A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}$ and $A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ as in Theorem 3.2.5

$$V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2} = S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}V_{\hat{x};m_1,m_2,m_3;\alpha_2},$$

where $\mathbb{C}_{\hat{x};m_3;m_1m_2} = \mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2}$. The recursive formula Eq. (3.60) yields a lower bound on entropy

$$\begin{aligned}&h(\mathbb{A}_{x;2\times 2\times 2}) \\ \geq& \lim_{m_2\to\infty}\frac{1}{m_1m_2P}\log\rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})\end{aligned} \tag{1.39}$$

such as in Theorem 3.2.12 and which implies $h(\mathbb{A}_{x;2\times 2\times 2})>0$ if a diagonal periodic cycle is applied with a limit in Eq. (1.39) that exceeds 0. This method powerfully yields the positivity of spatial entropy, which is useful in evaluating the complexity of patterns generation problems.

The method is very effective in elucidating the complexity of the set of mosaic patterns in 3DCNN. A typical 3DCNN is of the form

$$\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + \sum_{|\alpha|,|\beta|,|\gamma|\le 1} a_{\alpha,\beta,\gamma} f(u_{i+\alpha,j+\beta,k+\gamma}) + \sum_{|\alpha|,|\beta|,|\gamma|\le 1} b_{\alpha,\beta,\gamma} u_{i+\alpha,j+\beta,k+\gamma}, \quad (1.40)$$

where $(i,j,k) \in \mathbf{Z}^3$, $f(u)$ is a piecewise-linear output function, defined by

$$v = f(u) = \frac{1}{2}(|u+1| - |u-1|).$$

Here, $A = (a_{\alpha,\beta,\gamma})$ is a feedback template, a spatial-invariant template; $B = (b_{\alpha.\beta,\gamma})$ is a controlling template, and w is called a biased term or threshold. To elucidate the method, consider nonzero $a_{0,0,0} = a$, $a_{1,0,0} = a_x$, $a_{0,1,0} = a_y$, $a_{0,0,1} = a_z$ and zero other $a_{\alpha,\beta,\gamma}$ and $b_{\alpha,\beta,\gamma}$. Therefore, Eq. (1.40) can be rewritten as

$$\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + af(u_{i,j,k}) + a_x f(u_{i+1,j,k}) + a_y f(u_{i,j+1,k}) + a_z f(u_{i,j,k+1}). \quad (1.41)$$

The quantities $u_{i,j,k}$ represent the state of cell at (i,j,k). The stationary solution $\bar{u} = (\bar{u}_{i,j,k})$ of Eq. (1.41) satisfies

$$u_{i,j,k} = w + av_{i,j,k} + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1}, \quad (1.42)$$

where $v = f(u)$, which is very important in studying 3DCNNs: their outputs $\bar{v} = (\bar{v}_{i,j,k}) = f(\bar{u}_{i,j,k})$ are called patterns. A mosaic solution $\bar{u}$ satisfies $|\bar{u}_{i,j,k}| \ge 1$ and its corresponding pattern $\bar{v}$ is called a mosaic pattern here $|\bar{u}_{i,j,k}| \ge 1$ for all $(i,j,k) \in \mathbf{Z}^3$. Among the stationary solutions, the mosaic solutions are stable and are crucial to study the complexity of Eq. (1.41). Equation (1.42) has five parameters w, a, a_x, a_y and a_z. $a_x < a_y < a_z < 0$ and $|a_x| > |a_y| + |a_z|$ are considered to elucidate application of our work. In particular, region [4,8] in Fig. 4 in Sec. 3.3 is considered: the transition matrix can be written as

$$\mathbb{A}_{x;2\times2\times2} = G \otimes E \otimes E \otimes E,$$

where $G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Then, Steps (I), (II) and (III) yield the aforementioned admissible patterns in $\Sigma_{2\times m_2\times m_3}$; the corresponding transition matrix can be derived as in Proposition 3.1.15.

Step (I) $\Longrightarrow \mathbb{A}_{x;2\times m_2\times 2} = \otimes(G\otimes E)^{m_2-1} \otimes (\otimes E^2)$,
Step (II) $\Longrightarrow \mathbb{A}_{\hat{x};2\times m_2\times 2} = (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1})$,
Step (III) $\Longrightarrow \mathbb{A}_{\hat{x};2\times m_2\times m_3} = \otimes((\otimes G^{m_2-1})\otimes E)^{m_3-1} \otimes (\otimes E^{m_2})$.

The complexity of the 3DCNN model, as in Eq. (1.41), can be examined using the connecting operator defined in Sec. 3.2. Since the connecting operator

$$C_{z;m_1;m_2 2;11} = S_{z;m_1;m_2 2;11} = (\otimes G^{m_2-1}) \otimes E,$$

the maximum eigenvalue can be exactly computed as

$$\rho(S_{z;m_1;m_2 2;11}) = 2g^{m_2-1}$$

, where $g = \frac{1+\sqrt{5}}{2}$ is the golden-mean, as in Proposition 3.3.1. According to Eq. (1.39), the lower bound of spatial entropy in the region (VIII)-(i)-(1)-[4,8] can be estimated

$$h(\mathbb{A}_{x;2\times 2\times 2}) \geq \lim_{m_2\to\infty} \frac{1}{2m_2} \log \rho(S_{z;m_1;m_2 2;11}) = \frac{1}{2}\log g.$$

Moreover, in this case, spatial entropy can be solved exactly from the maximum eigenvalue of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$. Since

$$\rho(\mathbb{A}_{\hat{x};2\times m_2\times m_3}) = 2^{m_2+m_3-1} g^{(m_2-1)(m_3-1)},$$

the spatial entropy is

$$h(\mathbb{A}_{x;2\times 2\times 2}) = \lim_{m_2,m_3\to\infty} \frac{\rho(\mathbb{A}_{\hat{x};2\times m_2\times m_3})}{m_2 m_3} = \log g$$

as in Proposition 3.1.15.

In summary, in two-dimensional case, this study yields lower-bound estimates of entropy like (1.25) by introducing connecting operators $\mathbb{C}_m$, and upper-bound estimates of entropy like (1.34) by introducing trace operators $\mathbb{T}_m$. And in three-dimensional case, an ordering matrix for the set of all local patterns is established to derive a recursive formula for the ordering matrix for a larger finite lattice. For a given admissible set of local patterns, the transition matrix is defined and the recursive formula of high order transition matrix is presented. Then, the spatial entropy is obtained by computing the maximum eigenvalues of a sequence of transition matrices. This approach accurately and effectively yields the spatial entropy.

The rest of this dissertation is organized as follows. Section 2 derives the connecting operator $\mathbb{C}_m$ which can recursively reduce higher order elementary patterns to patterns of lower order in two-dimensional lattice models. Then, the lower-bound of spatial entropy can be found by computing the maximum eigenvalues of the diagonal periodic cycles of sequence $S_{m;\alpha\beta}$. Moreover, the trace operator $\mathbb{T}_m$ of $\mathbb{C}_m$ is addressed. The entropy can be calculated by computing the maximum eigenvalues of $\mathbb{T}_m$. When $\mathbb{A}_2$ is symmetric, the upper-bounds of entropy are also found. Finally, briefly discusses the theory for many symbols on larger lattices. In Section 3, in three-dimensional lattice models, a recursive formula for the ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$ can be derived from $\mathbb{X}_{2\times 2\times 2}$. The ordering $[x]$ is converted to $[\hat{x}]$. Then, a similar recursive formula is constructed for ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ from $\hat{\mathbb{X}}_{2\times m_2\times 2}$. Then, the recursive formula for the associated high order transition matrices $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be obtained from $\mathbb{A}_{x;2\times 2\times 2}$. Moreover, the connecting operator $\mathbb{C}_{\hat{x};m_3;m_1 m_2}$ can be defined, which can recursively reduce elementary patterns of high order to patterns of low order. Then, the lower-bound of spatial entropy is determined by computing the maximum eigenvalues of the diagonal periodic cycles of sequence $S_{\hat{x};m_3;m_1 m_2;\alpha\beta}$. Finally, an example of the application of our main results to 3DCNN is presented.

2 Two-dimensional Pattern Generation Problems

2.1 Connecting Operators

2.1.1 Connecting operators and ordering matrices

This section derives connecting operators and investigates their properties. For clarity, two symbols on 2×2 lattice $\mathbb{Z}_{2\times 2}$ are examined first. Section 2.3 addresses more general situations.

Let $\mathbb{A}_2$ and $\mathbb{B}_2$ be defined as in (1.1)∼(1.4). The column matrices $\widetilde{\mathbb{A}_2}$ and $\widetilde{\mathbb{B}_2}$ of $\mathbb{A}_2$ and $\mathbb{B}_2$ are defined by

$$\widetilde{\mathbb{A}_2} = \left[\begin{array}{cc|cc} a_{11} & a_{21} & a_{12} & a_{22} \\ a_{31} & a_{41} & a_{32} & a_{42} \\ \hline a_{13} & a_{23} & a_{14} & a_{24} \\ a_{33} & a_{43} & a_{34} & a_{44} \end{array}\right] = \begin{bmatrix} \tilde{A}_{2;1} & \tilde{A}_{2;2} \\ \tilde{A}_{2;3} & \tilde{A}_{2;4} \end{bmatrix} \tag{2.1}$$

and

$$\widetilde{\mathbb{B}_2} = \left[\begin{array}{cc|cc} b_{11} & b_{21} & b_{12} & b_{22} \\ b_{31} & b_{41} & b_{32} & b_{42} \\ \hline b_{13} & b_{23} & b_{14} & b_{24} \\ b_{33} & b_{43} & b_{34} & b_{44} \end{array}\right] = \begin{bmatrix} \tilde{B}_{2;1} & \tilde{B}_{2;2} \\ \tilde{B}_{2;3} & \tilde{B}_{2;4} \end{bmatrix} \tag{2.2}$$

, respectively.

For matrices of higher order $n \geq 2$, $\mathbb{A}_n$, $\mathbb{A}_{n+1}$ and $A_{n+1;\alpha}$ are defined as in (1.5)∼(1.7).

For matrix multiplication, the indices of $A_{n;\alpha}$ are conveniently expressed as

$$\mathbb{A}_n = \begin{bmatrix} A_{n;11} & A_{n;12} \\ A_{n;21} & A_{n;22} \end{bmatrix}. \tag{2.3}$$

Clearly, $A_{n;\alpha} = A_{n;j_1 j_2}$, where

$$\alpha = \alpha(j_1, j_2) = 2(j_1 - 1) + j_2. \tag{2.4}$$

For $m \geq 2$, the elementary pattern in the entries of $\mathbb{A}_n^m$ is represented by

$$A_{n;j_1 j_2} A_{n;j_2 j_3} \cdots A_{n;j_m j_{m+1}},$$

where $j_s \in \{1, 2\}$. A lexicographic order for multiple indices

$$J_{m+1} = (j_1 j_2 \cdots j_m j_{m+1})$$

is introduced, using

$$\chi(J_{m+1}) = 1 + \sum_{s=2}^{m} 2^{m-s}(j_s - 1). \tag{2.5}$$

Now,

$$A_{m,n;\alpha}^{(k)} = A_{n;j_1 j_2} A_{n;j_2 j_3} \cdots A_{n;j_m j_{m+1}}, \tag{2.6}$$

where

$$\alpha = \alpha(j_1, j_{m+1}) = 2(j_1 - 1) + j_{m+1} \tag{2.7}$$

and

$$k = \chi(J_{m+1}) \tag{2.8}$$

is given in (2.5). Notably, (2.5) and (2.8) do not involve j_{m+1} but (2.7)does.

Therefore, $\mathbb{A}_n^m$ can be expressed as

$$\mathbb{A}_n^m = \begin{bmatrix} A_{m,n;1} & A_{m,n;2} \\ A_{m,n;3} & A_{m,n;4} \end{bmatrix}, \tag{2.9}$$

where

$$A_{m,n;\alpha} = \sum_{k=1}^{2^{m-1}} A_{m,n;\alpha}^{(k)}. \tag{2.10}$$

Furthermore,

$$X_{m,n;\alpha} = (A_{m,n;\alpha}^{(k)})^t_{1\le k\le 2^{m-1}}. \tag{2.11}$$

$1 \le k \le 2^{m-1}$, $X_{m,n;\alpha}$ is a 2^{m-1} column-vector that consists of all elementary patterns in $A_{m,n;\alpha}$. The ordering matrix $\mathbb{X}_{m,n}$ of $\mathbb{A}_n^m$ is now defined by

$$\mathbb{X}_{m,n} = \begin{bmatrix} X_{m,n;1} & X_{m,n;2} \\ X_{m,n;3} & X_{m,n;4} \end{bmatrix}. \tag{2.12}$$

The ordering matrix $\mathbb{X}_{m,n}$ allows the elementary patterns to be tracked during the reduction from $\mathbb{A}_{n+1}^m$ to $\mathbb{A}_n^m$. This careful book-keeping provides a systematic way to generate the admissible patterns and later, lower-bound estimates of spatial entropy.

The following simplest example is studied first to illustrate the above concept.

Example 2.1.1. For $m = 2$, the following can easily be verified;

$$\mathbb{A}_n^2 = \begin{bmatrix} A_{n;11}^2 + A_{n;12}A_{n;21} & A_{n;11}A_{n;12} + A_{n;12}A_{n;22} \\ A_{n;21}A_{n;11} + A_{n;22}A_{n;21} & A_{n;21}A_{n;12} + A_{n;22}^2 \end{bmatrix}, \tag{2.13}$$

and

$$\left.\begin{array}{ll} A_{2,n;1}^{(1)} = A_{n;11}^2, & A_{2,n;1}^{(2)} = A_{n;12}A_{n;21}, \\ A_{2,n;2}^{(1)} = A_{n;11}A_{n;12}, & A_{2,n;2}^{(2)} = A_{n;12}A_{n;22}, \\ A_{2,n;3}^{(1)} = A_{n;21}A_{n;11}, & A_{2,n;3}^{(2)} = A_{n;22}A_{n;21}, \\ A_{2,n;4}^{(1)} = A_{n;21}A_{n;12}, & A_{2,n;4}^{(2)} = A_{n;22}^2. \end{array}\right\}. \tag{2.14}$$

Therefore,

$$\left.\begin{array}{l} X_{2,n;1} = \begin{bmatrix} A_{n;11}^2 \\ A_{n;12}A_{n;21} \end{bmatrix}, \quad X_{2,n;2} = \begin{bmatrix} A_{n;11}A_{n,12} \\ A_{n;12}A_{n;22} \end{bmatrix}, \\ \\ X_{2,n;3} = \begin{bmatrix} A_{n;21}A_{n;11} \\ A_{n;22}A_{n;21} \end{bmatrix}, \quad X_{2,n;4} = \begin{bmatrix} A_{n;21}A_{n,12} \\ A_{n;22}^2 \end{bmatrix}. \end{array}\right\}. \tag{2.15}$$

Applying (1.7), and by a straightforward computation,

$$X_{2,n+1;1} = \begin{bmatrix} A_{n+1;11}^2 \\ A_{n+1;12}A_{n+1;21} \end{bmatrix} \tag{2.16}$$

$$= \begin{bmatrix} \begin{bmatrix} b_{11}^2 A_{n;1}^2 + b_{12}b_{13}A_{n;2}A_{n;3} & b_{11}b_{12}A_{n;1}A_{n;2} + b_{12}b_{14}A_{n;2}A_{n;4} \\ b_{13}b_{11}A_{n;3}A_{n;1} + b_{14}b_{13}A_{n;4}A_{n;3} & b_{13}b_{12}A_{n;3}A_{n;2} + b_{14}^2 A_{n;4}^2 \end{bmatrix} \\ \\ \begin{bmatrix} b_{21}b_{31}A_{n;1}^2 + b_{22}b_{33}A_{n;2}A_{n;3} & b_{21}b_{32}A_{n;1}A_{n;2} + b_{22}b_{34}A_{n;2}A_{n;4} \\ b_{23}b_{31}A_{n;3}A_{n;1} + b_{24}b_{33}A_{n;4}A_{n;3} & b_{23}b_{32}A_{n;3}A_{n;2} + b_{24}b_{34}A_{n;4}^2 \end{bmatrix} \end{bmatrix}$$

Clearly, the $j_1 j_2$ entries of $A^2_{n+1;11}$ and $A_{n+1;12}A_{n+1;21}$ in (2.16) consist of entries of $X_{2,n;\alpha}$ in (2.14) with $\alpha = \alpha(j_1, j_2)$ in (2.4). Moreover, the terms in (2.16) can be rearranged in terms of $X_{2,n;\alpha}$ by exchanging the second row in the first matrix with the first row in the second matrix in (2.16) as follows.

$$\left[\begin{array}{cc} \begin{bmatrix} b_{11}^2 & b_{12}b_{13} \\ b_{21}b_{31} & b_{22}b_{33} \end{bmatrix}\begin{bmatrix} A^2_{n;1} \\ A_{n;2}A_{n;3} \end{bmatrix} & \begin{bmatrix} b_{11}b_{12} & b_{12}b_{14} \\ b_{21}b_{32} & b_{22}b_{34} \end{bmatrix}\begin{bmatrix} A_{n;1}A_{n;2} \\ A_{n;2}A_{n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{13}b_{11} & b_{14}b_{13} \\ b_{23}b_{31} & b_{24}b_{33} \end{bmatrix}\begin{bmatrix} A_{n;3}A_{n;1} \\ A_{n;4}A_{n;3} \end{bmatrix} & \begin{bmatrix} b_{13}b_{12} & b_{14}^2 \\ b_{23}b_{32} & b_{24}b_{34} \end{bmatrix}\begin{bmatrix} A_{n;3}A_{n;2} \\ A^2_{n;4} \end{bmatrix} \end{array}\right] \tag{2.17}$$

Applying (1.1), (1.2) and (2.1), (2.17) can be rewritten as

$$\left[\begin{array}{cc} \begin{bmatrix} a_{11}^2 & a_{12}a_{21} \\ a_{13}a_{31} & a_{14}a_{41} \end{bmatrix}\begin{bmatrix} A^2_{n;11} \\ A_{n;12}A_{n;21} \end{bmatrix} & \begin{bmatrix} a_{11}a_{12} & a_{12}a_{22} \\ a_{13}a_{32} & a_{14}a_{42} \end{bmatrix}\begin{bmatrix} A_{n;11}A_{n;12} \\ A_{n;12}A_{n;22} \end{bmatrix} \\ \\ \begin{bmatrix} a_{21}a_{11} & a_{22}a_{21} \\ a_{23}a_{31} & a_{24}a_{41} \end{bmatrix}\begin{bmatrix} A_{n;21}A_{n;11} \\ A_{n;22}A_{n;21} \end{bmatrix} & \begin{bmatrix} a_{21}a_{12} & a_{22}^2 \\ a_{23}a_{32} & a_{24}a_{42} \end{bmatrix}\begin{bmatrix} A_{n;21}A_{n;12} \\ A^2_{n;22} \end{bmatrix} \end{array}\right]$$

$$= \begin{bmatrix} (B_{2;11} \circ \tilde{A}_{2;11})X_{2,n;1} & (B_{2;11} \circ \tilde{A}_{2;12})X_{2,n;2} \\ (B_{2;12} \circ \tilde{A}_{2;11})X_{2,n;3} & (B_{2;12} \circ \tilde{A}_{2;12})X_{2,n;4} \end{bmatrix}. \tag{2.18}$$

Therefore, after the entries of $X_{2,n+1;1}$ as in (2.17) or (2.18) have been permuted, $X_{2,n+1;1}$ can be represented by a 2×2 matrix

$$\hat{X}_{2,n+1;1} \equiv \mathcal{P}(X_{2,n+1;1}) \equiv \begin{bmatrix} X_{2,n+1;1;1} & X_{2,n+1;1;2} \\ X_{2,n+1;1;3} & X_{2,n+1;1;4} \end{bmatrix}, \tag{2.19}$$

where

$$\left.\begin{array}{r} X_{2,n+1;1;1} = S_{2;11}X_{2,n;1}, \\ X_{2,n+1;1;2} = S_{2;12}X_{2,n;2}, \\ X_{2,n+1;1;3} = S_{2;13}X_{2,n;3}, \\ X_{2,n+1;1;4} = S_{2;14}X_{2,n;4} \end{array}\right\} \tag{2.20}$$

and

$$\left.\begin{array}{r} S_{2;11} = B_{2;11} \circ \tilde{A}_{2;11} \equiv C_{2;11}, \\ S_{2;12} = B_{2;11} \circ \tilde{A}_{2;12} \equiv C_{2;12}, \\ S_{2;13} = B_{2;12} \circ \tilde{A}_{2;11} \equiv C_{2;21}, \\ S_{2;14} = B_{2;12} \circ \tilde{A}_{2;12} \equiv C_{2;22}, \end{array}\right\}. \tag{2.21}$$

The above derivation indicates that $X_{2,n+1;\alpha}$ can be reduced to $X_{2,n;\beta}$ via multiplication with connecting matrices $C_{2;\alpha\beta}$. This procedure can be extended to introduce the connecting operator $\mathbb{C}_m = [\ C_{m;\alpha\beta}\ ]$, for all $m \geq 2$.

Before $\mathbb{C}_m$ is introduced, three products of matrices are defined as follows.

Definition 2.1.2. *For any two matrices* $\mathbb{M} = (M_{ij})$ *and* $\mathbb{N} = (N_{kl})$*, the Kronecker product (tensor product)* $\mathbb{M} \otimes \mathbb{N}$ *of* $\mathbb{M}$ *and* $\mathbb{N}$ *is defined by*

$$\mathbb{M} \otimes \mathbb{N} = (M_{ij}\mathbb{N}). \tag{2.22}$$

For any $n \geq 1$*,*

$$\otimes \mathbb{N}^n = \mathbb{N} \otimes \mathbb{N} \otimes \cdots \otimes \mathbb{N},$$

n-times in $\mathbb{N}$.
Next, for any two $m \times m$ *matrices*

$$\mathbb{P} = (P_{ij}) \text{ and } \mathbb{Q} = (Q_{ij})$$

where P_{ij} *and* Q_{ij} *are numbers or matrices, the Hadamard product* $\mathbb{P} \circ \mathbb{Q}$ *is defined by*

$$\mathbb{P} \circ \mathbb{Q} = (P_{ij} \cdot Q_{ij}), \tag{2.23}$$

where the product $P_{ij} \cdot Q_{ij}$ *of* P_{ij} *and* Q_{ij} *may be a multiplication between numbers, between numbers and matrices or between matrices whenever it is well-defined.*

Finally, product $\hat{\otimes}$ *is defined as follows. For any* 4×4 *matrix*

$$\mathbb{M}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{21} & m_{22} \\ m_{13} & m_{14} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{41} & m_{42} \\ m_{33} & m_{34} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} M_{2;1} & M_{2;2} \\ M_{2;3} & M_{2;4} \end{bmatrix} \tag{2.24}$$

and any 2×2 *matrix*

$$\mathbb{N} = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}, \tag{2.25}$$

where m_{ij} *are numbers and* N_k *are numbers or matrices, for* $1 \leq i, j, k \leq 4$, *define*

$$\mathbb{M}_2 \hat{\otimes} \mathbb{N} = \begin{bmatrix} m_{11}N_1 & m_{12}N_2 & m_{21}N_1 & m_{22}N_2 \\ m_{13}N_3 & m_{14}N_4 & m_{23}N_3 & m_{24}N_4 \\ m_{31}N_1 & m_{32}N_2 & m_{41}N_1 & m_{42}N_2 \\ m_{33}N_3 & m_{34}N_4 & m_{43}N_3 & m_{44}N_4 \end{bmatrix}. \tag{2.26}$$

Furthermore, for $n \geq 1$, *the* $n+1$ *th order of transition matrix of* $\mathbb{M}_2$ *is defined by*

$$\mathbb{M}_{n+1} \equiv \hat{\otimes}\mathbb{M}_2^n = \mathbb{M}_2 \hat{\otimes} \mathbb{M}_2 \hat{\otimes} \cdots \hat{\otimes} \mathbb{M}_2,$$

n-times in $\mathbb{M}_2$. *More precisely,*

$$\mathbb{M}_{n+1} = \mathbb{M}_2 \hat{\otimes} (\hat{\otimes}\mathbb{M}_2^{n-1}) = \begin{bmatrix} M_{2;1} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;2} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \\ M_{2;3} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;4} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} m_{11}M_{n;1} & m_{12}M_{n;2} & m_{21}M_{n;1} & m_{22}M_{n;2} \\ m_{13}M_{n;3} & m_{14}M_{n;4} & m_{23}M_{n;3} & m_{24}M_{n;4} \\ \hline m_{31}M_{n;1} & m_{32}M_{n;2} & m_{41}M_{n;1} & m_{42}M_{n;2} \\ m_{33}M_{n;3} & m_{34}M_{n;4} & m_{43}M_{n;3} & m_{44}M_{n;4} \end{array}\right] = \begin{bmatrix} M_{n+1;1} & M_{n+1;2} \\ M_{n+1;3} & M_{n+1;4} \end{bmatrix}, \tag{2.27}$$

where

$$\mathbb{M}_n = \hat{\otimes}\mathbb{M}_2^{n-1} = \begin{bmatrix} M_{n;1} & M_{n;2} \\ M_{n;3} & M_{n;4} \end{bmatrix}.$$

Here, the following convention is adopted,

$$\hat{\otimes}\mathbb{M}_2^0 = \mathbb{E}_{2\times 2}.$$

Definition 2.1.3. *For $m \geq 2$, define*

$$\mathbb{C}_m = \begin{bmatrix} C_{m;11} & C_{m;12} & C_{m;13} & C_{m;14} \\ C_{m;21} & C_{m;22} & C_{m;23} & C_{m;24} \\ C_{m;31} & C_{m;32} & C_{m;33} & C_{m;34} \\ C_{m;41} & C_{m;42} & C_{m;43} & C_{m;44} \end{bmatrix} = \begin{bmatrix} S_{m;11} & S_{m;12} & S_{m;21} & S_{m;22} \\ S_{m;13} & S_{m;14} & S_{m;23} & S_{m;24} \\ S_{m;31} & S_{m;32} & S_{m;41} & S_{m;42} \\ S_{m;33} & S_{m;34} & S_{m;43} & S_{m;44} \end{bmatrix}, \tag{2.28}$$

where

$$\begin{aligned} C_{m;\alpha\beta} &= \left(\begin{bmatrix} a_{\alpha 1} & a_{\alpha 2} \\ a_{\alpha 3} & a_{\alpha 4} \end{bmatrix} \circ \left(\hat{\otimes} \begin{bmatrix} B_{2;1} & B_{2;2} \\ B_{2;3} & B_{2;4} \end{bmatrix}^{m-2} \right)_{2\times 2} \right)_{2^{m-1}\times 2^{m-1}} \\ &\circ \left(E_{2^{m-2}\times 2^{m-2}} \otimes \left(\begin{bmatrix} a_{1\beta} & a_{2\beta} \\ a_{3\beta} & a_{4\beta} \end{bmatrix} \right) \right)_{2^{m-1}\times 2^{m-1}}. \end{aligned} \tag{2.29}$$

Similarly, for $\mathbb{B}_2$, define

$$\mathbb{U}_m = \begin{bmatrix} U_{m;11} & U_{m;12} & U_{m;13} & U_{m;14} \\ U_{m;21} & U_{m;22} & U_{m;23} & U_{m;24} \\ U_{m;31} & U_{m;32} & U_{m;33} & U_{m;34} \\ U_{m;41} & U_{m;42} & U_{m;43} & U_{m;44} \end{bmatrix} = \begin{bmatrix} W_{m;11} & W_{m;12} & W_{m;21} & W_{m;22} \\ W_{m;13} & W_{m;14} & W_{m;23} & W_{m;24} \\ W_{m;31} & W_{m;32} & W_{m;41} & W_{m;42} \\ W_{m;33} & W_{m;34} & W_{m;43} & W_{m;44} \end{bmatrix}, \tag{2.30}$$

where

$$\begin{aligned} U_{m;\alpha\beta} &= \left(\begin{bmatrix} b_{\alpha_1} & b_{\alpha_2} \\ b_{\alpha_3} & b_{\alpha_4} \end{bmatrix} \circ \left(\hat{\otimes} \begin{bmatrix} A_{2;1} & A_{2;2} \\ A_{2;3} & A_{2;4} \end{bmatrix}^{m-2} \right)_{2\times 2} \right)_{2^{m-1}\times 2^{m-1}} \\ &\circ \left(E_{2^{m-2}\times 2^{m-2}} \otimes \left(\begin{bmatrix} b_{1\beta} & b_{2\beta} \\ b_{3\beta} & b_{4\beta} \end{bmatrix} \right) \right)_{2^{m-1}\times 2^{m-1}}. \end{aligned} \tag{2.31}$$

$\mathbb{S}_m = [S_{m;\alpha\beta}]$ *and* $\mathbb{W}_m = [W_{m;\alpha\beta}]$.

Now $\mathbb{C}_{m+1}$ can be found from $\mathbb{C}_m$ by a recursive formula, as in (1.7).

Theorem 2.1.4. *For any $m \geq 2$ and $1 \leq \alpha, \beta \leq 4$,*

$$C_{m+1;\alpha\beta} = \begin{bmatrix} a_{\alpha_1} C_{m;1\beta} & a_{\alpha_2} C_{m;2\beta} \\ a_{\alpha_3} C_{m;3\beta} & a_{\alpha_4} C_{m;4\beta} \end{bmatrix}, \tag{2.32}$$

and

$$U_{m+1;\alpha\beta} = \begin{bmatrix} b_{\alpha_1} U_{m;1\beta} & b_{\alpha_2} U_{m;2\beta} \\ b_{\alpha_3} U_{m;3\beta} & b_{\alpha_4} U_{m;4\beta} \end{bmatrix}. \tag{2.33}$$

Proof. By (2.27),

$$\hat{\otimes}\mathbb{B}_2^{m-1} = \mathbb{B}_2 \hat{\otimes} (\hat{\otimes}\mathbb{B}_2^{m-2}) = \begin{bmatrix} B_{2;1} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) & B_{2;2} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) \\ B_{2;3} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) & B_{2;4} \circ (\hat{\otimes}\mathbb{B}_2^{m-2}) \end{bmatrix}.$$

Therefore,

$$C_{m+1;\alpha\beta} = (B_{2;\alpha} \circ (\hat{\otimes}\mathbb{B}_2^{m-1})) \circ (E_{2^{m-1}\times 2^{m-1}} \otimes \tilde{A}_{2;\beta})$$

$$= \begin{bmatrix} a_{\alpha 1}(B_{2;1} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) & a_{\alpha 2}(B_{2;2} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \\ a_{\alpha 3}(B_{2;3} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) & a_{\alpha 4}(B_{2;4} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \end{bmatrix} \circ (E_{2^{m-1}\times 2^{m-1}} \otimes \tilde{A}_{2;\beta})$$

$$= \begin{bmatrix} a_{\alpha 1}[(B_{2;1} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta})] & a_{\alpha 2}[(B_{2;2} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta}) \\ a_{\alpha 3}[(B_{2;3} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta})] & a_{\alpha 4}[(B_{2;4} \circ \hat{\otimes}\mathbb{B}_2^{m-2}) \circ (E_{2^{m-2}\times 2^{m-2}} \otimes \tilde{A}_{2;\beta})] \end{bmatrix}$$

$$= \begin{bmatrix} a_{\alpha 1} C_{m;1\beta} & a_{\alpha 2} C_{m;2\beta} \\ a_{\alpha 3} C_{m;3\beta} & a_{\alpha 4} C_{m;4\beta} \end{bmatrix}.$$

A similar result also holds for $U_{m;\alpha\beta}$; the details are omitted here. The proof is complete. $\square$

Notably, (2.32) implies $\mathbb{C}_{m;ij}$ is $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_m i_{m+1}}$ with $i_1 = i$ and $i_{m+1} = j$. $\mathbb{C}_{m;ij}$ consist of all words(or paths) of length m starting from i and ending at j. Indeed, the entries of $\mathbb{C}_m$ and $\mathbb{B}_{m+1}$ are the same. However, the arrangements are different. $\mathbb{C}_m$ can also be used to study the primitivity of $\mathbb{A}_n$, $n \geq 2$, as in [6].

That the recursive formula (1.24) holds remains to be shown. Indeed, in (2.6) substituting n for $n+1$ and using (1.7),

$$\begin{aligned} & A^{(k)}_{m,n+1;\alpha} \\ &= A_{n+1;j_1 j_2} A_{n+1;j_2 j_3} \cdots A_{n+1,j_m j_{m+1}} \\ &= \prod_{i=1}^{m} \begin{bmatrix} b_{\alpha_i 1} A_{n;11} & b_{\alpha_i 2} A_{n;12} \\ b_{\alpha_i 3} A_{n;21} & b_{\alpha_i 4} A_{n;22} \end{bmatrix} \end{aligned} \tag{2.34}$$

where $\alpha_i = \alpha(j_i, j_{i+1})$, for $1 \leq i \leq m$. After m matrix multiplications are executed in (2.34),

$$A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} A^{(k)}_{m,n+1;\alpha;1} & A^{(k)}_{m,n+1;\alpha;2} \\ A^{(k)}_{m,n+1;\alpha;3} & A^{(k)}_{m,n+1;\alpha;4} \end{bmatrix} \tag{2.35}$$

where

$$A^{(k)}_{m,n+1;\alpha;\beta} = \sum_{l=1}^{2^{m-1}} K(m;\alpha,\beta;k,l) A^{(l)}_{m,n;\beta} \tag{2.36}$$

is a linear combination of $A^{(l)}_{m,n;\beta}$ with the coefficients $K(m;\alpha,\beta;k,l)$ which are products of $b_{\alpha_l j}, 1 \leq l \leq m$. $K(m;\alpha,\beta;k,l)$ must be studied in more details.

Note that

$$\begin{aligned} \mathbb{A}^m_{n+1} &= \begin{bmatrix} A_{m,n+1;1} & A_{m,n+1;2} \\ A_{m,n+1;3} & A_{m,n+1;4} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4} \end{bmatrix} \end{aligned} \tag{2.37}$$

$$= \begin{bmatrix} \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;2} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;2} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;1;4} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;2;4} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;2} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;1} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;2} \\ \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;3;4} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;3} & \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n+1;4;4} \end{bmatrix}$$

Now, $X_{m,n+1;\alpha;\beta}$ is defined as

$$X_{m,n+1;\alpha;\beta} = (A^{(k)}_{m,n+1;\alpha;\beta})^t. \tag{2.38}$$

As in (2.17), the entries of $X_{m,n+1;\alpha}$ are rearranged into a new matrix

$$\hat{X}_{m,n+1;\alpha} \equiv \mathcal{P}(X_{m,n+1;\alpha}) \equiv \begin{bmatrix} X_{m,n+1;\alpha;1} & X_{m,n+1;\alpha;2} \\ X_{m,n+1;\alpha;3} & X_{m,n+1;\alpha;4} \end{bmatrix}. \tag{2.39}$$

From (2.36) and (2.38),

$$X_{m,n+1;\alpha;\beta} = \mathbb{K}(m;\alpha,\beta)X_{m,n;\beta} \tag{2.40}$$

where

$$\mathbb{K}(m;\alpha,\beta) = (K(m;\alpha,\beta;k,l)),\ 1 \le k,l \le 2^{m-1},$$

is a $2^{m-1} \times 2^{m-1}$ matrix. Now, $\mathbb{K}(m;\alpha,\beta) = S_{m;\alpha\beta}$ must be shown as follows.

Theorem 2.1.5. *For any $m \ge 2$ and $n \ge 2$, let $S_{m;\alpha\beta}$ be given as in (2.28) and (2.29). Then,*

$$\mathbb{K}(m;\alpha,\beta) = S_{m;\alpha\beta},$$

i.e.,

$$X_{m,n+1;\alpha;\beta} = S_{m;\alpha\beta}X_{m,n;\beta}, \tag{2.41}$$

or equivalently, the recursive formula (1.24) holds. That is,

$$A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 1})_{kl} A^{(l)}_{m,n;1} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 2})_{kl} A^{(l)}_{m,n;2} \\ \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 3})_{kl} A^{(l)}_{m,n;3} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 4})_{kl} A^{(l)}_{m,n;4} \end{bmatrix}. \tag{2.42}$$

Moreover, for $n = 1$,

$$A^{(k)}_{m,2;\alpha} = \begin{bmatrix} \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 1})_{kl} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 2})_{kl} \\ \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 3})_{kl} & \sum_{l=1}^{2^{m-1}} (S_{m;\alpha 4})_{kl} \end{bmatrix} \tag{2.43}$$

for any $1 \le k \le 2^{m-1}$ and $\alpha \in \{1,2,3,4\}$.

Proof. The result is proven by the induction on m.

When $m = 2$, and $\alpha = 1$, (2.41) was proven as in Example 2.1.1. The case with $\alpha = 2,\ 3$ and 4 can also be proved analogously; the details are omitted.

Now, (2.41) ia assumed to hold for m; the goal is to show that it also holds for $m+1$. Since

$$\mathbb{A}^{m+1}_{n+1} = \mathbb{A}_{n+1} \cdot \mathbb{A}^{m}_{n+1} = \begin{bmatrix} A_{n+1;1} & A_{n+1;2} \\ A_{n+1;3} & A_{n+1;4} \end{bmatrix} \begin{bmatrix} A_{m,n+1,1} & A_{m,n+1;2} \\ A_{m,n+1,3} & A_{m,n+1;4} \end{bmatrix},$$

(2.11) implies

$$X_{m+1,n+1;1} = \begin{bmatrix} A_{n+1;1}X_{m,n+1;1} \\ A_{n+1;2}X_{m,n+1;3} \end{bmatrix},\ X_{m+1,n+1;2} = \begin{bmatrix} A_{n+1;1}X_{m,n+1;2} \\ A_{n+1;2}X_{m,n+1;4} \end{bmatrix},$$

$$X_{m+1,n+1;3} = \begin{bmatrix} A_{n+1;3}X_{m,n+1;1} \\ A_{n+1;4}X_{m,n+1;3} \end{bmatrix},\ \text{and } X_{m+1,n+1;4} = \begin{bmatrix} A_{n+1;3}X_{m,n+1;2} \\ A_{n+1;4}X_{m,n+1;4} \end{bmatrix}.$$

For $\alpha = 1$, by induction on m,

$$(A_{n+1;1}\mathcal{P}(X_{m,n+1;1}),\ A_{n+1;2}\mathcal{P}(X_{m,n+1;3}))^t$$

$$= \begin{bmatrix} \begin{bmatrix} b_{11}A_{n;1} & b_{12}A_{n;2} \\ b_{13}A_{n;3} & b_{14}A_{n;4} \end{bmatrix} & \begin{bmatrix} S_{m;11}X_{m,n;1} & S_{m;12}X_{m,n;2} \\ S_{m;13}X_{m,n;3} & S_{m;14}X_{m,n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{21}A_{n;1} & b_{22}A_{n;2} \\ b_{23}A_{n;3} & b_{24}A_{n;4} \end{bmatrix} & \begin{bmatrix} S_{m;31}X_{m,n;1} & S_{m;32}X_{m,n;2} \\ S_{m;33}X_{m,n;3} & S_{m;34}X_{m,n;4} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} b_{11}S_{m;11}A_{n;1}X_{m,n;1} + b_{12}S_{m;13}A_{n;2}X_{m,n;3} & b_{11}S_{m;12}A_{n;1}X_{m,n;2} + b_{12}S_{m;14}A_{n;2}X_{m,n;4} \\ b_{13}S_{m;11}A_{n;3}X_{m,n;1} + b_{14}S_{m;13}A_{n;4}X_{m,n;3} & b_{13}S_{m;12}A_{n;3}X_{m,n;2} + b_{14}S_{m;14}A_{n;4}X_{m,n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{21}S_{m;31}A_{n;1}X_{m,n;1} + b_{22}S_{m;33}A_{n;2}X_{m,n;3} & b_{21}S_{m;32}A_{n;1}X_{m,n;2} + b_{22}S_{m;34}A_{n;2}X_{m,n;4} \\ b_{23}S_{m;31}A_{n;3}X_{m,n;1} + b_{24}S_{m;33}A_{n;4}X_{m,n;3} & b_{23}S_{m;32}A_{n;3}X_{m,n;2} + b_{24}S_{m;34}A_{n;4}X_{m,n;4} \end{bmatrix} \end{bmatrix}$$

Hence $X_{m+1,n+1;1}$ can be represented by a matrix

$$\hat{X}_{m+1,n+1;1} \equiv \mathcal{P}(X_{m+1,n+1;1}) \equiv \begin{bmatrix} X_{m+1,n+1;1,1} & X_{m+1,n+1;1,2} \\ X_{m+1,n+1;1,3} & X_{m+1,n+1;1,4} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} b_{11}S_{m;11} & b_{12}S_{m;13} \\ b_{21}S_{m;31} & b_{22}S_{m;33} \end{bmatrix} \begin{bmatrix} A_{n;1}X_{m,n;1} \\ A_{n;2}X_{m,n;3} \end{bmatrix} & \begin{bmatrix} b_{11}S_{m;12} & b_{12}S_{m;14} \\ b_{21}S_{m;32} & b_{22}S_{m;34} \end{bmatrix} \begin{bmatrix} A_{n;1}X_{m,n;2} \\ A_{n;2}X_{m,n;4} \end{bmatrix} \\ \\ \begin{bmatrix} b_{13}S_{m;11} & b_{14}S_{m;13} \\ b_{23}S_{m;31} & b_{24}S_{m;33} \end{bmatrix} \begin{bmatrix} A_{n;3}X_{m,n;1} \\ A_{n;4}X_{m,n;3} \end{bmatrix} & \begin{bmatrix} b_{13}S_{m;12} & b_{14}S_{m;14} \\ b_{23}S_{m;32} & b_{24}S_{m;34} \end{bmatrix} \begin{bmatrix} A_{n;3}X_{m,n;2} \\ A_{n;4}X_{m,n;4} \end{bmatrix} \end{bmatrix}$$

Once again, (1.1), (1.2) and (2.1) can be used to recast the matrix $\hat{X}_{m+1,n+1;1}$ as

$$\begin{bmatrix} \begin{bmatrix} a_{11}C_{m;11} & a_{12}C_{m;21} \\ a_{13}C_{m;31} & a_{14}C_{m;41} \end{bmatrix} X_{m+1,n;1} & \begin{bmatrix} a_{11}C_{m;12} & a_{12}C_{m;22} \\ a_{13}C_{m;32} & a_{14}C_{m;42} \end{bmatrix} X_{m+1,n;2} \\ \\ \begin{bmatrix} a_{21}C_{m;11} & a_{22}C_{m;21} \\ a_{23}C_{m;31} & a_{24}C_{m;41} \end{bmatrix} X_{m+1,n;3} & \begin{bmatrix} a_{21}C_{m;12} & a_{22}C_{m;22} \\ a_{23}C_{m;32} & a_{24}C_{m;42} \end{bmatrix} X_{m+1,n;4} \end{bmatrix}$$

According to Theorem 2.1.4, the above matrix becomes

$$= \begin{bmatrix} C_{m+1;11}X_{m+1,n;1} & C_{m+1;12}X_{m+1,n;2} \\ C_{m+1;21}X_{m+1,n;3} & C_{m+1;22}X_{m+1,n;4} \end{bmatrix} = \begin{bmatrix} S_{m+1;11}X_{m+1,n;1} & S_{m+1;12}X_{m+1,n;2} \\ S_{m+1;13}X_{m+1,n;3} & S_{m+1;14}X_{m+1,n;4} \end{bmatrix}.$$

The cases with $\alpha = 2, 3$ and 4 can also be considered analogously (2.41) follows.
Next, (2.42) follows easily from (2.35), (2.36) and (2.41).
Equation (2.43) remains to be shown. If the 2×2 matrix

$$\mathbb{A}_1 \equiv \begin{bmatrix} A_{1;11} & A_{1;12} \\ A_{1;21} & A_{1;22} \end{bmatrix} \equiv \begin{bmatrix} A_{1;1} & A_{1;2} \\ A_{1;3} & A_{1;4} \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.44}$$

is introduced, then the previous argument also hold for $n = 1$. Hence, (2.43) holds. The proof is complete. □

For any positive integer $p \geq 2$, applying Theorem 2.1.5 p times permits the elementary patterns of $\mathbb{A}^m_{n+p}$ to be expressed as the product of a sequence of $S_{m;\beta_i\beta_{i+1}}$ and the elementary patterns in $\mathbb{A}^m_n$. The elementary pattern in $\mathbb{A}^m_{n+p}$ is first studied.

For any $p \geq 2$ and $1 \leq q \leq p-1$, define

$$A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q} = \begin{bmatrix} A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;1} & A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;2} \\ A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;3} & A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_q;4} \end{bmatrix}. \tag{2.45}$$

Then

$$A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} = \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} (\prod_{i=1}^{p} K(m;\beta_{i-1},\beta_i;l_{i-1},l_i))A^{(l_p)}_{m,n;\beta_p}, \tag{2.46}$$

where $\beta_0 = \alpha$ and $l_0 = k$ can be easily verified. Therefore, for any $p \geq 1$, a generalization for (2.37) can be found for $\mathbb{A}^m_{n+p}$ as a $2^{p+1} \times 2^{p+1}$ matrix

$$\mathbb{A}^m_{n+p} = \left[A_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p}\right] \tag{2.47}$$

where

$$A_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p} = \sum_{k=1}^{2^{m-1}} A^{(k)}_{m,n;\alpha;\beta_1;\beta_2\cdots;\beta_p}. \tag{2.48}$$

In particular, if $\alpha;\beta_1,\beta_2\cdots,\beta_p \in \{1,4\}$, then $A_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p}$ lies on the diagonal of $\mathbb{A}^m_{n+p}$ in (2.47).

Now, define

$$X_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} = (A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p})^t. \tag{2.49}$$

Therefore, Theorem 2.1.5 can be generalized to

Theorem 2.1.6. *For any $m \geq 2$, $n \geq 2$ and $p \geq 1$,*

$$X_{m,n+p;\alpha;\beta_1;\beta_2\cdots;\beta_p} = S_{m;\alpha\beta_1}S_{m;\beta_1\beta_2}\cdots S_{m;\beta_{p-1}\beta_p}X_{m,n;\beta_p} \tag{2.50}$$

where $\alpha,\beta_i \in \{1,2,3,4\}$ and $1 \leq i \leq p$.

Proof. From (2.46), (2.40) and (2.42),

$$\begin{aligned}
A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} &= \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} (\prod_{i=1}^{p} K(m;\beta_{i-1},\beta_i;l_{i-1},l_i))A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} (\prod_{i=1}^{p} (S_{m;\beta_{i-1}\beta_i})_{l_{i-1}l_i})A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_1=1}^{2^{m-1}} \cdots \sum_{l_p=1}^{2^{m-1}} (S_{m;\beta_0\beta_1})_{l_0l_1}(S_{m;\beta_1\beta_2})_{l_1l_2}\cdots(S_{m;\beta_{p-1}\beta_p})_{l_{p-1}l_p}A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_p=1}^{2^{m-1}} (S_{m;\beta_0\beta_1}S_{m;\beta_1\beta_2}\cdots S_{m;\beta_{p-1}\beta_p})_{l_0l_p}A^{(l_p)}_{m,n;\beta_p} \\
&= \sum_{l_p=1}^{2^{m-1}} (S_{m;\alpha\beta_1}S_{m;\beta_1\beta_2}\cdots S_{m;\beta_{p-1}\beta_p})_{kl_p}A^{(l_p)}_{m,n;\beta_p}
\end{aligned}$$

is derived. By (2.49), then

$$\begin{aligned}
&X_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p} = (A^{(k)}_{m,n+p;\alpha;\beta_1;\beta_2;\cdots;\beta_p})^t \\
&= (\sum_{l_p=1}^{2^{m-1}} (S_{m;\alpha\beta_1} S_{m;\beta_1\beta_2} \cdots S_{m;\beta_{p-1}\beta_p})_{kl_p} A^{(l_p)}_{m,n;\beta_p})^t \\
&= S_{m;\alpha\beta_1} S_{m;\beta_1\beta_2} \cdots S_{m;\beta_{p-1}\beta_p} X_{m,n;\beta_p}.
\end{aligned}$$

The proof is complete. □

2.1.2 Lower bound of entropy

In this subsection, the connecting operator $\mathbb{C}_m$ is employed to estimate the lower bound of entropy, and in particular, to verify the positivity of entropy.

First, recall some properties of $\Gamma_{m,n}$ and spatial entropy.

$\Gamma_{m,n}$ satisfies the subadditivity in m and n:

$$\Gamma_{m_1+m_2,n} \leq \Gamma_{m_1,n}\Gamma_{m_2,n}, \tag{2.51}$$

and

$$\Gamma_{m,n_1+n_2} \leq \Gamma_{m,n_1}\Gamma_{m,n_2}, \tag{2.52}$$

or equivalently,

$$|\mathbb{A}_n^{m_1+m_2}| \leq |\mathbb{A}_n^{m_1}||\mathbb{A}_n^{m_2}| \tag{2.53}$$

and

$$|\mathbb{A}_{n_1+n_2}^{m}| \leq |\mathbb{A}_{n_1}^{m}||\mathbb{A}_{n_2}^{m}|, \tag{2.54}$$

for positive integers m, n, m_1, n_1, m_2 and n_2. Here

$$\mathbb{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.55}$$

is applied.

The subadditivity property implies

$$\limsup_{m,n\to\infty} \frac{1}{mn} \log|\mathbb{A}_n^m| \leq \frac{1}{pq} \log|\mathbb{A}_q^{p-1}| \tag{2.56}$$

for any p and $q \geq 2$. Therefore,

$$h(\mathbb{A}_2) = \lim_{m,n\to\infty} \frac{1}{mn} \log|\mathbb{A}_n^m|$$

exists, and equals

$$\inf_{p,q\geq 2} \frac{1}{pq} \log|\mathbb{A}_q^{p-1}|. \tag{2.57}$$

In particular, $h(\mathbb{A}_2)$ has an upper bound

$$h(\mathbb{A}_2) \leq \frac{1}{pq} \log|\mathbb{A}_q^{p-1}| \tag{2.58}$$

for any p and $q \geq 2$.

Similarly, when $\mathbb{A}_2$ is horizontal (or vertical) transition matrix for any $m \geq 1$ and $q \geq 2$,

$$\limsup_{n\to\infty} \frac{1}{n} \log|\mathbb{A}_n^m| \leq \frac{1}{q} \log|\mathbb{A}_q^m|. \tag{2.59}$$

Hence, the spatial entropy is $h_m(\mathbb{A}_2)$ on an infinite lattice $\mathbb{Z}_{m+1\times\infty}$ (or $\mathbb{Z}_{\infty\times m+1}$) and

$$h_m(\mathbb{A}_2) \equiv \lim_{n\to\infty} \frac{1}{n} \log|\mathbb{A}_n^m| = \inf_{q\geq 2} \frac{1}{q} \log|\mathbb{A}_q^m|. \tag{2.60}$$

For the proof of the above results, see [18].

Furthermore, by Perron-Frobenius theorem,

$$\lim_{m\to\infty}\frac{1}{m}\log|\mathbb{A}_n^m| = \log\rho(\mathbb{A}_n). \tag{2.61}$$

Therefore, for any $n \geq 2$

$$h(\mathbb{A}_2) \leq \frac{1}{n}\log\rho(\mathbb{A}_n). \tag{2.62}$$

For a proof of (2.61), see [4], [35].

The following notation is adopted.

Definition 2.1.7. *Let $X = (X_1, \cdots, X_M)^t$, where X_k are $N \times N$ matrices. Define the summation of X_k by*

$$|X| = \sum_{k=1}^{N} X_k. \tag{2.63}$$

If $\mathbb{M} = [M_{ij}]$ is a $M \times M$ matrix, then

$$|\mathbb{M}X| = \sum_{i=1}^{M}\sum_{j=1}^{M} M_{ij}X_j. \tag{2.64}$$

Note that, (2.63) implies

$$|X_{m,n;\alpha}| = \sum_{k=1}^{2^{m-1}} A_{m,n;\alpha}^{(k)} = A_{m,n;\alpha}. \tag{2.65}$$

As usual, the set of all matrices with the same order can be partially ordered.

Definition 2.1.8. *Let $\mathbb{M} = [M_{ij}]$ and $\mathbb{N} = [N_{ij}]$ be two $M \times M$ matrices, $\mathbb{M} \geq \mathbb{N}$ if $M_{ij} \geq N_{ij}$ for all $1 \leq i, j \leq M$.*

Notably, if $\mathbb{A}_2 \geq \mathbb{A}_2'$ then $\mathbb{A}_n \geq \mathbb{A}_n'$ for all $n \geq 2$. Therefore, $h(\mathbb{A}_2) \geq h(\mathbb{A}_2')$. Hence, the spatial entropy as a function of $\mathbb{A}_2$ is monotonic with respect to the partial order $\geq$.

Definition 2.1.9. *A $K+1$ multiple index*

$$\mathcal{B}_K \equiv (\beta_1\beta_2\cdots\beta_K\beta_{K+1}) \tag{2.66}$$

is called a (periodic) cycle if

$$\beta_{K+1} = \beta_1. \tag{2.67}$$

It is called a diagonal cycle if (2.67) holds and

$$\beta_k \in \{1, 4\} \tag{2.68}$$

for each $1 \leq k \leq K+1$.

For a diagonal cycle (2.66), denote

$$\bar{\beta}_K = \beta_1; \beta_2; \cdots; \beta_K \tag{2.69}$$

and

$$\bar{\beta}_K^n = \bar{\beta}_K; \bar{\beta}_K; \cdots; \bar{\beta}_K. \text{ (n times)} \tag{2.70}$$

First, prove the following Lemma.

Lemma 2.1.10. *Let $m \geq 2$, $K \geq 1$, $\mathcal{B}_K$ be a diagonal cycle. Then, for any $n \geq 1$,*

$$\rho(\mathbb{A}^m_{nK+2}) \geq \rho(|(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_{K+1}})^n X_{m,2;\beta_1}|) \tag{2.71}$$

Proof. Since $\mathcal{B}_K$ is a periodic cycle, Theorem 2.1.6 implies

$$X_{m,nK+2;\bar{\beta}^n_K} = (S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_{K+1}})^n X_{m,2;\beta_1}. \tag{2.72}$$

Furthermore $\mathcal{B}_K$ is diagonal, and $|X_{m,nK+2;\bar{\beta}^n_k}| = A_{m,nK+2;\bar{\beta}^n_k}$ lies on the diagonal part as in (2.47) with $n+p = nK+2$, therefore

$$\rho(\mathbb{A}^m_{nK+2}) \geq \rho(|X_{m,nK+2;\bar{\beta}^n_K}|). \tag{2.73}$$

Therefore, (2.71) follows from (2.72) and (2.73).

The proof is complete. □

The following lemma is valuable in studying maximum eigenvalue of $(S_{m;\beta_1\beta_2}\cdots S_{m;\beta_K\beta_{K+1}})^n X_{m,2;\beta_1}$ in (2.71).

Lemma 2.1.11. *For any $m \geq 2$, $1 \leq k \leq 2^{m-1}$ and $\alpha \in \{1,4\}$, if*

$$tr(A^{(k)}_{m,2;\alpha}) = 0, \tag{2.74}$$

then for all $1 \leq l \leq 2^{m-1}$,

$$(S_{m,\alpha 1})_{kl} = 0 \text{ and } (S_{m;\alpha 4})_{kl} = 0, \tag{2.75}$$

i.e., the k-th rows of matrices $S_{m;\alpha 1}$ and $S_{m;\alpha 4}$ are zeros. Furthermore, for any diagonal cycle $\mathcal{B}_K$, let $U = (u_1, u_2, \cdots, u_{2^{m-1}})$ be an eigenvector of $S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}$, if $u_k \neq 0$ for some $1 \leq k \leq 2^{m-1}$, then

$$tr(A^{(k)}_{m,2;\alpha}) > 0. \tag{2.76}$$

Proof. Since $A^{(k)}_{m,2;\alpha}$ can be expressed as in (2.43). Therefore, $tr(A^{(k)}_{m,2;\alpha}) = 0$ if and only if (2.75) holds for all $1 \leq l \leq 2^{m-1}$. The second part of the lemma follows easily from the first part.

The proof is complete. □

By Lemma 2.1.10 and Lemma 2.1.11, the lower bound of entropy can be obtained as follows.

Theorem 2.1.12. *Let $\beta_1\beta_2\cdots\beta_K\beta_1$ be a diagonal cycle. Then for any $m \geq 2$,*

$$h(\mathbb{A}_2) \geq \frac{1}{mK}\log\rho(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}). \tag{2.77}$$

and

$$h(\mathbb{A}_2) \geq \frac{1}{mK}\log\rho(W_{m;\beta_1\beta_2}W_{m;\beta_2\beta_3}\cdots W_{m;\beta_K\beta_1}). \tag{2.78}$$

In particular, if a diagonal cycle $\beta_1\beta_2\cdots\beta_K\beta_1$ exists and $m \geq 2$ such that

$$\rho(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}) > 1,$$

or

$$\rho(W_{m;\beta_1\beta_2}W_{m;\beta_2\beta_3}\cdots W_{m;\beta_K\beta_1}) > 1$$

then $h(\mathbb{A}_2) > 0$.

Proof. First, show that

$$h(\mathbb{A}_2) \geq \frac{1}{mK} \limsup_{n\to\infty} (\log \rho(|(S_{m;\beta_1\beta_2} S_{m;\beta_2\beta_3} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}|). \tag{2.79}$$

Indeed, from (1.11) and (2.71),

$$\begin{aligned} h(\mathbb{A}_2) &= \lim_{n\to\infty} \frac{1}{nK+2} \log \rho(\mathbb{A}_{nK+2}) \\ &= \lim_{n\to\infty} \frac{1}{m(nK+2)} \log \rho(\mathbb{A}^m_{nK+2}) \\ &\geq \frac{1}{mK} \limsup_{n\to\infty} \frac{1}{n} (\log \rho(|(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}|)). \end{aligned}$$

Now, the following remains to be shown

$$\limsup_{n\to\infty} \frac{1}{n} (\log \rho(|(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}|) = \log \rho(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1}). \tag{2.80}$$

Since $X_{m,2;\beta_1} = (A^{(k)}_{m,2;\beta_1})^t$, if $tr(A^{(k)}_{m,2;\beta_1}) = 0$ then Lemma 2.1.11 implies the k-th row of $S_{m;\beta_1\beta_2}$ is zero which implies that the k-th row of $(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n$ is also zero for any $n \geq 1$.

If $tr(A^{(k)}_{m,2;\beta_1}) = 0$ for all $1 \leq k \leq 2^{m-1}$, then $S_{m;\beta_1\beta_2} \equiv 0$. (2.80) holds trivially.

Now, assume that $1 \leq k' \leq 2^{m-1}$ exists such that $tr(A^{(k')}_{m,2;\beta_1}) > 0$. Define

$$\hat{X} = (A^{(k')}_{m,2;\beta_1})^t = (\hat{X}_1, \cdots, \hat{X}_M), \tag{2.81}$$

where $tr(A^{(k')}_{m,2;\beta_1}) > 0$ for $1 \leq k' \leq M \leq 2^{m-1}$. Then $\rho(\hat{X}_j) > 0$ for $1 \leq j \leq M$.

Let $\mathbb{M}$ be the $M \times M$ sub-matrix of $S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1}$ from which the k-th row and k-th column have been removed whenever $tr(A^{(k)}_{m,2;\beta_1}) = 0$ for $1 \leq k \leq 2^{m-1}$.

Clearly,

$$|(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1})^n X_{m,2;\beta_1}| = |\mathbb{M}^n \hat{X}|, \tag{2.82}$$

and

$$\rho(S_{m;\beta_1\beta_2} \cdots S_{m;\beta_K\beta_1}) = \rho(\mathbb{M}). \tag{2.83}$$

The proof of (2.80) comprise three steps, according to

(i) $\mathbb{M}$ is primitive,

(ii) $\mathbb{M}$ is irreducible, and

(iii) $\mathbb{M}$ is reducible.

(i) $\mathbb{M}$ is primitive. Then by Perron-Frobenius Theorem the maximum eigenvalue $\rho(\mathbb{M})$ of $\mathbb{M}$ is unique with maximum modulus, i.e.

$$\rho(\mathbb{M}) = \lambda_1 > |\lambda_j|, \tag{2.84}$$

for all $2 \leq j \leq M$, where λ_j are eigenvalues of $\mathbb{M}$. Moreover, a positive eigenvector $\mathbf{v}_1 = (v_1, v_2, \cdots, v_M)^t$ is associated with λ_1 [29], [30]. Furthermore, Jordan canonical form theorem states that a non-singular matrix $\mathbb{P} = [P_{ij}]_{M\times M}$ exists, such that the real Jordan canonical form of $\mathbb{M}$ is

$$\hat{\mathbb{M}} \equiv \mathbb{P}\mathbb{M}\mathbb{P}^{-1} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & J_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & J_{n_q} \end{bmatrix}, \tag{2.85}$$

where J_{n_k}, $2 \leq k \leq q$ are real Jordan blocks and the associated eigenvalue λ_k of J_{n_k} satisfies (2.84). Moreover, the positivity of eigenvector $\mathbf{v}_1$ implies that $\mathbb{P}$ can be chosen such that

$$\sum_{i=1}^{M} P_{ij} = 1 \tag{2.86}$$

and

$$P_{1j} > 0 \tag{2.87}$$

for all $1 \leq j \leq M$. Therefore, by (2.86)

$$\begin{aligned} &|\mathbb{M}^n \hat{X}| = |\mathbb{P}\mathbb{M}^n \hat{X}| = |\mathbb{P}\mathbb{M}^n\mathbb{P}^{-1}\mathbb{P}\hat{X}| \\ &= |(\mathbb{P}\mathbb{M}\mathbb{P}^{-1})^n\mathbb{P}\hat{X}| = |\hat{\mathbb{M}}^n\mathbb{P}\hat{X}| \\ &= {\lambda_1}^n\{\sum_{j=1}^{M} P_{1j}\hat{X}_j + \sum_{j=1}^{M} q_{n,j}\hat{X}_j\} \end{aligned}$$

where

$$\lim_{n\to\infty} q_{n,j} = 0, \tag{2.88}$$

for all $1 \leq j \leq M$, by (2.84).

Hence, by (2.87) and (2.88),

$$\lim_{n\to\infty} \frac{1}{n} \log \rho(|\mathbb{M}^n \hat{X}|) = \log \lambda_1. \tag{2.89}$$

Combining with (2.82), (2.83) and (2.89), (2.80) follows.

(ii) $\mathbb{M}$ is irreducible.

If $\mathbb{M}$ is irreducible but imprimitive, then $k \geq 2$ exists, such that

$$\lambda_1 = |\lambda_2| = \cdots = |\lambda_k| > |\lambda_j|$$

for all $j > k$. Then, by applying a permutation, $\mathbb{M}$ can be expressed as

$$\mathbb{M} = \begin{bmatrix} 0 & M_{12} & 0 & \cdots & 0 \\ 0 & 0 & M_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \vdots & 0 & M_{k-1,k} \\ M_{k1} & 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{2.90}$$

and,

$$\mathbb{M}^k = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & M_k \end{bmatrix}, \tag{2.91}$$

where $M_j = M_{j,j+1}M_{j+1,j+2}\cdots M_{j-1,j}$ is primitive with the maximum eigenvalue λ_1^k, see [29], [30]. Hence, by the same argument as in (i)

$$\lim_{n\to\infty}\frac{1}{n}\log\rho(|\mathbb{M}^{nk}\hat{X}|) = \lambda_1^k,$$

(2.80) follows.

(iii) $\mathbb{M}$ is reducible.

In this case, by applying a permutation, $\mathbb{M}$ can be expressed as a block upper triangular matrix:

$$\mathbb{M} = \begin{bmatrix} M_{11} & M_{12} & \cdots & \cdots & M_{1k} \\ 0 & M_{22} & \cdots & \cdots & M_{2k} \\ 0 & 0 & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 & M_{kk} \end{bmatrix}, \tag{2.92}$$

where M_{ii} is either irreducible or zero. Furthermore,

$$\sigma(\mathbb{M}) = \bigcup_{j=1}^{k}\sigma(\mathbb{M}_{jj}),$$

where $\sigma(\mathbb{M})$ and $\sigma(\mathbb{M}_{jj})$ are the sets of eigenvalues of $\mathbb{M}$ and $\mathbb{M}_{jj}$, respectively. In particular, $1 \le j \le k$ exists, such that

$$\rho(\mathbb{M}_{jj}) = \rho(\mathbb{M}) = \lambda_1. \tag{2.93}$$

[29], [30]. Therefore, applying (2.83), (2.93) and the same argument as in (ii) yields (2.80).

The proof is complete.

□

Definition 2.1.13. *Let $\mathcal{D}$ denote the set of all diagonal cycle:*

$$\mathcal{D} = \{\beta_1\beta_2\cdots\beta_K\beta_{K+1}|\beta_1\beta_2\cdots\beta_K\beta_{K+1} \text{ satisfies (2.67) and (2.68)}\},$$

define

$$h_*(\mathbb{A}_2) = \sup_{m\ge 2,\beta_1\beta_2\cdots\beta_{K+1}\in\mathcal{D}}\frac{1}{mK}\log\rho(S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_K\beta_1}). \tag{2.94}$$

and

$$h'_*(\mathbb{A}_2) = \sup_{m\ge 2,\ \beta_1\cdots\beta_K\in D}\frac{1}{mK}\log\rho(W_{m;\beta_1\beta_2}W_{m;\beta_2\beta_3}\cdots W_{m;\beta_K\beta_1}). \tag{2.95}$$

Then Theorem 2.1.12 implies

$$h(\mathbb{A}_2) \geq h_*(\mathbb{A}_2) \text{ and } h(\mathbb{A}_2) \geq h'_*(\mathbb{A}_2). \tag{2.96}$$

Knowing whether the equality holds for $\mathbb{A}_2$ is of interest, since $h_*(\mathbb{A}_2)$ and $h'_*(\mathbb{A}_2)$ are more manageable than $h(\mathbb{A}_2)$. However, a class of $\mathbb{A}_2$ has been found for what equality (2.96) holds; details can be found in Example 2.1.14. of the next subsection.

2.1.3 Examples of transition matrices with positive entropy

In this subsection, various examples are studied to elucidate the power of Theorem 2.1.12 in verifying that the entropies are positive. First, Golden-Mean type transition matrices are studied.

Example 2.1.14. (A) Golden-Mean type

When two symbols on two-cell horizontal lattice $\mathbb{Z}_{2\times 1}$ and vertical lattice $\mathbb{Z}_{1\times 2}$ are considered and both transition matrices are given by golden-mean type, i.e.,

$$\mathbb{H}_1 = \mathbb{V}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

then the (horizontal) transition matrix $\mathbb{A}_2$ on $\mathbb{Z}_{2\times 2}$ is

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.97}$$

as in [46]. Verifying

$$\mathbb{B}_2 = \widetilde{\mathbb{A}}_2 = \widetilde{\mathbb{B}}_2 = \mathbb{A}_2. \tag{2.98}$$

is also easy. Furthermore, for any $n \geq 2$,

$$\mathbb{A}_{n+1} = \begin{bmatrix} A_{n+1} & B_{n+1} \\ C_{n+1} & 0 \end{bmatrix} = \begin{bmatrix} A_n & B_n & A_n & 0 \\ C_n & 0 & C_n & 0 \\ A_n & B_n & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.99}$$

where

$$A_{n+1} = \begin{bmatrix} A_n & B_n \\ C_n & 0 \end{bmatrix}$$

with $C_n = {B_n}^t$ and ${A_n}^t = A_n$, i.e., $\mathbb{A}_n$ are symmetric for all $n \geq 2$.

Moreover, the following two properties hold:

(i) For any $m \geq 2$,

$$C_{m;11} = \mathbb{A}_{m-1}, \tag{2.100}$$

where

$$\mathbb{A}_1 \equiv \begin{bmatrix} a_{11}a_{11} & a_{12}a_{21} \\ a_{13}a_{31} & a_{14}a_{41} \end{bmatrix}, \tag{2.101}$$

and

(ii) for any $m \geq 2$,

$$\frac{1}{m}\log\rho(\mathbb{A}_{m-1}) \leq h(\mathbb{A}_2) \leq \frac{1}{m}\log\rho(\mathbb{A}_m). \tag{2.102}$$

Therefore,

$$h(\mathbb{A}_2) = h_*(\mathbb{A}_2) > 0. \tag{2.103}$$

The numerical results appears in Example 2.2.12.

(B) Simplified Golden-Mean type.

Consider

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.104}$$

(2.104) cannot be generated from one-dimensional transition matrices $\mathbb{H}_1$ and $\mathbb{V}_1$, as in the Golden-Mean type (2.97). Equation (2.104) is obtained by letting $a_{23} = a_{32} = 0$ in the Golden-Mean type (2.97). (2.98) is easily verified, and for any $n \geq 2$,

$$\mathbb{A}_{n+1} = \begin{bmatrix} \mathbb{A}_n & & \begin{matrix} \mathbb{A}_{n-1} & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} \mathbb{A}_{n-1} & 0 \\ 0 & 0 \end{matrix} & & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}. \tag{2.105}$$

Furthermore, (i), (ii) and (2.103) hold as in (A).

(C) Generally, if $\mathbb{A}_2$ satisfies the following three conditions

(C1) $\mathbb{B}_2 = \mathbb{A}_2$,

(C2) $a_{1j} = 1$ if $A_{2;j} \neq 0$ for $1 \leq j \leq 4$,

(C3) $\widetilde{A}_{2;1} \geq A_{2;j}$ for $1 \leq j \leq 4$,

then (i), (ii) and (2.103) hold. The matrices $\mathbb{A}_2$, which satisfy (C1), (C2) and (C3) can be listed as

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & a_{23} & 0 \\ 1 & a_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.106}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & a_{23} & a_{24} \\ 1 & a_{32} & 1 & a_{34} \\ 1 & a_{34} & a_{43} & a_{44} \end{bmatrix}, \tag{2.107}$$

where a_{ij} is either 0 or 1 in (2.106) and (2.107).

Notably, if (C2) and (C3) are replaced by

(C2)′ $a_{4j} = 1$ if $A_{2;j} \neq 0$ for $1 \leq j \leq 4$,

(C3)′ $\widetilde{A}_{2;4} \geq A_{2;j}$ for $1 \leq j \leq 4$,

then for any $m \geq 2$,

$$C_{m;44} = \mathbb{A}_{m-1} \tag{2.108}$$

with

$$\mathbb{A}_1 = \begin{bmatrix} a_{41}a_{14} & a_{42}a_{24} \\ a_{43}a_{34} & a_{44}a_{44} \end{bmatrix}, \tag{2.109}$$

and property (ii) and equation (2.103) hold.

In Example 2.1.14, the diagonal parts $A_{2;1}$ or $A_{2;4}$ are dominant. In this case, only $C_{m;11}$ or $C_{m;44}$ is required to apply Theorem 2.1.12. In contrast, when $\mathbb{A}_{2;1}$ and $\mathbb{A}_{2;4}$ are no longer dominant as in the following examples, $A_{2;2}$ and $A_{2;3}$ can complement each other to establish that the entropy is positive.

Example 2.1.15. (A) Consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.110}$$

that (2.98) holds can be verified and

$$C_{2;11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_{2;22} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$
$$C_{2;33} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_{2;44} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore,

$$S_{2;14}S_{2;41} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$h(\mathbb{A}_2) \geq \frac{1}{4}\log 2.$$

(B) Consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{2.111}$$

Then verifying

$$\mathbb{B}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \widetilde{\mathbb{B}}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \text{and } \widetilde{\mathbb{A}}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

is simple.

Furthermore,

$$C_{2;11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_{2;22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C_{2;33} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_{2;44} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$U_{2;11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U_{2;22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$U_{2;33} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U_{2;44} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now, for any diagonal cycle, $\beta_1 \cdots \beta_K \beta_1$, $\rho(S_{2;\beta_1\beta_2} \cdots S_{2;\beta_K\beta_1}) = 1$, $h(\mathbb{A}_2) > 0$ cannot be established.
However,

$$W_{2;11}W_{2;14}W_{2;41} = U_{2;11}U_{2;22}U_{2;33} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

which implies

$$h(\mathbb{A}_2) \geq \frac{1}{6} \log g,$$

where

$$g = \frac{1}{2}(1 + \sqrt{5}) \tag{2.112}$$

is the golden mean, which is a root of $\lambda^2 - \lambda - 1 = 0$.

This example demonstrates the asymmetry of $\mathbb{A}_2$ and $\mathbb{B}_2$ in applying Theorem 2.1.12, to verify the entropy is positive. Both $\mathbb{C}_m$ and $\mathbb{U}_m$ are typically checked for completeness.

Example 2.1.16. Consider

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.113}$$

Then it is easy to check that

$$W_{2;11}W_{2;14}W_{2;41} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_{3;44} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$S_{4;44} = \begin{bmatrix} G & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.114}$$

Therefore,

$$h(\mathbb{A}_2) \geq \max\{\frac{1}{6}\log 2, \frac{1}{3}\log g, \frac{1}{4}\log g\} = \frac{1}{3}\log g.$$

Example 2.1.17. Consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.115}$$

Then

$$\mathbb{B}_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{\mathbb{A}}_2 \text{ and } \tilde{\mathbb{B}}_2 = \mathbb{A}_2.$$

Therefore

$$C_{2,11} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \equiv G'.$$

Furthermore,

$$C_{4;11} = G' \otimes e_1 \otimes G'$$

and

$$C_{2m;11} = G' \otimes (\otimes(e_1 \otimes G')^{m-1})$$

can be proved, and which implies

$$\frac{1}{2m} \log \rho(C_{2m;11}) = \frac{1}{2} \log g. \tag{2.116}$$

for all $m \geq 1$. Hence, $h(\mathbb{A}_2) \geq \frac{1}{2} \log g$. Moreover, in Remark 2.2.10 (ii), it can be shown that $h(\mathbb{A}_2) = \frac{1}{2} \log g$

2.2 Trace operators $\mathbb{T}_m$

The preceding section introduces connecting operators $\mathbb{C}_m$, which can be used to find lower bounds of spatial entropy. This section studies the diagonal part of $\mathbb{C}_m$, which can be used to investigate the trace of $\mathbb{A}_n^m$. When $\mathbb{A}_2$ is symmetric, $\mathbb{T}_{2m}$ gives the upper bound of spatial entropy.

The trace operator is defined first.

Definition 2.2.1. *For $m \geq 2$, the m-th order trace operator $\mathbb{T}_m$ of $\mathbb{A}_2$ is defined by*

$$\mathbb{T}_m = \begin{bmatrix} C_{m;11} & C_{m;22} \\ C_{m;33} & C_{m;44} \end{bmatrix} = \begin{bmatrix} S_{m;11} & S_{m;14} \\ S_{m;41} & S_{m;44} \end{bmatrix}, \tag{2.117}$$

where $C_{m;ij}$ is as given in (1.23) or (2.29).

Similarly, the m-th order trace operator $\mathbb{T}'_m$ of $\mathbb{B}_2$ is defined by

$$\mathbb{T}'_m = \begin{bmatrix} U_{m;11} & U_{m;22} \\ U_{m;33} & U_{m;44} \end{bmatrix} = \begin{bmatrix} W_{m;11} & W_{m;14} \\ W_{m;41} & W_{m;44} \end{bmatrix} \tag{2.118}$$

where $U_{m;ij}$ is as given in (2.31).

The relationships between the trace operator $\mathbb{T}_m$, $\mathbb{T}'_m$ and $\mathbb{A}_m$, $\mathbb{B}_m$ are given as follows.

Theorem 2.2.2. *For any $m \geq 2$,*

$$\mathbb{T}_m = (\mathbb{B}_m)_{2^m\times 2^m} \circ \begin{bmatrix} E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{11} & a_{21} \\ a_{31} & a_{41} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{12} & a_{22} \\ a_{32} & a_{42} \end{bmatrix} \\ E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{13} & a_{23} \\ a_{33} & a_{43} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{14} & a_{24} \\ a_{34} & a_{44} \end{bmatrix} \end{bmatrix} \tag{2.119}$$

and

$$\mathbb{T}'_m = (\mathbb{A}_m)_{2^m\times 2^m} \circ \begin{bmatrix} E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{11} & b_{21} \\ b_{31} & b_{41} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{12} & b_{22} \\ b_{32} & b_{42} \end{bmatrix} \\ E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{13} & b_{23} \\ b_{33} & b_{43} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} b_{14} & b_{24} \\ b_{34} & b_{44} \end{bmatrix} \end{bmatrix}. \tag{2.120}$$

In particular,

$$\mathbb{T}_m \leq \mathbb{B}_m \text{ and } \mathbb{T}'_m \leq \mathbb{A}_m. \tag{2.121}$$

Proof. By (2.117) and (2.29),

$$\mathbb{T}_m = (\mathbb{B}_m)_{2^m\times 2^m} \circ \begin{bmatrix} E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{11} & a_{21} \\ a_{31} & a_{41} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{12} & a_{22} \\ a_{32} & a_{42} \end{bmatrix} \\ E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{13} & a_{23} \\ a_{33} & a_{43} \end{bmatrix} & E_{2^{m-2}\times 2^{m-2}} \otimes \begin{bmatrix} a_{14} & a_{24} \\ a_{34} & a_{44} \end{bmatrix} \end{bmatrix}.$$

A similar result also holds for $\mathbb{T}'_m$. Hence, (2.121) follows immediately.

The proof is complete. □

Notably, the trace operator $\mathbb{T}_m$ (or $\mathbb{T}'_m$) preserves all periodic words $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_mi_{m+1}}$ ($b_{i_1i_2}b_{i_2i_3}\cdots b_{i_mi_{m+1}}$) with $i_{m+1}=i_1$ of length m systematically as $\mathbb{B}_m$ (or $\mathbb{A}_m$).

The traces of the elementary patterns are defined accordingly.

Definition 2.2.3. *For $m,n\geq 2$ and $1\leq\alpha\leq 4$, define*

$$t^{(k)}_{m,n;\alpha}=tr(A^{(k)}_{m,n;\alpha}), \tag{2.122}$$

$$tr(X_{m,n;\alpha})=(t^{(k)}_{m,n;\alpha})_{1\leq k\leq 2^{m-1}}, \tag{2.123}$$

and

$$t_{m,n}=(tr(X_{m,n;1}),tr(X_{m,n;4}))^t, \tag{2.124}$$

which are 2^{m-1} and 2^m vectors, respectively.

Note that

$$\begin{aligned} tr(\mathbb{A}^m_n) &= tr(\textstyle\sum_{k=1}^{2^{m-1}}A^{(k)}_{m,n;1}+\sum_{k=1}^{2^{m-1}}A^{(k)}_{m,n;4}) \\ &= |tr(X_{m,n;1})|+|tr(X_{m,n;4})| \\ &= |t_{m,n}|. \end{aligned} \tag{2.125}$$

First prove that $\mathbb{T}_m$ can reduce the traces of higher-order to lower-order.

Proposition 2.2.4. *For $m\geq 2$ and $n\geq 2$,*

$$t_{m,n+1}=\mathbb{T}_m t_{m,n} \tag{2.126}$$

Proof. By Theorem 2.1.5, it is easy to see

$$\begin{pmatrix} tr(X_{m,n+1;1}) \\ \\ tr(X_{m,n+1;4}) \end{pmatrix}=\begin{pmatrix} C_{m;11}tr(X_{m,n;1})+C_{m;22}tr(X_{m,n;4}) \\ \\ C_{m;33}tr(X_{m,n;1})+C_{m;44}tr(X_{m,n;4}) \end{pmatrix}.$$

Then, (2.126) follows immediately.

The proof is complete. □

Repeatedly applying Proposition 2.2.4 yields the following result.

Theorem 2.2.5. *For $m\geq 2$ and $n\geq 1$,*

$$tr(\mathbb{A}^m_{n+2})=|\mathbb{T}^n_m t_{m,2}| \tag{2.127}$$

$$\equiv\sum_{\beta_k\in\{1,4\}}|S_{m;\beta_1\beta_2}S_{m;\beta_2\beta_3}\cdots S_{m;\beta_n\beta_{n+1}}tr(X_{m,2;\beta_{n+1}})|. \tag{2.128}$$

Proof.

$$\begin{aligned} &tr(\mathbb{A}^m_n) \\ &=\sum_{k=1}^{2^{m-1}}tr(A^{(k)}_{m,n;1;1})+\sum_{k=1}^{2^{m-1}}tr(A^{(k)}_{m,n;1;4})+\sum_{k=1}^{2^{m-1}}tr(A^{(k)}_{m,n;4;1})+\sum_{k=1}^{2^{m-1}}tr(A^{(k)}_{m,n;4;4}) \\ &=|tr(X_{m,n;1;1})|+|tr(X_{m,n;1;4})|+|tr(X_{m,n;4;1})|+|tr(X_{m,n;4;4})| \\ &=|tr(S_{m;11}X_{m,n-1;1})|+|tr(S_{m;14}X_{m,n-1;4})|+|tr(S_{m;41}X_{m,n-1;1})|+|tr(S_{m;44}X_{m,n-1;4})| \\ &=|\mathbb{T}_m t_{m,n-1}|, \end{aligned}$$

here Theorem 2.1.4 is used.

Reduction on n, yields

$$tr(\mathbb{A}^m_n)=|\mathbb{T}^{n-2}_m t_{m,2}|.$$

Finally, (2.128) follows from (2.117) and (2.124).

The proof is complete. □

The following lemma is needed to show (1.33).

Lemma 2.2.6. *Let V_m be a nonnegative eigenvector of $\mathbb{T}_m$ with respect to the maximum eigenvalue $\rho(\mathbb{T}_m)$. If $\rho(\mathbb{T}_m) > 0$, then*

$$\langle V_m, t_{m,2} \rangle > 0,$$

where $\langle\ ,\ \rangle$ denotes the standard inner product of $\mathbb{C}^{2^m}$.

Proof. Let $V_m = (u_1, \cdots, u_M, u'_1, \cdots, u'_M)$ be a nonnegative eigenvector of $\mathbb{T}_m$, where $M = 2^{m-1}$. Since $\rho(\mathbb{T}_m) > 0$, by Lemma 2.1.11, if $u_k > 0$ (or $u'_l > 0$) then $tr(A^{(k)}_{m,2;1}) > 0$ (or $tr(A^{(l)}_{m,2;4}) > 0$). The result follows by (2.124).

The proof is complete. □

Now, (1.33) can be proved.

Theorem 2.2.7. *For any $m \geq 2$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log tr(\mathbb{A}_n^m) = \log \rho(\mathbb{T}_m), \tag{2.129}$$

and

$$h(\mathbb{A}_2) = \limsup_{m\to\infty} \frac{1}{m} \log \rho(\mathbb{T}_m). \tag{2.130}$$

Furthermore, if $\mathbb{A}_n$ are primitive for all $n \geq 2$, then limsup in (2.129) and (2.130) can be replaced by lim, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log tr(\mathbb{A}_n^m) = \log \rho(\mathbb{T}_m) \tag{2.131}$$

and

$$h(\mathbb{A}_2) = \lim_{m\to\infty} \frac{1}{m} \log \rho(\mathbb{T}_m). \tag{2.132}$$

Proof. By Perron-Frobenius theorem, for all $n \geq 2$, we have

$$\limsup_{m\to\infty} \frac{1}{m} \log tr(\mathbb{A}_n^m) = \log \rho(\mathbb{A}_n). \tag{2.133}$$

Therefore, by (2.133) and Theorem 2.2.5, we have

$$h(\mathbb{A}_2) = \lim_{n\to\infty} \frac{1}{n} \log \rho(\mathbb{A}_n) = \limsup_{n,m\to\infty} \frac{1}{mn} \log tr(\mathbb{A}_n^m) = \limsup_{n,m\to\infty} \frac{1}{mn} \log |\mathbb{T}_m^n t_{m,2}|.$$

By Lemma 2.2.6 and by argument used to prove Theorem 2.1.12,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\mathbb{T}_m^n t_{m,2}| = \log \rho(\mathbb{T}_m) \tag{2.134}$$

can be shown, and (2.129) and (2.130) follow immediately.

When $\mathbb{A}_n$ are primitive for all $n \geq 2$, (2.131) and (2.132) follow.

The proof is complete. □

Now, the symmetry of $\mathbb{A}_2$ is established to be able to be inherited by the higher order matrices.

Proposition 2.2.8. *If $\mathbb{A}_2$ is symmetric, then $\mathbb{A}_n$ is also symmetric for each $n \geq 3$.*

Proof. The proposition is proven by induction on n.

Let $\mathbb{M} = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$ be a square matrix and $M_i,\ 1 \leq i \leq 4$, all be square matrices. Then, the transpose matrix $\mathbb{M}^t$ of $\mathbb{M}$ is

$$\mathbb{M}^t = \begin{bmatrix} {M_1}^t & {M_3}^t \\ {M_2}^t & {M_4}^t \end{bmatrix}.$$

Therefore, $\mathbb{M}$ is symmetric if and only if

$${M_1}^t = M_1,\ {M_3}^t = M_2 \text{ and } {M_4}^t = M_4.$$

In particular, $\mathbb{A}_2$ is symmetric if and only if

$$A_{2;1}^t = A_{2;1},\ A_{2;3}^t = A_{2;2} \text{ and } A_{2;4}^t = A_{2;4}. \tag{2.135}$$

Now, $\mathbb{A}_n$ is assumed to be symmetric, such that

$$A_{n;1}^t = A_{n;1},\ A_{n;3}^t = A_{n;2} \text{ and } A_{n;4}^t = A_{n;4}. \tag{2.136}$$

Since

$$A_{n+1;\alpha} = [A_{2;\alpha}]_{2\times 2} \circ \begin{bmatrix} A_{n;1} & A_{n;2} \\ A_{n;3} & A_{n;4} \end{bmatrix},$$

(2.135) and (2.136) imply

$$A_{n+1;1}^t = A_{n+1;1},\ A_{n+1;3}^t = A_{n+1;2} \text{ and } A_{n+1;4}^t = A_{n+1;4}.$$

Hence, $\mathbb{A}_{n+1}$ is symmetric.

The proof is complete. □

Now, upper estimates of spatial entropy $h(\mathbb{A}_2)$ are obtained when $\mathbb{A}_2$ is symmetric.

Theorem 2.2.9. *If $\mathbb{A}_2$ is symmetric then for any $m \geq 1$,*

$$h(\mathbb{A}_2) \leq \frac{1}{2m} \log \rho(\mathbb{T}_{2m}). \tag{2.137}$$

Proof. By Proposition 2.2.8, $\mathbb{A}_n^{2m}$ is symmetric for any $m \geq 1$. The symmetry of $\mathbb{A}_n^{2m}$ implies that all eigenvalues of $\mathbb{A}_n^{2m}$ are non-negative. Hence,

$$\rho(\mathbb{A}_n)^{2m} = \rho(\mathbb{A}_n^{2m}) \leq tr(\mathbb{A}_n^{2m}). \tag{2.138}$$

On the other hand, the subadditivity of (2.58) implies

$$h(\mathbb{A}_2) \leq \frac{1}{(2mk+1)n} \log |\mathbb{A}_n^{2mk}|. \tag{2.139}$$

Therefore, (2.138), (2.139) and (2.127) imply

$$\begin{aligned} h(\mathbb{A}_2) &\leq \lim_{n,k\to\infty} \frac{1}{(2mk+1)n} \log |\mathbb{A}_n^{2mk}| = \lim_{n\to\infty} \frac{1}{2mn} \log \rho(\mathbb{A}_n^{2m}) \\ &\leq \lim_{n\to\infty} \frac{1}{2mn} \log tr(\mathbb{A}_n^{2m}) = \lim_{n\to\infty} \frac{1}{2mn} \log |\mathbb{T}_{2m}^{n-2} t_{2m,2}| \\ &\leq \frac{1}{2m} \log \rho(\mathbb{T}_{2m}). \end{aligned}$$

The proof is complete. □

Notably, $\mathbb{T}_m$ (or $\mathbb{T}'_m$) yields a better estimate than $\mathbb{B}_n$ (or $\mathbb{A}_n$) whenever

$$h(\mathbb{A}_2) \leq \frac{1}{m} \log \rho(\mathbb{T}_m) \tag{2.140}$$

holds.

Remark 2.2.10. *(i) The problem in which $\mathbb{A}_n$ are primitive for all $n \geq 2$ has already been investigated [6]. In [6], various sufficient conditions have been found to ensure that $\mathbb{A}_n$ are primitive for all $n \geq 2$. Notably, limit in (2.131) and (2.132), instead of limsup in (2.129) and (2.130), causes $\mathbb{A}_n$ to have a unique maximum eigenvalue with a maximum modulus. Therefore, $\mathbb{A}_n$ may be imprimitive but (2.131) and (2.132) still hold. For example, Golden-Mean type and simplified Golden-Mean type in Example 2.1.14 are imprimitive but (2.131) and (2.132) still hold. The remaining matrices of these $\mathbb{A}_n$ are primitive if their rows and columns with zero entries are removed.*

(ii) In general, limsup cannot be replaced by limit. For example, consider

$$\mathbb{A}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.141}$$

Further computation shows that

$$\mathbb{T}_{2m+1} = 0$$

and

$$\mathbb{T}_{2m} = \begin{bmatrix} (\otimes(G' \otimes e_1)^{m-1}) \otimes G' & e_1 \otimes (\otimes(G' \otimes e_1)^{m-1}) \\ e_1 \otimes (\otimes(G' \otimes e_1)^{m-1}) & e_1 \otimes (\otimes(G' \otimes e_1)^{m-1}) \end{bmatrix}$$

for all $m \geq 1$, where $G' = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Therefore, $\rho(\mathbb{T}_{2m+1}) = 0$. Furthermore, it can be shown that

$$\rho(\mathbb{T}_{2m}) \leq g^m + g^{m-1}. \tag{2.142}$$

Combining (2.116) and (2.142), $h(\mathbb{A}_2) = \frac{1}{2}\log g$. Hence (2.130) holds only for limsup. Unlike (2.62) this example demonstrates that (2.140) does not hold for any $n = 2m+1$. This phenomenon is a disadvantage in determining the upper estimate of entropy associated with replacing $\mathbb{A}_n$ with $\mathbb{T}_n$.

Example 2.2.11. Consider

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

which was studied as in Example 2.16. Now, $\mathbb{A}_2$ is asymmetric. Furthermore,

$$tr(\mathbb{A}_n^2) = 3$$

can be obtained for all $n \geq 2$. Hence, (2.138) and then (2.137) fail when $m = 1$. However,

$$C_{4;44} = \begin{bmatrix} G & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $tr(\mathbb{A}_n^4)$ grows at least exponentially with exponent $\rho(G) = g$, the golden-mean.

Whether (2.137) holds for some $m \geq 2$ is of interest.

Example 2.2.12. Consider the Golden-Mean type

$$\mathbb{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which was studied as in Example 2.1.14. $\mathbb{A}_2$ is symmetric, so the numerical results can be obtained as follows.

m	$\rho(\mathbb{A}_{m-1})^{\frac{1}{m}}$	$\rho(\mathbb{T}_m)^{\frac{1}{m}}$	$\rho(\mathbb{A}_m)^{\frac{1}{m}}$
2	1.3415037626	1.5537739740	1.5537739740
3	1.3804413572	1.4892228485	1.5370592754
4	1.4041128626	1.5069022259	1.5284545258
5	1.4201397131	1.5017251916	1.5233415461
6	1.4316975290	1.5035148094	1.5199401525
7	1.4404277508	1.5028716910	1.5175154443
8	1.4472546963	1.5031163748	1.5156994341
9	1.4527395436	1.5030208210	1.5142884861
10	1.4572426033	1.5030591603	1.5131606734
11	1.4610058138	1.5030435026	1.5122385423
12	1.4641976583	1.5030500001	1.5114705290
13	1.4669390746	1.5030472703	1.5108209763
14	1.4693191202	1.5030484295	1.5102644390
15	1.4714048275	1.5030479329	1.5097822725
16	1.4732476160	1.5030481473	1.5093605030

Notably, both $\rho(\mathbb{A}_m)^{\frac{1}{m}}$ and $\rho(\mathbb{T}_{2m})^{\frac{1}{2m}}$ are monotonically decreasing in m. In contrast, $\rho(\mathbb{A}_{m-1})^{\frac{1}{m}}$ and $\rho(\mathbb{T}_{2m+1})^{\frac{1}{2m+1}}$ are monotonically increasing in m, that $\rho(\mathbb{T}_{2m})^{\frac{1}{2m}}$ gives better upper bound than $\rho(\mathbb{A}_m)^{\frac{1}{m}}$. That $\rho(\mathbb{T}_{2m+1})^{\frac{1}{2m+1}}$ are lower bounds is conjectured. If they were, then $\rho(\mathbb{T}_m)^{\frac{1}{m}}$ would yield a very sharp estimates.

2.3 More symbols on larger lattice

As mentioned in the introduction, many physical and engineering problems involve many (more than two) symbols and larger $k \times k$ lattices, where $k \geq 3$. Therefore, the results found in the previous sections must be extended to any finite number of symbols $p \geq 2$ on any finite square lattice $\mathbb{Z}_{2l\times 2l,\ l\geq 1}$, where

$$2l = \begin{cases} k & \text{if } \ k \text{ is even} \\ 2k-2 & \text{if } \ k \text{ is odd} \end{cases}$$

. The results are only outlined here, and the details are left to the readers. Proofs of theorems are omitted for brevity.

For fixed $p \geq 2$ and $l \geq 1$, denote by

$$q = p^{l^2}. \tag{2.143}$$

The horizontal and vertical transition matrices are given by

$$\mathbb{A}_2 = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,q^2} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,q^2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q^2,1} & a_{q^2,2} & \cdots & a_{q^2,q^2} \end{bmatrix} \tag{2.144}$$

and

$$\mathbb{B}_2 = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,q^2} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,q^2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{q^2,1} & b_{q^2,2} & \cdots & b_{q^2,q^2} \end{bmatrix}, \tag{2.145}$$

respectively.
Now, $\mathbb{A}_2$ and $\mathbb{B}_2$ are related to each other by

$$\mathbb{A}_2 = \begin{bmatrix} A_{2;1} & A_{2;2} & \cdots & A_{2;q} \\ A_{2;q+1} & A_{2;q+2} & \cdots & A_{2;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{2;q(q-1)+1} & \cdots & \cdots & A_{2;q^2} \end{bmatrix} \tag{2.146}$$

where

$$A_{2;\alpha} = \begin{bmatrix} b_{\alpha,1} & b_{\alpha,2} & \cdots & b_{\alpha,q} \\ b_{\alpha,q+1} & b_{\alpha,q+2} & \cdots & b_{\alpha,2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\alpha,q(q-1)+1} & b_{\alpha,q(q-1)+2} & \cdots & b_{\alpha,q^2} \end{bmatrix}, \tag{2.147}$$

and

$$\mathbb{B}_2 = \begin{bmatrix} B_{2;1} & B_{2;2} & \cdots & B_{2;q} \\ B_{2;q+1} & B_{2;q+2} & \cdots & B_{2;2q} \\ \vdots & \vdots & \ddots & \vdots \\ B_{2;q(q-1)+1} & \cdots & \cdots & B_{2;q^2} \end{bmatrix} \tag{2.148}$$

where

$$B_{2;\alpha} = \begin{bmatrix} a_{\alpha,1} & a_{\alpha,2} & \cdots & a_{\alpha,q} \\ a_{\alpha,q+1} & a_{\alpha,q+2} & \cdots & a_{\alpha,2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha,q(q-1)+1} & a_{\alpha,q(q-1)+2} & \cdots & a_{\alpha,q^2} \end{bmatrix}, \tag{2.149}$$

respectively, where $1 \le \alpha \le q^2$. The column matrices $\widetilde{\mathbb{A}_2}$ and $\widetilde{\mathbb{B}_2}$, $\mathbb{A}_2$ and $\mathbb{B}_2$ are defined as in (2.1) and (2.2). For higher order transition matrices $\mathbb{A}_n,\ n \ge 3$, are defined as

$$\mathbb{A}_n = \begin{bmatrix} A_{n;1} & A_{n;2} & \cdots & A_{n;q} \\ A_{n;q+1} & A_{n;q+2} & \cdots & A_{n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n;q(q-1)+1} & A_{n;(q-1)q+2} & \cdots & A_{n;q^2} \end{bmatrix} \tag{2.150}$$

where

$$\mathbb{A}_{n;\alpha} = \begin{bmatrix} b_{\alpha,1}A_{n-1;1} & b_{\alpha,2}A_{n-1;2} & \cdots & b_{\alpha,q}A_{n-1;q} \\ b_{\alpha,q+1}A_{n-1;q+1} & b_{\alpha,q+2}A_{n-1;q+2} & \cdots & b_{\alpha,2q}A_{n-1;2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\alpha,q(q-1)+1}A_{n-1;q(q-1)+1} & b_{\alpha,q(q-1)+2}A_{n-1;q(q-1)+2} & \cdots & b_{\alpha,q^2}A_{n;q^2} \end{bmatrix}. \tag{2.151}$$

Rewriting the indices of $A_{n;\alpha}$ as follows, facilitates matrix multiplication.

$$\mathbb{A}_n = \begin{bmatrix} A_{n;11} & A_{n;12} & \cdots & A_{n;1q} \\ A_{n;21} & A_{n;22} & \cdots & A_{n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n;q1} & A_{n;q2} & \cdots & A_{n;qq} \end{bmatrix}. \tag{2.152}$$

Clearly, $A_{n;\alpha} = A_{n;j_1j_2}$, where

$$\alpha = \alpha(j_1, j_2) = q(j_1 - 1) + j_2. \tag{2.153}$$

For $m \ge 2$, the elementary pattern in the entries of $\mathbb{A}_n^m$ is given by

$$A_{n;j_1j_2}A_{n;j_2j_3}\cdots A_{n;j_mj_{m+1}},$$

where $j_s \in \{1, 2, \cdots, q\}$.

The lexicographic order for multiple indices

$$J_{m+1} = (j_1j_2\cdots j_mj_{m+1})$$

is introduced by

$$\chi(J_{m+1}) = 1 + \sum_{l=2}^{m} q^{m-l}(j_l - 1). \tag{2.154}$$

Specify

$$A^{(k)}_{m,n;\alpha} = A_{n;j_1j_2}A_{n;j_2j_3}\cdots A_{n;j_mj_{m+1}},$$

where $\alpha = \alpha(j_1, j_{m+1})$ satisfies (2.153) and $k = \chi(J_{m+1})$ is as given in (2.154). Based on this arrangement, $\mathbb{A}_n^m$ can be written as

$$\mathbb{A}_n^m = \begin{bmatrix} A_{m,n;1} & A_{m,n;2} & \cdots & A_{m,n;q} \\ A_{m,n;q+1} & A_{m,n;q+2} & \cdots & A_{m,n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,n;q(q-1)+1} & A_{m,n;q(q-1)+2} & \cdots & A_{m,n;q^2} \end{bmatrix},$$

where

$$A_{m,n;\alpha} = \sum_{k=1}^{q^{m-1}} A_{m,n;\alpha}^{(k)}.$$

Moreover, $X_{m,n;\alpha} = (A_{m,n;\alpha}^{(k)})^t$, where $1 \le k \le q^{m-1}$ and $X_{m,n;\alpha}$ is a q^{m-1}-vector that comprise all elementary patterns in $A_{m,n;\alpha}$. The ordering matrix $\mathbb{X}_{m,n}$ of $\mathbb{A}_n^m$ is now defined as

$$\mathbb{X}_{m,n} = \begin{bmatrix} X_{m,n;1} & X_{m,n;2} & \cdots & X_{m,n;q} \\ X_{m,n;q+1} & X_{m,n;q+2} & \cdots & X_{m,n;2q} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,n;q(q-1)+1} & X_{m,n;q(q-1)+2} & \cdots & X_{m,n;q^2} \end{bmatrix},$$

and $X_{m,n+1;\beta}$ can be reduced to $X_{2,n;\beta}$ by multiplication with connecting matrices $C_{m;\alpha,\beta}$. The connecting operator $\mathbb{C}_m$ is defined as follows.

Definition 2.3.1. *For $m \ge 2$, define*

$$\mathbb{C}_m = \begin{bmatrix} C_{m;1,1} & C_{m;1,2} & \cdots & C_{m;1,q^2} \\ C_{m;2,1} & C_{m;2,2} & \cdots & C_{m;2,q^2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m;q^2,1} & C_{m;q^2,2} & \cdots & C_{m;q^2,q^2} \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c|ccc} S_{m;1,1} & \cdots & S_{m;1,q} & & S_{m;q,1} & \cdots & S_{m;q,q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ S_{m;1,q(q-1)+1} & \cdots & S_{m;1,q^2} & & S_{m;q,q(q-1)+1} & \cdots & S_{m;q,q^2} \\ \hline & \vdots & & \ddots & & \vdots & \\ \hline S_{m;q(q-1)+1,1} & \cdots & S_{m;q(q-1)+1,q} & & S_{m;q^2,1} & \cdots & S_{m;q^2,q} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ S_{m;q(q-1)+1,q(q-1)+1} & \cdots & S_{m;q(q-1)+1,q^2} & & S_{m;q^2,q(q-1)+1} & \cdots & S_{m;q^2,q^2} \end{array}\right] \tag{2.155}$$

where

$$C_{m;\alpha,\beta} = ((B_{2;\alpha})_{q\times q} \circ (\hat{\otimes}\mathbb{B}_2^{m-2})_{q\times q})_{q^{m-1}\times q^{m-1}} \circ (E_{q^{m-2}\times q^{m-2}} \otimes \tilde{A}_{2;\beta})_{q^{m-1}\times q^{m-1}}. \tag{2.156}$$

Like Theorem 2.1.4, $C_{m+1;\alpha,\beta}$ can be obtained in terms of $C_{m;\gamma,\beta}$.

Theorem 2.3.2. *For any $m \ge 2$ and $1 \le \alpha, \beta \le q^2$*

$$C_{m+1;\alpha,\beta} = \begin{bmatrix} a_{\alpha;1}C_{m;1,\beta} & a_{\alpha;2}C_{m;2,\beta} & \cdots & a_{\alpha;q}C_{m;q,\beta} \\ a_{\alpha;q+1}C_{m;q+1,\beta} & a_{\alpha;q+2}C_{m;q+2,\beta} & \cdots & a_{\alpha;2q}C_{m;2q,\beta} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha;q(q-1)+1}C_{m;q(q-1)+1,\beta} & a_{\alpha;q(q-1)+2}C_{m;q(q-1)+2,\beta} & \cdots & a_{\alpha;q^2}C_{m;q^2,\beta} \end{bmatrix}.$$

Denote by

$$A^{(k)}_{m,n+1;\alpha} = \begin{bmatrix} A^{(k)}_{m,n+1;\alpha;1} & A^{(k)}_{m,n+1;\alpha;2} & \cdots & A^{(k)}_{m,n+1;\alpha;q} \\ A^{(k)}_{m,n+1;\alpha;q+1} & A^{(k)}_{m,n+1;\alpha;q+2} & \cdots & A^{(k)}_{m,n+1;\alpha;2q} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(k)}_{m,n+1;\alpha;q(q-1)+1} & A^{(k)}_{m,n+1;\alpha;q(q-1)+2} & \cdots & A^{(k)}_{m,n+1;\alpha;q^2} \end{bmatrix}$$

and $X_{m,n+1;\alpha;\beta} = (A^{(k)}_{m,n+1;\alpha;\beta})^t$ where $A^{(k)}_{m,n+1;\alpha;\beta}$ is a linear combination of $A^{(l)}_{m,n;\gamma}$. Now, Theorem 2.1.5 can be generalized to the following theorem.

Theorem 2.3.3. *For any $m \geq 2$ and $n \geq 2$, let $S_{m;\alpha,\beta}$ be as given in (2.155) and (2.156). Then $X_{m,n+1;\alpha;\beta} = S_{m;\alpha,\beta} X_{m,n;\beta}$.*

3 Three-dimensional Patterns Generation Problems

3.1 Ordering Matrices and Transition Matrices

This section describes three-dimensional patterns generation problems. Here, m_1, m_2, $m_3 \geq 2$ are fixed and indices are omitted for brevity. Let $\mathcal{S}$ be a set of p colors, and $\mathbf{Z}_{m_1\times m_2\times m_3}$ be a fixed finite rectangular sublattice of $\mathbf{Z}^3$, where $\mathbf{Z}^3$ denotes the integer lattice on $\mathbb{R}^3$ and (m_1, m_2, m_3) a three-tuple of positive integers. Functions $U : \mathbf{Z}^3 \to \mathcal{S}$ and $U_{m_1\times m_2\times m_3} : \mathbf{Z}_{m_1\times m_2\times m_3} \to \mathcal{S}$ are called global patterns and local patterns on $\mathbf{Z}_{m_1\times m_2\times m_3}$ respectively. The set of all patterns U is denoted by $\Sigma_p^3 \equiv \mathcal{S}^{\mathbf{Z}^3}$, such that Σ_p^3 is the set of all patterns with p different colors in a three-dimensional lattice. For clarity, two symbols, $\mathcal{S} = \{0, 1\}$ are considered. Let x, y and z coordinate represent the 1st-, 2nd- and 3rd-coordinates respectively as Fig. 1.

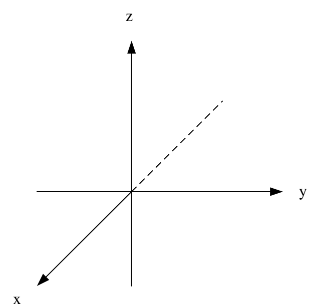


Figure 1: Three-dimensional coordinate system.

Six orderings $[\omega]$ ordering are represented as Eq. (3.1)

$$\begin{array}{lcccccc} [x] & : & [1] & \succ & [2] & \succ & [3] \\ [y] & : & [2] & \succ & [1] & \succ & [3] \\ [z] & : & [3] & \succ & [1] & \succ & [2] \\ [\hat{x}] & : & [1] & \succ & [3] & \succ & [2] \\ [\hat{y}] & : & [2] & \succ & [3] & \succ & [1] \\ [\hat{z}] & : & [3] & \succ & [2] & \succ & [1] \end{array} \tag{3.1}$$

On a fixed finite lattice $\mathbf{Z}_{m_1\times m_2\times m_3}$, an ordering $[\omega] : [i] \succ [j] \succ [k]$ is obtained on $\mathbf{Z}_{m_1\times m_2\times m_3}$, which is any one of the above orderings on $\mathbf{Z}_{m_1\times m_2\times m_3}$

$$\psi_\omega(\alpha_1, \alpha_2, \alpha_3) = m_j m_k(\alpha_i - 1) + m_k(\alpha_j - 1) + \alpha_k,$$

where $1 \leq \alpha_\ell \leq m_\ell$ and $1 \leq \ell \leq 3$. The ordering $[\omega]$ on $\mathbf{Z}_{m_1\times m_2\times m_3}$ can now be applied to $\Sigma_{m_1\times m_2\times m_3}$. Indeed, for each $U = (u_{\alpha_1\alpha_2\alpha_3}$[1]$) \in \Sigma_{m_1\times m_2\times m_3}$, define

$$\begin{aligned} \psi_\omega(U) &\equiv \psi_{\omega;m_1,m_2,m_3}(U) \\ &\equiv 1 + \sum_{\alpha_i=1}^{m_i}\sum_{\alpha_j=1}^{m_j}\sum_{\alpha_k=1}^{m_k} u_{\alpha_1\alpha_2\alpha_3}\omega_{m_i,m_j,m_k}^{\alpha_i,\alpha_j,\alpha_k}, \end{aligned}$$

where

$$\begin{aligned} \omega_{m_i,m_j,m_k}^{\alpha_i,\alpha_j,\alpha_k} &= 2^{m_i m_j m_k - \psi_\omega(\alpha_1,\alpha_2,\alpha_3)} \\ &= 2^{m_k m_j(m_i-\alpha_i)+m_k(m_j-\alpha_j)+(m_k-\alpha_k)}. \end{aligned}$$

[1]Use $u_{\alpha_1\alpha_2\alpha_3}$ to substitute $u_{\alpha_1,\alpha_2,\alpha_3}$ for simplicity afterward.

U is referred to herein as the $\psi_\omega(U)$-th element in $\Sigma_{m_1\times m_2\times m_3}$ by ordering $[\omega]$. Identifying the pictorial patterns using $\psi_\omega(U)$ is very effective in proving theorems since computations can now be performed on $\psi_\omega(U)$. For instance, the orderings on $\mathbf{Z}_{2\times2\times2}$ can be represented as Fig. 2.

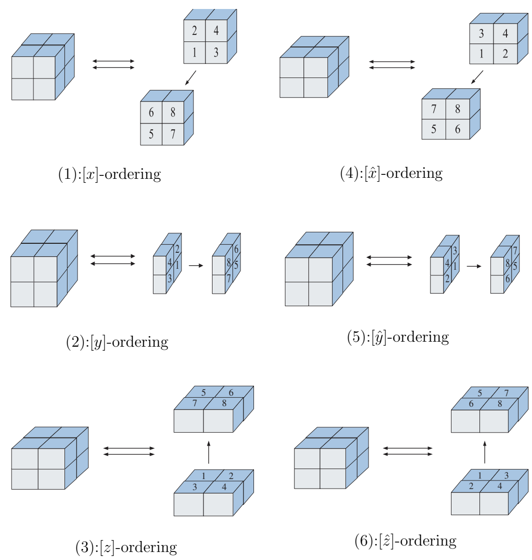


Figure 2: The orderings of $\mathbf{Z}_{2\times2\times2}$.

3.1.1 Ordering matrices

The cube $\mathbf{Z}_{m_1\times m_2\times m_3}$ can be decomposed by m_1-many (m_2-many and m_3-many) parallel 2-dimensional rectangles in $\mathbf{Z}_{1\times m_2\times m_3}$ ($\mathbf{Z}_{m_1\times 1\times m_3}$ and $\mathbf{Z}_{m_1\times m_2\times 1}$). Any patterns $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{m_1\times m_2\times m_3}$ can be decomposed accordingly. For example, in $[x]$-ordering, define the α_1-th layer of rectangle as

$$\mathbf{Z}_{\alpha_1;m_2\times m_3} = \{(\alpha_1,\alpha_2,\alpha_3)|1\le\alpha_2\le m_2, 1\le\alpha_3\le m_3\}.$$

Pattern U in α_1-th layer is assigned the number

$$i_{\alpha_1} \equiv 1 + \sum_{\alpha_2=1}^{m_2}\sum_{\alpha_3=1}^{m_3} u_{\alpha_1\alpha_2\alpha_3} x^{1,\alpha_2,\alpha_3}_{1,m_2,m_3}, \tag{3.2}$$

where $x^{1,\alpha_2,\alpha_3}_{1,m_2,m_3} = 2^{m_2m_3-m_3(\alpha_2-1)-\alpha_3}$. As denoted by the $1\times m_2\times m_3$ pattern

$$x_{1\times m_2\times m_3;i_{\alpha_1}} = \begin{array}{|c|c|c|c|}\hline u_{\alpha_1 1 m_3} & u_{\alpha_1 2 m_3} & \cdots & u_{\alpha_1 m_2 m_3} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline u_{\alpha_1 12} & u_{\alpha_1 22} & \cdots & u_{\alpha_1 m_2 2} \\ \hline u_{\alpha_1 11} & u_{\alpha_1 21} & \cdots & u_{\alpha_1 m_2 1} \\ \hline \end{array}.$$

In particular, when $m_2 = 2$ and $m_3 = 2$, as denoted by $x_{1\times2\times2;i_{\alpha_1}}$, where

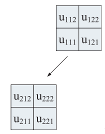
$$i_{\alpha_1} = 1 + 2^3u_{\alpha_1 11} + 2^2u_{\alpha_1 12} + 2u_{\alpha_1 21} + u_{\alpha_1 22} \tag{3.3}$$

and

$$x_{1\times2\times2;i_{\alpha_1}} \equiv x_{i_{\alpha_1}} = \begin{array}{|c|c|}\hline u_{\alpha_1 12} & u_{\alpha_1 22} \\ \hline u_{\alpha_1 11} & u_{\alpha_1 21} \\ \hline \end{array} ,$$

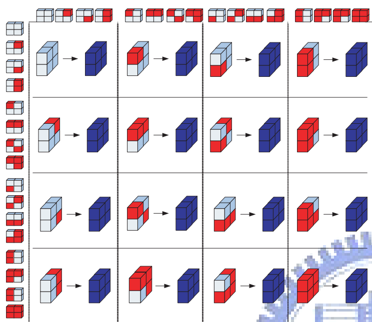
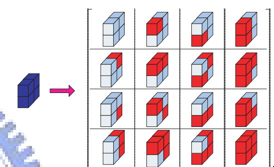
where $\alpha_1 \in \{1,2\}$. A $2\times2\times2$ pattern $U = (u_{\alpha_1\alpha_2\alpha_3})$ can now be obtained from the $[x]$-direct sum of two $1\times2\times2$ patterns using $[x]$-ordering:

$$\begin{aligned} x_{2\times2\times2;i_1 i_2} &\equiv x_{i_1 i_2} \\ &\equiv x_{i_1} \oplus x_{i_2} \\ &= \end{aligned}$$



,

where i_{α_1} as in Eq. (3.3) and $\alpha_1 \in \{1,2\}$. Therefore, the complete set of 2^8 patterns in $\Sigma_{2\times2\times2}$ is given by a 16×16 matrix $\mathbb{X}_{2\times2\times2} = [x_{i_1 i_2}{}^2]$ as its entries in

 where . (3.4)

That

$$\psi_x(x_{i_1 i_2}) = 2^4(i_1 - 1) + i_2$$

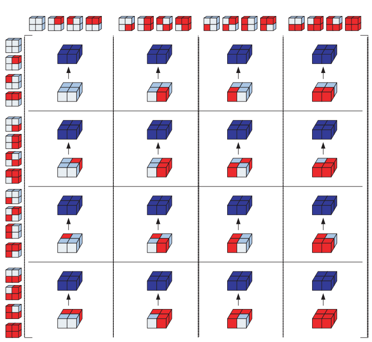
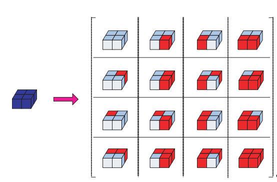
is easily verified, and local patterns in $\Sigma_{2\times2\times2}$ are thus counted by going through each row successively in Eq. (3.4). Correspondingly, $\mathbb{X}_{2\times2\times2}$ can be referred to as an ordering matrix for $\Sigma_{2\times2\times2}$. A $2\times2\times2$ pattern can also be regarded as an $[x]$-direct sum of two $1\times2\times2$ patterns using $[\hat{x}]$-ordering,

$$\hat{x}_{2\times2\times2;\hat{i}_1\hat{i}_2} \equiv \hat{x}_{\hat{i}_1\hat{i}_2} \equiv \hat{x}_{\hat{i}_1} \oplus \hat{x}_{\hat{i}_2}$$

where

$$\hat{i}_{\alpha_1} = 1 + 2^3 u_{\alpha_1 11} + 2^2 u_{\alpha_1 21} + 2u_{\alpha_1 12} + u_{\alpha_1 22},\ \alpha_1 \in \{1,2\}.$$

The ordering matrix $\hat{\mathbb{X}}_{2\times2\times2}$ can be represented as

 where .

[2]Use $x_{i_1 i_2}$ to substitute x_{i_1,i_2} for simplicity afterward.

Now,

$$\psi_{\hat{x}}(\hat{x}_{\hat{i}_1\hat{i}_2}) = 2^4(\hat{i}_1 - 1) + \hat{i}_2$$

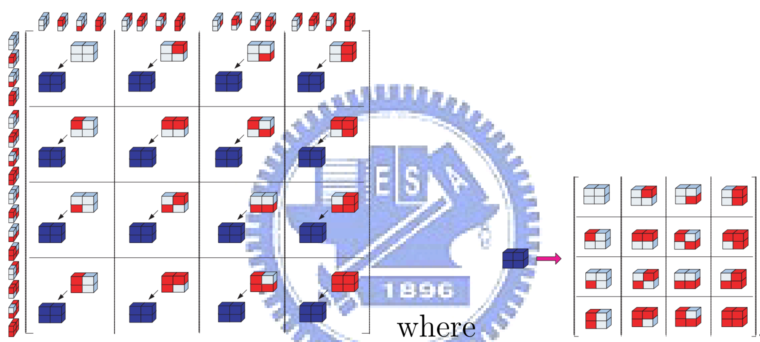
can be verified. Similarly, a $2\times2\times2$ pattern can also be viewed as a $[y]$-direct ($[\hat{y}]$-direct) and $[z]$-direct ($[\hat{z}]$-direct) sum of $2\times1\times2$ and $2\times2\times1$ patterns:

$$\begin{aligned} y_{j_1j_2} &\equiv y_{j_1} \oplus y_{j_2}, \\ \hat{y}_{\hat{j}_1\hat{j}_2} &\equiv \hat{y}_{\hat{j}_1} \oplus \hat{y}_{\hat{j}_2}, \\ z_{k_1k_2} &\equiv z_{k_1} \oplus z_{k_2}, \\ \hat{z}_{\hat{k}_1\hat{k}_2} &\equiv \hat{z}_{\hat{k}_1} \oplus \hat{z}_{\hat{k}_2}, \end{aligned}$$

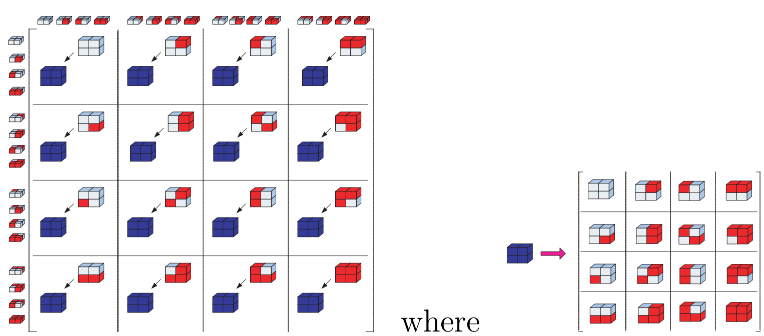
where

$$\begin{aligned} j_{\alpha_2} &= 1 + 2^3 u_{1\alpha_2 1} + 2^2 u_{1\alpha_2 2} + 2u_{2\alpha_2 1} + u_{2\alpha_2 2},\ \alpha_2 \in \{1,2\}, \\ \hat{j}_{\alpha_2} &= 1 + 2^3 u_{1\alpha_2 1} + 2^2 u_{2\alpha_2 1} + 2u_{1\alpha_2 2} + u_{2\alpha_2 2},\ \alpha_2 \in \{1,2\}, \\ k_{\alpha_3} &= 1 + 2^3 u_{11\alpha_3} + 2^2 u_{12\alpha_3} + 2u_{21\alpha_3} + u_{22\alpha_3},\ \alpha_3 \in \{1,2\}, \\ \hat{k}_{\alpha_3} &= 1 + 2^3 u_{11\alpha_3} + 2^2 u_{21\alpha_3} + 2u_{12\alpha_3} + u_{22\alpha_3},\ \alpha_3 \in \{1,2\}. \end{aligned} \tag{3.5}$$

A 16×16 matrix $\mathbb{Y}_{2\times2\times2} = [y_{j_1j_2}]$ or $\mathbb{Z}_{2\times2\times2} = [z_{k_1k_2}]$ can also be obtained for $\Sigma_{2\times2\times2}$, such that $\mathbb{Y}_{2\times2\times2} =$



or $\mathbb{Z}_{2\times2\times2}$



The relationship between $\mathbb{W}_{2\times2\times2}$ must be studied, where $\mathbb{W} \in \{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \hat{\mathbb{X}}, \hat{\mathbb{Y}}, \hat{\mathbb{Z}}\}$. Before the relations are explained, the column matrix and the row matrix are must be given. Let $\mathbb{A} = [a_{ij}]$ be a $m^2 \times m^2$ matrix, the column matrix $\mathbb{A}^{(c)}$ of $\mathbb{A}$ is defined as

$$\mathbb{A}^{(c)} = \begin{bmatrix} A_1^{(c)} & A_2^{(c)} & \cdots & A_m^{(c)} \\ A_{m+1}^{(c)} & A_{m+2}^{(c)} & \cdots & A_{2m}^{(c)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(m-1)m+1}^{(c)} & A_{(m-1)m+2}^{(c)} & \cdots & A_{m^2}^{(c)} \end{bmatrix},$$

$$A_\alpha^{(c)} = \begin{bmatrix} a_{1\alpha} & a_{2\alpha} & \cdots & a_{m\alpha} \\ a_{(m+1)\alpha} & a_{(m+2)\alpha} & \cdots & a_{(2m)\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ a_{((m-1)m+1)\alpha} & a_{((m-1)m+2)\alpha} & \cdots & a_{m^2\alpha} \end{bmatrix},$$

where $1 \le \alpha \le m^2$.

The row matrix $\mathbb{A}^{(r)}$ of $\mathbb{A}$ is defined as

$$\mathbb{A}^{(r)} = \begin{bmatrix} A_1^{(r)} & A_2^{(r)} & \cdots & A_m^{(r)} \\ A_{m+1}^{(r)} & A_{m+2}^{(r)} & \cdots & A_{2m}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(m-1)m+1}^{(r)} & A_{(m-1)m+2}^{(r)} & \cdots & A_{m^2}^{(r)} \end{bmatrix}, \tag{3.6}$$

$$A_\alpha^{(r)} = \begin{bmatrix} a_{\alpha 1} & a_{\alpha 2} & \cdots & a_{\alpha m} \\ a_{\alpha(m+1)} & a_{\alpha(m+2)} & \cdots & a_{\alpha(2m)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha((m-1)m+1)} & a_{\alpha((m-1)m+2)} & \cdots & a_{\alpha m^2} \end{bmatrix}, \tag{3.7}$$

where $1 \le \alpha \le m^2$. Hence, based on some observations, $\mathbb{X}_{2\times2\times2}$ can be represented in terms of $y_{j_1 j_2}$ as

$$\mathbb{X}_{2\times2\times2} = \mathbb{Y}^{(r)}_{2\times2\times2}. \tag{3.8}$$

Furthermore, $\mathbb{Y}_{2\times2\times2} = \mathbb{X}^{(r)}_{2\times2\times2}$, $\mathbb{Z}_{2\times2\times2} = \hat{\mathbb{X}}^{(r)}_{2\times2\times2}$, $\hat{\mathbb{X}}_{2\times2\times2} = \mathbb{Z}^{(r)}_{2\times2\times2}$, $\hat{\mathbb{Y}}_{2\times2\times2} = \hat{\mathbb{Z}}^{(r)}_{2\times2\times2}$ and $\hat{\mathbb{Z}}_{2\times2\times2} = \hat{\mathbb{Y}}^{(r)}_{2\times2\times2}$ can also be obtained. The remainder of this subsection addresses the construction of $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ from $\mathbb{X}_{2\times2\times2}$ in the following three steps, where $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ represents the ordering matrix of $\Sigma_{2\times m_2\times m_3}$ according to $[\hat{x}]$-ordering generated from $\Sigma_{2\times2\times2}$.

Step I : Apply $[x]$-ordering to $\mathbf{Z}_{1\times m_2\times 2}$ using

2	4	...	2k	...	2m$_2$-2	2m$_2$
1	3	...	2k-1	...	2m$_2$-3	2m$_2$-1

(3.9)

→ y

and introduce ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$.

Step II : Convert $[x]$-ordering into $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times 2}$ by

z ↑

m$_2$+1	m$_2$+2	...	m$_2$+k	...	2m$_2$
1	2	...	k	...	m$_2$

(3.10)

and introduce ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$.

Step III : Define $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times m_3}$ by

z ↑

$(m_3-1)m_2+1$	$(m_3-1)m_2+2$	...	m_3m_2-1	m_3m_2
:	:	:	:	:
m_2+1	m_2+2	...	$2m_2-1$	$2m_2$
1	2	...	m_2-1	m_2

(3.11)

and introduce ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$.

To introduce $\mathbb{X}_{2\times m_2\times 2}$, define

$$\begin{aligned} y_{2\times m_2\times 2;j_1j_2\dots j_{m_2}} &\equiv y_{2\times 2\times 2;j_1j_2}\hat{\oplus}y_{2\times 2\times 2;j_2j_3}\hat{\oplus}\cdots\hat{\oplus}y_{2\times 2\times 2;j_{m_2-1}j_{m_2}} \\ &\equiv y_{j_1}\oplus y_{j_2}\oplus\cdots\oplus y_{j_{m_2}}, \end{aligned} \tag{3.12}$$

where $1\leq j_k\leq 2^4$ and $1\leq k\leq m_2$. Herein, a wedge direct sum $\hat{\oplus}$ is applied to $2\times 2\times 2$ patterns whenever they can be attached to each other.

Now, $\mathbb{X}_{2\times m_2\times 2}$ can be obtained as follows.

Theorem 3.1.1. *For any $m_2\geq 2$, $\Sigma_{2\times m_2\times 2}=\{y_{j_1j_2\dots j_{m_2}}\}$, where $y_{j_1j_2\dots j_{m_2}}$ is given in Eq. (3.12). Furthermore, the ordering matrix $\mathbb{X}_{2\times m_2\times 2}=[y_{j_1j_2\dots j_{m_2}}]$ which is a $2^{2m_2}\times 2^{2m_2}$ matrix can be decomposed into following matrices*

$$\mathbb{X}_{2\times m_2\times 2}=[X_{2\times m_2\times 2;j_1}]_{2^2\times 2^2},$$

where $1\leq j_1\leq 2^4$. For fixed $j_1,j_2,\dots,j_k\in\{1,2,\dots,2^4\}$,

$$X_{2\times m_2\times 2;j_1j_2\dots j_k}=[X_{2\times m_2\times 2;j_1j_2\dots j_kj_{k+1}}]_{2^2\times 2^2},$$

where $1\leq j_{k+1}\leq 2^4$ and $k\in\{1,2,\cdots,m_2-2\}$. For fixed $j_1,j_2,\cdots,j_{m_2-1}$,

$$X_{2\times m_2\times 2;j_1j_2\dots j_{m_2-1}}=[y_{2\times m_2\times 2;j_1j_2\dots j_{m_2-1}j_{m_2}}]_{2^2\times 2^2},$$

where $y_{2\times m_2\times 2;j_1j_2\dots j_{m_2}}$ is defined as in Eq. (3.12).

Proof. From Eq. (3.5), $u_{\alpha_1\alpha_2\alpha_3}$ can be solved in terms of j_{α_2}, yielding

$$u_{1\alpha_2 1} = [\frac{j_{\alpha_2}-1}{2^3}], \tag{3.13}$$

$$u_{1\alpha_2 2} = [\frac{j_{\alpha_2}-1-2^3u_{1\alpha_2 1}}{2^2}], \tag{3.14}$$

$$u_{2\alpha_2 1} = [\frac{j_{\alpha_2}-1-2^3u_{1\alpha_2 1}-2^2u_{1\alpha_2 2}}{2}], \tag{3.15}$$

$$u_{2\alpha_2 2} = j_{\alpha_2}-1-2^3u_{1\alpha_2 1}-2^2u_{1\alpha_2 2}-2u_{2\alpha_2 1}, \tag{3.16}$$

where [ ] is the Gauss symbol. Equations (3.13)-(3.16), yield the following table.

j_{α_2}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$u_{1\alpha_2 1}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$u_{1\alpha_2 2}$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
$u_{2\alpha_2 1}$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$u_{2\alpha_2 2}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

For any $m_2 \geq 2$, we have

$$i_{m_2;1} = 1 + \sum_{\alpha_2=1}^{m_2} (2^{2(m_2-\alpha_2)+1} u_{1\alpha_2 1} + 2^{2(m_2-\alpha_2)} u_{1\alpha_2 2}),$$
$$i_{m_2;2} = 1 + \sum_{\alpha_2=1}^{m_2} (2^{2(m_2-\alpha_2)+1} u_{2\alpha_2 1} + 2^{2(m_2-\alpha_2)} u_{2\alpha_2 2}).$$

From the above formulae,

$$i_{m_2+1;1} = 2^2(i_{m_2;1} - 1) + 2u_{1(m_2+1)1} + u_{1(m_2+1)2} + 1,$$
$$i_{m_2+1;2} = 2^2(i_{m_2;2} - 1) + 2u_{2(m_2+1)1} + u_{2(m_2+1)2} + 1.$$

Now, by induction on m_2 the theorem follows from the last two formulae and the above table. The proof is complete. □

Remark 3.1.2. *By the same method, the following relations can be derived. The details of the proof are omitted here for brevity.*

$$\begin{aligned}
\hat{\mathbb{X}}_{2\times 2\times m_3} &= [z_{2\times 2\times m_3;k_1k_2\ldots k_{m_3-1}k_{m_3}}]_{2^{2m_3}\times 2^{2m_3}} \\
\mathbb{Y}_{m_1\times 2\times 2} &= [x_{m_1\times 2\times 2;i_1i_2\ldots i_{m_1-1}i_{m_1}}]_{2^{2m_1}\times 2^{2m_1}} \\
\hat{\mathbb{Y}}_{2\times 2\times m_3} &= [\hat{z}_{2\times 2\times m_3;\hat{k}_1\hat{k}_2\ldots\hat{k}_{m_3-1}\hat{k}_{m_3}}]_{2^{2m_3}\times 2^{2m_3}} \\
\mathbb{Z}_{m_1\times 2\times 2} &= [\hat{x}_{m_1\times 2\times 2;\hat{i}_1\hat{i}_2\ldots\hat{i}_{m_1-1}\hat{i}_{m_1}}]_{2^{2m_1}\times 2^{2m_1}} \\
\hat{\mathbb{Z}}_{2\times m_2\times 2} &= [\hat{y}_{2\times m_2\times 2;\hat{j}_1\hat{j}_2\ldots\hat{j}_{m_2-1}\hat{j}_{m_2}}]_{2^{2m_2}\times 2^{2m_2}}
\end{aligned}$$

Next, $[x]$-ordering is converted into $[\hat{x}]$-ordering for $\mathbf{Z}_{1\times m_2\times 2}$. Since $\mathbf{Z}_{1\times m_2\times 2} = \{(1, \alpha_2, \alpha_3) : 1 \leq \alpha_2 \leq m_2, 1 \leq \alpha_3 \leq 2\}$, the position (α_2, α_3) is the α-th in Eq. (3.9), where

$$\alpha = 2(\alpha_2 - 1) + \alpha_3. \tag{3.17}$$

In Eq. (3.10), the position of $(1, \alpha_2, \alpha_3)$ is the $\hat{\alpha}$-th, where

$$\hat{\alpha} = m_2(\alpha_3 - 1) + \alpha_2.$$

The relation

$$\hat{\alpha} = m_2\alpha + (1 - 2m_2)[\frac{\alpha - 1}{2}] + (1 - m_2),$$

or

$$\hat{\alpha} = k \quad if \ \ \alpha = 2k - 1,$$

and

$$\hat{\alpha} = m_2 + k \quad if \ \ \alpha = 2k,$$

$1 \leq k \leq m_2$ is easily verified.

Now, the ordering $[\hat{x}]$ in Eq. (3.10) on $\mathbf{Z}_{1\times m_2\times 2}$ can be extended to $\mathbf{Z}_{1\times m_2\times m_3}$ by Eq. (3.11). For a fixed m_2, $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times m_3}$ is clearly one-dimensional; it grows in the z-direction. Given ordering Eq. (3.11) on $\mathbf{Z}_{1\times m_2\times m_3}$, for $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{2\times m_2\times m_3}$, denoted by

$$\hat{i}_{\alpha_1} = 1 + \sum_{\alpha_2=1}^{m_2}\sum_{\alpha_3=1}^{m_3} u_{\alpha_1\alpha_2\alpha_3} 2^{m_2(m_3-\alpha_3)+(m_2-\alpha_2)},$$

where $\alpha_1 = 1, 2$,

$$\psi_{\hat{x}}(U) = 2^{m_2 m_3}(\hat{i}_1 - 1) + \hat{i}_2.$$

Now, let $\hat{x}_{\hat{i}_1\hat{i}_2} = U = (u_{\alpha_1\alpha_2\alpha_3})$, yielding the new ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times 2} = [\hat{x}_{2\times m_2\times 2;\hat{i}_1\hat{i}_2}]$ for $\Sigma_{2\times m_2\times 2}$. The relationship between $\mathbb{X}_{2\times m_2\times 2}$ and $\hat{\mathbb{X}}_{2\times m_2\times 2}$ is established before $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ is constructed from $\hat{\mathbb{X}}_{2\times m_2\times 2}$ for $m_3 \geq 3$.

A conversion sequence of orderings can be obtained from Eqs. (3.9)-(3.10). Where P_k represents the permutation of $\mathbb{N}_{2m} = \{1, 2, \cdots, 2m_2\}$ such that $P_k(k+1) = k$, $P_k(k) = k+1$ and the other numbers are fixed. P_k is also the permutation on $\mathbf{Z}_{1\times m_2\times 2}$ such that it exchanges k and k+1 while keeping the other positions fixed, i.e.,

$\cdot$		$k+1$		$\cdot$		$\cdot$
$\cdot$		$\cdot$		k		$\cdot$

$\xrightarrow{P_k}$

$\cdot$		k		$\cdot$		$\cdot$
$\cdot$		$\cdot$		$k+1$		$\cdot$

.

Clearly, equation (3.9) can be converted into Eq. (3.10) in many ways using the sequence of P_k. A systematic approach is proposed here.

Lemma 3.1.3. *For $m_2 \geq 2$, equation (3.9) can be converted into Eq. (3.10) using the following sequences of $\frac{m_2(m_2-1)}{2}$ permutations successively*

$$\begin{aligned}&(P_2P_4\cdots P_{2m_2-2})(P_3P_5\cdots P_{2m_2-3})\cdots\\&(P_kP_{k+2}\cdots P_{2m_2-k})\cdots(P_{m_2-1}P_{m_2+1})P_{m_2},\end{aligned} \tag{3.18}$$

$2 \leq k \leq m_2$.

Proof. When $m_2 = 2$ and 3, verifying that Eq. (3.18) can convert Eq. (3.9) into Eq. (3.10) is relatively simple.

When $m_2 \geq 4$, and for any $2 \leq k \leq m_2$, applying

$$(P_2P_4\cdots P_{2m_2-2})(P_3P_5\cdots P_{2m_2-3})\cdots(P_kP_{k+2}\cdots P_{2m_2-k})$$

to Eq. (3.9), yields two intermediate cases:
(i) when $2 \leq k \leq [\frac{m_2}{2}]$,

$k+1$	$k+3$	$\cdots$	$3k-1$	$\cdots$	$\cdots$	$\cdots$	$3k-1+2\ell$	$\cdots$	$2m_2-k-1$	$2m_2-k+1$	$\cdots$	$2m_2-1$	$2m_2$
1	2	$\cdots$	k	$k+2$	$k+4$	$\cdots$	$k+2\ell$	$\cdots$	$\cdots$	$2m_2-3k+1$	$\cdots$	$2m_2-k-2$	$2m_2-k$

, (3.19)

where $0 \leq \ell \leq m_2 - 2k$.
(ii) when $[\frac{m_2}{2}] + 1 \leq k \leq m_2 - 1$,

$k+1$	$\cdots$	$2m_2-k-1$	$2m_2-k+1$	$2m_2-k+2$	$\cdots$	$\cdots$	$\cdots$	$2m_2-1$	$2m_2$
1	2	$\cdots$	$\cdots$	$\cdots$	$k-1$	k	$k+2$	$\cdots$	$2m_2-k$

. (3.20)

When $k = m_2$ in Eq. (3.20), equation (3.10) holds. Equation (3.19) and Eq. (3.20) are established by mathematical induction on k. When k=2, verifying that Eq. (3.9) is converted into

3	5	$\cdots$	$\cdots$	$\cdots$	$2m_2-3$	$2m_2-1$	$2m_2$
1	2	4	$\cdots$	$\cdots$	$\cdots$	$2m_2-4$	$2m_2-2$

by $P_2P_4\cdots P_{2m_2-2}$ is relatively easy such that Eq. (3.19) holds for k=2. Next, assume that Eq. (3.19) holds for $k \leq [\frac{m}{2}]$. Then, by applying $P_{k+1}P_{k+3}\cdots P_{2m_2-k-1}$ to Eq. (3.19), equation (3.19) can be verified to hold for $k+1$ when $k+1 \leq [\frac{m_2}{2}]$ or becomes Eq. (3.20) when $k+1 \geq [\frac{m_2}{2}]$. When $k \geq [\frac{m_2}{2}]+1$, $P_{k+1}P_{k+3}\cdots P_{2m_2-k-1}$ is applied to Eq. (3.20). Equation (3.20) can also be confirmed to hold for $k+1$. Finally, equation (3.10) is concluded to hold for $k = m_2$. The proof is thus complete. □

Based on Lemma 3.1.3, $\mathbb{X}_{2\times m_2\times 2}$ can be converted into $\hat{\mathbb{X}}_{2\times m_2\times 2}$ as follows. Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.21}$$

and for $2 \leq j \leq 2m_2-2$, as denoted by

$$P_{2m_2;j} = I_{2^{j-1}} \otimes P \otimes I_{2^{2m_2-j-1}},$$

where I_k is the $k\times k$ identity matrix. Moreover, let

$$\begin{aligned} \mathbb{P}_{x;2\times m_2\times 2} &= (P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2}) \\ &\quad \cdots (P_{2m_2;k}\cdots P_{2m_2;2m_2-k})\cdots(P_{2m_2;m_2}), \end{aligned} \tag{3.22}$$

$2 \leq k \leq m_2$. Then, the following theorem holds.

Theorem 3.1.4. *For any $m_2 \geq 2$,*

$$\hat{\mathbb{X}}_{2\times m_2\times 2} = \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{X}_{2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}. \tag{3.23}$$

Proof. From Eq. (3.17), in $\mathbf{Z}_{1\times m_2\times 2}$ the position (α_2,α_3) is the α-th in Eq. (3.9), where $\alpha = 2(\alpha_2-1)+\alpha_3$. Define

$$\ell_\alpha \equiv 1 + 2u_{1\alpha_2\alpha_3} + u_{2\alpha_2\alpha_3},$$

$1 \leq \ell_\alpha \leq 4$ and $1 \leq \alpha \leq 2m_2$. For $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{2\times m_2\times 2}$, from Theorem 3.1.1 it can be denoted by $y_{2\times m_2\times 2;j_1j_2\ldots j_{m_2}}$ and by Eq. (3.5) for fixed $1 \leq \alpha_2 \leq m_2$:

$$j_{\alpha_2} = 1 + 2^3u_{1\alpha_21} + 2^2u_{1\alpha_22} + 2u_{2\alpha_21} + u_{2\alpha_22},$$

where $1 \leq j_{\alpha_2} \leq 16$. Accordingly, $y_{j_{\alpha_2}}$ can be represented by $y_{\ell_{2\alpha_2-1}\ell_{2\alpha_2}}$ and the relation is

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \\ y_9 & y_{10} & y_{11} & y_{12} \\ y_{13} & y_{14} & y_{15} & y_{16} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{21} & y_{22} \\ y_{13} & y_{14} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{41} & y_{42} \\ y_{33} & y_{34} & y_{43} & y_{44} \end{bmatrix}.$$

Therefore, from Eq. (3.12) patterns in ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ can be specified by

$$\begin{aligned} y_{2\times m_2\times 2;j_1j_2\ldots j_{m_2}} &= y_{j_1}\oplus y_{j_2}\oplus\cdots\oplus y_{j_{m_2}} \\ &= y_{\ell_1\ell_2}\oplus y_{\ell_3\ell_4}\oplus\cdots\oplus y_{\ell_{2m_2-1}\ell_{2m_2}} \\ &\equiv y_{\ell_1\ell_2\ldots\ell_{2m_2}}. \end{aligned}$$

For any $1\le k\le 2m_2-1$,

$$\begin{aligned} &P^t_{2m_2;k}\mathbb{X}_{2\times m_2\times 2}P_{2m_2;k} \\ &= P^t_{2m_2;k}[y_{\ell_1\ell_2\ldots\ell_k\ell_{k+1}\ldots\ell_{2m_2}}]P_{2m_2;k} \\ &= [y_{\ell_1\ell_2\ldots\ell_{k+1}\ell_k\ldots\ell_{2m_2}}] \end{aligned}$$

is easily verified, such that $P_{2m_2;k}$ exchanges ℓ_k and ℓ_{k+1} in $\mathbb{X}_{2\times m_2\times 2}$. Therefore, Eq. (3.23) follows from Eq. (3.22) and Lemma 3.1.3. □

Now, according to Theorem 3.1.4,

$$\hat{\mathbb{X}}_{2\times m_2\times 2} = [\hat{x}_{2\times m_2\times 2;\hat{i}_1\hat{i}_2}],$$

$1\le \hat{i}_1,\hat{i}_2\le 2m_2$. From some observations as Eq. (3.8), $\hat{\mathbb{X}}_{2\times m_2\times 2}$ can be represented as $z_{2\times m_2\times 2;k_1k_2}$, where $1\le k_1,k_2\le 2^{2m_2}$. The $[\hat{x}]$-expression

$$\hat{\mathbb{X}}_{2\times m_2\times 2} = \mathbb{Z}^{(r)}_{2\times m_2\times 2} \tag{3.24}$$

for $\Sigma_{2\times m_2\times 2}$ enables $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ to be constructed for $\Sigma_{2\times m_2\times m_3}$. Indeed, for fixed $m_2\ge 2$ and $m_3\ge 2$, let

$$\begin{aligned} \hat{x}_{2\times m_2\times m_3;\hat{i}_1\hat{i}_2} &\equiv z_{2\times m_2\times m_3;k_1k_2\ldots k_{m_3}} \\ &\equiv z_{2\times m_2\times 2;k_1k_2}\hat{\oplus} z_{2\times m_2\times 2;k_2k_3}\hat{\oplus}\cdots\hat{\oplus} z_{2\times m_2\times 2;k_{m_3-1}k_{m_3}}. \end{aligned} \tag{3.25}$$

Therefore, by a similar argument as was used to establish Theorem 3.1.1 the following theorem holds for $\hat{\mathbb{X}}_{2\times m_2\times m_3}$, the detailed proofs are omitted for brevity.

Theorem 3.1.5. *For fixed $m_2\ge 2$ and for any $m_3\ge 2$, the ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ with respect to $[\hat{x}]$-ordering can be expressed as*

$$\hat{\mathbb{X}}_{\hat{x};2\times m_2\times m_3} = [\hat{X}_{2\times m_2\times m_3;k_1}]_{2^{m_2}\times 2^{m_2}},$$

where $1\le k_1\le 2^{2m_2}$. For fixed $1\le k_1,k_2,\cdots,k_l\le 2^{2m_2}$,

$$\hat{X}_{2\times m_2\times m_3;k_1k_2\cdots k_l} = [\hat{X}_{2\times m_2\times m_3;k_1k_2\cdots k_lk_{l+1}}]_{2^{m_2}\times 2^{m_2}}$$

where $1\le k_{l+1}\le 2^{2m_2}$ and $1\le l\le m_3-2$. For fixed $k_1,k_2,\cdots,k_{m_3-1}$,

$$\hat{X}_{2\times m_2\times m_3;k_1k_2\cdots k_{m_3-1}} = [z_{2\times m_2\times m_3;k_1k_2\ldots k_{m_3}}],$$

where $z_{2\times m_2\times m_3;k_1k_2\ldots k_{m_3}}$ is given by Eq. (3.25).

Remark 3.1.6. *Similarly, according to other orderings, the following relations can be derived*

$$\begin{aligned} \mathbb{X}_{2\times m_2\times m_3} &= [y_{2\times m_2\times m_3;j_1j_2\ldots j_{m_2}}]_{2^{m_2m_3}\times 2^{m_2m_3}} \\ \hat{\mathbb{Y}}_{m_1\times 2\times m_3} &= [\hat{z}_{m_1\times 2\times m_3;\hat{k}_1\hat{k}_2\ldots\hat{k}_{m_3}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\ \mathbb{Y}_{m_1\times 2\times m_3} &= [x_{m_1\times 2\times m_3;i_1i_2\ldots i_{m_1}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\ \hat{\mathbb{Z}}_{m_1\times m_2\times 2} &= [\hat{y}_{m_1\times m_2\times 2;\hat{j}_1\hat{j}_2\ldots\hat{j}_{m_2}}]_{2^{m_1m_2}\times 2^{m_1m_2}} \\ \mathbb{Z}_{m_1\times m_2\times 2} &= [\hat{x}_{m_1\times m_2\times 2;\hat{i}_1\hat{i}_2\ldots\hat{i}_{m_1}}]_{2^{m_1m_2}\times 2^{m_1m_2}}. \end{aligned}$$

3.1.2 Transition matrices

Based on the definitions of the ordering matrices $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$ having been defined, high order transition matrices $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can now be derived from $\mathbb{A}_{x;2\times 2\times 2}$. As in the two-dimensional case [4], a basic set $\mathcal{B}\subset\Sigma_{2\times 2\times 2}$ is assumed to give. Define the transition matrix $\mathbb{A}_{x;2\times 2\times 2}=\mathbb{A}_{x;2\times 2\times 2}(\mathcal{B})$ by

$$\mathbb{A}_{x;2\times 2\times 2}=[a_{x;2\times 2\times 2;i_1i_2}]_{2^4\times 2^4}, \tag{3.26}$$

where

$$\begin{aligned} a_{x;2\times 2\times 2;i_1i_2} &= 1 \quad \text{if } x_{i_1i_2}\in\mathcal{B}, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{3.27}$$

Then, the transition matrix $\mathbb{A}_{x;2\times m_2\times 2}$ is a $2^{2m_2}\times 2^{2m_2}$ matrix with entries $a_{x;2\times m_2\times 2;i_1i_2}$[3], where

$$\begin{aligned} a_{x;2\times m_2\times 2;i_1i_2} &= a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}} \\ &= \prod_{k=1}^{m_2-1} a_{y;2\times 2\times 2;j_kj_{k+1}}. \end{aligned} \tag{3.28}$$

Before $\mathbb{A}_{x;2\times m_2\times 2}$ is introduced, three products of the matrices are defined as follows.

Definition 3.1.7. *For any two matrices $\mathbb{M}=(M_{ij})$ and $\mathbb{N}=(N_{kl})$, the Kronecker product (tensor product) $\mathbb{M}\otimes\mathbb{N}$ of $\mathbb{M}$ and $\mathbb{N}$ is defined by*

$$\mathbb{M}\otimes\mathbb{N}=(M_{ij}\mathbb{N}).$$

For any $n\geq 1$,

$$\otimes\mathbb{N}^n=\mathbb{N}\otimes\mathbb{N}\otimes\cdots\otimes\mathbb{N},$$

n-times in $\mathbb{N}$.
Next, for any two $m\times m$ matrices

$$\mathbb{P}=(P_{ij}) \text{ and } \mathbb{Q}=(Q_{ij})$$

where P_{ij} and Q_{ij} are numbers or matrices, the Hadamard product $\mathbb{P}\circ\mathbb{Q}$ is defined by

$$\mathbb{P}\circ\mathbb{Q}=(P_{ij}\cdot Q_{ij}),$$

where the product $P_{ij}\cdot Q_{ij}$ of P_{ij} and Q_{ij} may be a multiplication of numbers, of numbers and matrices or of matrices whenever it is well-defined.

Finally, product $\hat{\otimes}$ is defined as follows. For any 4×4 matrix

$$\mathbb{M}_2=\begin{bmatrix} m_{11} & m_{12} & m_{21} & m_{22} \\ m_{13} & m_{14} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{41} & m_{42} \\ m_{33} & m_{34} & m_{43} & m_{44} \end{bmatrix}=\begin{bmatrix} M_{2;1} & M_{2;2} \\ M_{2;3} & M_{2;4} \end{bmatrix}$$

and any 2×2 matrix

$$\mathbb{N}=\begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix},$$

[3]Use $a_{x;2\times 2\times 2;i_1i_2}$ to substitute $a_{x;2\times 2\times 2;i_1,i_2}$ for simplicity afterward.

where m_{ij} are numbers and N_k are numbers or matrices, for $1 \leq i, j, k \leq 4$, define

$$\mathbb{M}_2 \hat{\otimes} \mathbb{N} = \begin{bmatrix} m_{11}N_1 & m_{12}N_2 & m_{21}N_1 & m_{22}N_2 \\ m_{13}N_3 & m_{14}N_4 & m_{23}N_3 & m_{24}N_4 \\ m_{31}N_1 & m_{32}N_2 & m_{41}N_1 & m_{42}N_2 \\ m_{33}N_3 & m_{34}N_4 & m_{43}N_3 & m_{44}N_4 \end{bmatrix}.$$

Furthermore, for $n \geq 1$, the $n+1$-th order of transition matrix of $\mathbb{M}_2$ is defined by

$$\mathbb{M}_{n+1} \equiv \hat{\otimes}\mathbb{M}_2^n = \mathbb{M}_2 \hat{\otimes} \mathbb{M}_2 \hat{\otimes} \cdots \hat{\otimes} \mathbb{M}_2,$$

n-times in $\mathbb{M}_2$. More precisely,

$$\mathbb{M}_{n+1} = \mathbb{M}_2 \hat{\otimes} (\hat{\otimes}\mathbb{M}_2^{n-1}) = \begin{bmatrix} M_{2;1} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;2} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \\ M_{2;3} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;4} \circ (\hat{\otimes}\mathbb{M}_2^{n-1}) \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} m_{11}M_{n;1} & m_{12}M_{n;2} & m_{21}M_{n;1} & m_{22}M_{n;2} \\ m_{13}M_{n;3} & m_{14}M_{n;4} & m_{23}M_{n;3} & m_{24}M_{n;4} \\ \hline m_{31}M_{n;1} & m_{32}M_{n;2} & m_{41}M_{n;1} & m_{42}M_{n;2} \\ m_{33}M_{n;3} & m_{34}M_{n;4} & m_{43}M_{n;3} & m_{44}M_{n;4} \end{array}\right] = \begin{bmatrix} M_{n+1;1} & M_{n+1;2} \\ M_{n+1;3} & M_{n+1;4} \end{bmatrix},$$

where

$$\mathbb{M}_n = \hat{\otimes}\mathbb{M}_2^{n-1} = \begin{bmatrix} M_{n;1} & M_{n;2} \\ M_{n;3} & M_{n;4} \end{bmatrix}.$$

Here, the following convention is adopted,

$$\hat{\otimes}\mathbb{M}_2^0 = \mathbb{E}_2.$$

where where $\mathbb{E}_2$ is the 2×2 matrix with 1 as its entries.

Theorem 3.1.1, yields results for $\mathbb{A}_{x;2\times m_2\times 2}$ as $\mathbb{T}_n$ in Theorem 3.1 in [4]. Indeed,

Theorem 3.1.8. *Let $\mathbb{A}_{x;2\times 2\times 2}$ be a transition matrix that is given by Eq. (3.26) and Eq. (3.27). Then, for high order transition matrices $\mathbb{A}_{x;2\times m_2\times 2}$, $m_2 \geq 3$, the following three equivalent statements hold:*
(I) $\mathbb{A}_{x;2\times m_2\times 2}$ can be decomposed into m_2 successive 4×4 matrices

$$\mathbb{A}_{x;2\times m_2\times 2} \quad = \quad [A_{x;2\times m_2\times 2;j_1}]_{4\times 4},$$

where $1 \leq j_1 \leq 16$. For fixed $1 \leq j_1, j_2, \ldots, j_k \leq 16$,

$$A_{x;2\times m_2\times 2;j_1j_2\ldots j_k} = [A_{x;2\times m_2\times 2;j_1j_2\ldots j_kj_{k+1}}]_{4\times 4},$$

where $1 \leq j_{k+1} \leq 16$ and $1 \leq k \leq m_2 - 1$. For fixed $j_1, j_2, \ldots, j_{m_2-1} \in \{1, 2, \ldots, 16\}$,

$$A_{x;2\times m_2\times 2;j_1j_2\ldots j_{m_2-1}} = [a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}}]_{4\times 4},$$

where $a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}}$ is defined in Eq. (3.28).
(II) Starting from

$$\mathbb{A}_{x;2\times 2\times 2} = [A_{x;2\times 2\times 2;j_1}]_{4\times 4}$$

and

$$A_{x;2\times2\times2;j_1} = [a_{y;2\times2\times2;j_1j_2}]_{4\times4},$$

for $m_2 \geq 3$, $\mathbb{A}_{x;2\times m_2\times2}$ *can be obtained from* $\mathbb{A}_{x;2\times(m_2-1)\times2}$ *by replacing* $A_{x;2\times2\times2;j_1}$ *with*

$$(A_{x;2\times2\times2;j_1})_{4\times4} \circ (\mathbb{A}_{x;2\times2\times2})_{4\times4}.$$

(III) For $m_2 \geq 3$,

$$\mathbb{A}_{x;2\times m_2\times2} = (\mathbb{A}_{x;2\times(m_2-1)\times2})_{2^{2(m_2-1)}\times2^{2(m_2-1)}} \circ (E_{2^{2(m_2-2)}} \otimes \mathbb{A}_{x;2\times2\times2}), \qquad (3.29)$$

where E_{2^k} *is the* $2^k \times 2^k$ *matrix with* 1 *as its entries.*

Proof. (I) The proof involves simply replacing $X_{2\times m_2\times2;j_1j_2\ldots j_k}$ and $y_{2\times m_2\times2;j_1j_2\ldots j_{m_2}}$ by $A_{x;2\times m_2\times2;j_1j_2\ldots j_k}$ and $a_{y;2\times m_2\times2;j_1j_2\ldots j_{m_2}}$ in Theorem 3.1.1, respectively.
(II) follows directly from (I).
(III) follows from (I); $\mathbb{A}_{x;2\times m_2\times2} = [A_{x;2\times m_2\times2;j_1}]$, $1 \leq j_1 \leq 2^4$. (I) yields the following formula;

$$\begin{aligned}\mathbb{A}_{x;2\times m_2\times2} &= [a_{y;2\times2\times2;j_1j_2}A_{x;2\times(m_2-1)\times2;j_2}]\\ &= (\mathbb{A}_{x;2\times(m_2-1)\times2})_{2^{2(m_2-1)}\times2^{2(m_2-1)}}\hat{\otimes}[E_{2^{2(m_2-2)}} \otimes \mathbb{A}_{x;2\times2\times2}].\end{aligned}$$

The proof is complete. □

Remark 3.1.9. *As stated in Remark 3.1.2, the following formulae apply*

$$\begin{aligned}\mathbb{A}_{\hat{x};2\times2\times m_3} &= [a_{z;2\times2\times m_3;k_1k_2\ldots k_{m_3-1}k_{m_3}}]_{2^{2m_3}\times2^{2m_3}}\\ \mathbb{A}_{y;m_1\times2\times2} &= [a_{x;m_1\times2\times2;i_1i_2\ldots i_{m_1-1}i_{m_1}}]_{2^{2m_1}\times2^{2m_1}}\\ \mathbb{A}_{\hat{y};2\times2\times m_3} &= [a_{\hat{z};2\times2\times m_3;\hat{k}_1\hat{k}_2\ldots\hat{k}_{m_3-1}\hat{k}_{m_3}}]_{2^{2m_3}\times2^{2m_3}}\\ \mathbb{A}_{z;m_1\times2\times2} &= [a_{\hat{x};m_1\times2\times2;\hat{i}_1\hat{i}_2\ldots\hat{i}_{m_1-1}\hat{i}_{m_1}}]_{2^{2m_1}\times2^{2m_1}}\\ \mathbb{A}_{\hat{z};2\times m_2\times2} &= [a_{\hat{y};2\times m_2\times2;\hat{j}_1\hat{j}_2\ldots\hat{j}_{m_2-1}\hat{j}_{m_2}}]_{2^{2m_2}\times2^{2m_2}}.\end{aligned}$$

Now, the transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times2}$, with respect to the ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times2}$ can be obtained. Additionally, by using Theorem 3.1.4 yields

Theorem 3.1.10.

$$\mathbb{A}_{\hat{x};2\times m_2\times2} = \mathbb{P}^t_{x;2\times m_2\times2}\mathbb{A}_{x;2\times m_2\times2}\mathbb{P}_{x;2\times m_2\times2}.$$

Proof. The proof involves simply replacing $y_{2\times m_2\times2;j_1j_2\ldots j_{m_2}}$ by $a_{y;2\times m_2\times2;j_1j_2\ldots j_{m_2}}$ in Theorem 3.1.4. □

Theorem 3.1.5 yields transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{\hat{x};2\times m_2\times2}$. Equation (3.24) yields the transition matrix

$$\mathbb{A}_{\hat{x};2\times m_2\times2} = [A_{\hat{x};2\times m_2\times2;k_1}] \qquad (3.30)$$

and

$$A_{\hat{x};2\times m_2\times2;k_1} = [a_{z;2\times m_2\times2;k_1k_2}]. \qquad (3.31)$$

Therefore,

Theorem 3.1.11. *Let $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ be a transition matrix given by Eq. (3.30) and Eq. (3.31). Then, for high order transition matrices $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, $m_2 \geq 3$, we have the following three equivalent statements hold,*
(I) $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be decomposed into m_3 successive $2^{m_2} \times 2^{m_2}$ matrices:

$$\mathbb{A}_{\hat{x};2\times m_2\times m_3} = [A_{\hat{x};2\times m_2\times m_3;k_1}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_1 \leq 2^{2m_2}$. For fixed $1 \leq k_1, k_2, \ldots, k_\ell \leq 2^{2m_2}$,

$$A_{\hat{x};2\times m_2\times m_3;k_1k_2\ldots k_\ell} = [A_{\hat{x};2\times m_2\times m_3;k_1k_2\ldots k_\ell k_{\ell+1}}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_{\ell+1} \leq 2^{2m_2}$ and $1 \leq \ell \leq m_3 - 2$,

$$A_{\hat{x};2\times m_2\times m_3;k_1k_2\ldots k_{m_3-1}} = [a_{z;2\times m_2\times m_3;k_1k_2\ldots k_{m_3}}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_{m_3} \leq 2^{2m_2}$ and by Eq. (3.25)

$$a_{z;2\times m_2\times m_3;k_1k_2\ldots k_{m_3}} = \prod_{\ell=1}^{m_3-1} a_{z;2\times m_2\times 2;k_\ell k_{\ell+1}}.$$

(II) For any $m_3 \geq 3$, $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be obtained from $\mathbb{A}_{\hat{x};2\times m_2\times (m_3-1)}$ by replacing $A_{\hat{x};2\times m_2\times 2;k_1}$ with

$$(A_{\hat{x};2\times m_2\times 2;k_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{A}_{\hat{x};2\times m_2\times 2})_{2^{m_2}\times 2^{m_2}}.$$

(III) Furthermore, for $m_3 \geq 3$,

$$\begin{aligned}\mathbb{A}_{\hat{x};2\times m_2\times m_3} &= (\mathbb{A}_{\hat{x};2\times m_2\times (m_3-1)})_{2^{m_2(m_3-1)}\times 2^{m_2(m_3-1)}} \\ &\quad \circ (E_{2^{m_2(m_3-2)}} \otimes \mathbb{A}_{\hat{x};2\times m_2\times 2}).\end{aligned} \tag{3.32}$$

The proof closely resembles that of Theorem 3.1.1 and Theorem 3.1.8. Details of the proof can be omitted since obvious and repeated.

Remark 3.1.12. *As in Remark 3.1.6, the following formulae are obtained*

$$\begin{aligned}
\mathbb{A}_{x;2\times m_2\times m_3} &= [a_{y;2\times m_2\times m_3;j_1j_2\ldots j_{m_2}}]_{2^{m_2m_3}\times 2^{m_2m_3}} \\
\mathbb{A}_{\hat{y};m_1\times 2\times m_3} &= [a_{\hat{z};m_1\times 2\times m_3;\hat{k}_1\hat{k}_2\ldots\hat{k}_{m_3}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\
\mathbb{A}_{y;m_1\times 2\times m_3} &= [a_{x;m_1\times 2\times m_3;i_1i_2\ldots i_{m_1}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\
\mathbb{A}_{\hat{z};m_1\times m_2\times 2} &= [a_{\hat{y};m_1\times m_2\times 2;\hat{j}_1\hat{j}_2\ldots\hat{j}_{m_2}}]_{2^{m_1m_2}\times 2^{m_1m_2}} \\
\mathbb{A}_{z;m_1\times m_2\times 2} &= [a_{\hat{x};m_1\times m_2\times 2;\hat{i}_1\hat{i}_2\ldots\hat{i}_{m_1}}]_{2^{m_1m_2}\times 2^{m_1m_2}}.
\end{aligned}$$

Finally, the spatial entropy $h(\mathcal{B})$ can be computed from the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$ of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$. Indeed,

Theorem 3.1.13. *Let $\lambda_{\hat{x};2,m_2,m_3}$ be the maximum eigenvalue of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, then*

$$h(\mathcal{B}) = \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{\hat{x};2,m_2,m_3}}{m_2m_3}. \tag{3.33}$$

Proof. By the same arguments as in [17], the limit Eq. (1.36) is well-defined and exists. From $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, for $m_2 \geq 2$ and $m_3 \geq 2$,

$$\begin{aligned}\Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B}) &= \sum_{1\leq i,j\leq 2^{m_2 m_3}} (\mathbb{A}^{m_1-1}_{\hat{x};2\times m_2\times m_3})_{ij} \\ &= |(\mathbb{A}^{m_1-1}_{\hat{x};2\times m_2\times m_3})|.\end{aligned}$$

As in the one-dimensional case,

$$\lim_{m_1\to\infty} \frac{\log |(\mathbb{A}^{m_1-1}_{\hat{x};2\times m_2\times m_3})|}{m_1} = \log \lambda_{\hat{x};2,m_2,m_3},$$

as in for example [4]. Hence,

$$\begin{aligned}h(\mathcal{B}) &= \lim_{m_1,m_2,m_3\to\infty} \frac{\log \Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B})}{m_1 m_2 m_3} \\ &= \lim_{m_2,m_3\to\infty} \frac{1}{m_2 m_3}\left(\lim_{m_1\to\infty} \frac{\log \Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B})}{m_1}\right) \\ &= \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{\hat{x};2,m_2,m_3}}{m_2 m_3}.\end{aligned}$$

The proof is complete. □

Remark 3.1.14. *Let $\lambda_{x;2,m_2,m_3}$, $\lambda_{\hat{y};m_1,2,m_3}$, $\lambda_{y;m_1,2,m_3}$, $\lambda_{\hat{z};m_1,m_2,2}$ and $\lambda_{z;m_1,m_2,2}$ be the maximum eigenvalue of $\mathbb{A}_{x;2\times m_2\times m_3}$, $\mathbb{A}_{\hat{y};m_1\times 2\times m_3}$, $\mathbb{A}_{y;m_1\times 2\times m_3}$, $\mathbb{A}_{\hat{z};m_1\times m_2\times 2}$ and $\mathbb{A}_{z;m_1\times m_2\times 2}$ respectively. Then,*

$$\begin{aligned}h(\mathcal{B}) &= \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{x;2,m_2,m_3}}{m_2 m_3} \\ &= \lim_{m_1,m_3\to\infty} \frac{\log \lambda_{\hat{y};m_1,2,m_3}}{m_1 m_3} \\ &= \lim_{m_1,m_3\to\infty} \frac{\log \lambda_{y;m_1,2,m_3}}{m_1 m_3} \\ &= \lim_{m_1,m_2\to\infty} \frac{\log \lambda_{\hat{z};m_1,m_2,2}}{m_1 m_2} \\ &= \lim_{m_1,m_2\to\infty} \frac{\log \lambda_{z;m_1,m_2,2}}{m_1 m_2}.\end{aligned}$$

The detailed proofs are as above.

3.1.3 Computation of $\lambda_{m,n}$ and entropies

The last subsection provided a systematic means of writing down $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{x;2\times 2\times 2}$. As in a two-dimensional case [4], a recursive formula for $\lambda_{\hat{x};2,m_2,m_3}$ can be obtained in a special structure. An illustrative example is presented in which $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ and $\lambda_{\hat{x};2,m_2,m_3}$ can be derived explicitly to demonstrate the methods developed in the preceding subsection. More complete results will be presented later.

Let

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } E = E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{3.34}$$

and

$$\mathbb{A}_{x;2\times 2\times 2} = G \otimes E \otimes E \otimes E. \tag{3.35}$$

Proposition 3.1.15. *Substitute $\mathbb{A}_{x;2\times 2\times 2}$ into Eq. (3.34) and Eq. (3.35). Then,*

$$(i) \quad \mathbb{A}_{x;2\times m_2\times 2} = \otimes(G\otimes E)^{m_2-1} \otimes (\otimes E^2), \tag{3.36}$$

$$(ii) \quad \mathbb{A}_{\hat{x};2\times m_2\times 2} = (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1}), \tag{3.37}$$

$$(iii) \quad \mathbb{A}_{\hat{x};2\times m_2\times m_3} = \otimes((\otimes G^{m_2-1})\otimes E)^{m_3-1} \otimes (\otimes E^{m_2}). \tag{3.38}$$

Furthermore, for the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$ of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, the following recursive formulae apply:

$$\lambda_{\hat{x};2,m_2+1,m_3} = 2g^{m_3-1}\lambda_{\hat{x};2,m_2,m_3} \tag{3.39}$$

and

$$\lambda_{\hat{x};2,m_2,m_3+1} = 2g^{m_2-1}\lambda_{\hat{x};2,m_2,m_3} \tag{3.40}$$

for $m_2, m_3 \geq 2$ with

$$\lambda_{\hat{x};2,2,2} = 2^3 g. \tag{3.41}$$

The spatial entropy is

$$h(\mathbb{A}_{x;2\times 2\times 2}) = \log g, \tag{3.42}$$

where $g = \frac{1+\sqrt{5}}{2}$, the golden-mean.

Proof. The proof is described only briefly, and the details are omitted for brevity. (i) can be proven by Theorem 3.1.8 and induction on m_2. Indeed, by Eq. (3.29),

$$\begin{aligned} \mathbb{A}_{x;2\times 3\times 2} &= (\mathbb{A}_{x;2\times 2\times 2})_{4\times 4} \circ (E_{2^2} \otimes \mathbb{A}_{x;2\times 2\times 2})_{4\times 4} \\ &= (G\otimes E\otimes E\otimes E)_{4\times 4} \circ (E\otimes E\otimes (G\otimes E\otimes E\otimes E))_{4\times 4} \\ &= (G\circ E)\otimes(E\circ E)\otimes(E\circ G)\otimes(E_{2\times 2}\circ(E\otimes E\otimes E))_{2\times 2} \\ &= G\otimes E\otimes G\otimes E\otimes E\otimes E. \end{aligned}$$

Assume that $\mathbb{A}_{x;2\times(m_2-1)\times 2} = \otimes(G\otimes E)^{m_2-2}\otimes(\otimes E^2)$. Then by Eq. (3.29) again,

$$
\begin{aligned}
&\mathbb{A}_{x;2\times m_2\times 2} = (\mathbb{A}_{x;2\times(m_2-1)\times 2})\circ((\otimes E^{2(m_2-2)})\otimes \mathbb{A}_{x;2\times 2\times 2})\\
&= (\otimes(G\otimes E)^{m_2-2}\otimes(\otimes E^2))_{2^{2m_2-2}\times 2^{2m_2-2}}\circ\ ((\otimes E^{2(m_2-2)})\otimes(G\otimes E\otimes E\otimes E)_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&= (\otimes(G\otimes E)^{m_2-2}\otimes(E\otimes E))_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&\quad\circ(\otimes(E\otimes E)^{m_2-2}\otimes(G\otimes E)\otimes(E\otimes E))_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&= \otimes[(G\circ E)\otimes(E\circ E)]^{m_2-2}\otimes(E\circ G)\otimes(E\circ(E\otimes E\otimes E))\\
&= \otimes(G\otimes E)^{m_2-2}\otimes(G\otimes E)\otimes(E\otimes E)\\
&= \otimes(G\otimes E)^{m_2-1}\otimes(\otimes E^2).
\end{aligned}
$$

(ii) The following property of matrices is required and detailed proofs are omitted: For any two 2×2 matrices A and B,

$$
P(A\otimes B)P = B\otimes A, \tag{3.43}
$$

where P is given by Eq. (3.21). Equation (3.37) is proven by induction on m_2. When $m_2 = 2$, by Theorem 3.1.10,

$$
\begin{aligned}
\mathbb{A}_{\hat{x};2\times 2\times 2} &= \mathbb{P}^t_{x;2\times 2\times 2}\mathbb{A}_{x;2\times 2\times 2}\mathbb{P}_{x;2\times 2\times 2}\\
&= (P_{4;2})^t\mathbb{A}_{x;2\times 2\times 2}P_{4;2}\\
&= (I_2\otimes P\otimes I_2)((G\otimes E)\otimes(E\otimes E))(I_2\otimes P\otimes I_2)\\
&= G\otimes(P(E\otimes E)P)\otimes E\\
&= G\otimes E\otimes E\otimes E
\end{aligned}
$$

by Eq. (3.43).

Now, Eq. (3.37) is assumed to hold for m_2-1;

$$
\mathbb{A}_{\hat{x};2\times(m_2-1)\times 2} = (\otimes G^{m_2-2})\otimes(\otimes E^{m_2}).
$$

Then

$$
\begin{aligned}
&\mathbb{A}_{\hat{x};2\times m_2\times 2}\\
&= \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{A}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}\\
&= [(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m_2;m_2})]^t\\
&\quad \mathbb{A}_{x;2\times m_2\times 2}[(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m_2;m_2})]\\
&= (P_{2m_2;m_2})\cdots(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})[(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})\\
&\quad\big(\otimes(G\otimes E)^{m_2-1}\otimes(\otimes E^2)\big)(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})](P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m;m_2})\\
&= (P_{2m_2;m_2})\cdots(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})[G\otimes(\otimes(G\otimes E)^{m_2-2}\otimes(\otimes E^2))\otimes E]\\
&\quad(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m_2;m_2})\\
&= G\otimes\{(P_{2(m_2-1);m_2-1})\cdots(P_{2(m_2-1);2}P_{2(m_2-1);4}\cdots P_{2(m_2-1);2(m_2-1)-2})[\otimes(G\otimes E)^{m_2-1}]\\
&\quad(P_{2(m_2-1);2}P_{2(m_2-1);4}\cdots P_{2(m_2-1);2(m_2-1)-2})\cdots(P_{2(m_2-1);m_2-1})\}\otimes E\\
&= G\otimes(\mathbb{P}^t_{x;2\times(m_2-1)\times 2}\mathbb{A}_{x;2\times(m_2-1)\times 2}\mathbb{P}_{x;2\times(m_2-1)\times 2})\otimes E\\
&= G\otimes\mathbb{A}_{\hat{x};2\times(m_2-1)\times 2}\otimes E\\
&= G\otimes((\otimes G^{m_2-2})\otimes(\otimes E^{m_2}))\otimes E\\
&= (\otimes G^{m_2-1})\otimes(\otimes E^{m_2+1}).
\end{aligned}
$$

(iii) For a fixed m_2, these results are proven by induction on $m_3\geq 2$. Assume that Eq. (3.38) holds for m_3-1;

$$
\mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)} = \otimes((\otimes G^{m_2-1})\otimes E)^{m_3-2}\otimes(\otimes E^{m_2}).
$$

Then, by Eq. (3.32),

$$\begin{aligned}
\mathbb{A}_{\hat{x};2\times m_2\times m_3} &= \mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)} \circ \left((\otimes E^{m_2(m_3-2)}) \otimes \mathbb{A}_{\hat{x};2\times m_2\times 2}\right) \\
&= [\otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-2} \otimes (\otimes E^{m_2})] \circ [(\otimes E^{m_2(m_3-2)}) \otimes (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1})] \\
&= \otimes\big((\otimes G^{m_2-1}) \otimes E\big)^{m_3-2} \otimes \big((\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1})\big) \\
&= \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2}).
\end{aligned}$$

For the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$, equation (3.41) is easily verified. Equation (3.39) is established for fixed m_3 using Eq. (3.38), yielding

$$\begin{aligned}
\mathbb{A}_{\hat{x};2\times(m_2+1)\times m_3} &= \otimes((\otimes G^{m_2}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2+1}) \\
&= \big(G \otimes (\otimes G^{m_2-1}) \otimes E\big)^{m_3-1} \otimes \big(\otimes E^{m_2} \otimes E\big),
\end{aligned}$$

which implies

$$\lambda_{\hat{x};2,m_2+1,m_3} = 2g^{m_3-1}\lambda_{\hat{x};2,m_2,m_3},$$

see [13], [29] and [30].
Similarly, for a fixed m_2, equation (3.40) is proven using Eq. (3.38) again:

$$\begin{aligned}
\mathbb{A}_{\hat{x};2\times m_2\times(m_3+1)} &= \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3} \otimes (\otimes E^{m_2}) \\
&= \otimes((\otimes G^{m_2-1}) \otimes E) \otimes \mathbb{A}_{\hat{x};2\times m_2\times m_3},
\end{aligned}$$

which implies

$$\lambda_{\hat{x};2,m_2,m_3+1} = 2g^{m_2-1}\lambda_{\hat{x};2,m_2,m_3}.$$

Finally, equation (3.42) follows from Eqs. (3.39), (3.40) and Theorem 3.1.13. The proof is thus complete. □

3.2 Connecting Operator

This section introduces the connecting operator and employs it to derive a recursive formula between an elementary pattern of order (m_1, m_2, m_3+1) and that of order (m_1, m_2, m_3). It is also adopted to obtain a lower bound on entropy.

3.2.1 Connecting operator in z-direction

This subsection derives connecting operators and studies their properties. For brevity, only the connecting operator in the z-direction is discussed but the other cases are similar, and will be considered in the following Remarks. For clarity, as in the former section, two symbols on lattice $\mathbf{Z}_{2\times 2\times 2}$ are examined first.

According to Theorem 3.1.11, the transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be represented as $A_{\hat{x};2\times m_2\times m_3;\alpha}$, where $1 \leq \alpha \leq 2^{2m_2}$, is a $2^{m_2(m_3-1)} \times 2^{m_2(m_3-1)}$ matrix.

For matrix multiplication, the indices of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ are conveniently expressed as

$$\begin{bmatrix} A_{\hat{x};2\times m_2\times m_3;11} & A_{\hat{x};2\times m_2\times m_3;12} & \cdots & A_{\hat{x};2\times m_2\times m_3;12^{m_2}} \\ A_{\hat{x};2\times m_2\times m_3;21} & A_{\hat{x};2\times m_2\times m_3;22} & \cdots & A_{\hat{x};2\times m_2\times m_3;22^{m_2}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\hat{x};2\times m_2\times m_3;2^{m_2}1} & A_{\hat{x};2\times m_2\times m_3;2^{m_2}2} & \cdots & A_{\hat{x};2\times m_2\times m_3;2^{m_2}2^{m_2}} \end{bmatrix}.$$

Clearly, $A_{\hat{x};2\times m_2\times m_3;\alpha} = A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}$, where $\alpha = \alpha(\beta_1,\beta_2) = 2^{m_2}(\beta_1 - 1) + \beta_2$. For $m_1 \geq 2$, the elementary pattern in the entries of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ is given by

$$A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}A_{\hat{x};2\times m_2\times m_3;\beta_2\beta_3}\cdots A_{\hat{x};2\times m_2\times m_3;\beta_{m_1}\beta_{m_1+1}}$$

where $\beta_r \in \{1, 2, \cdots, 2^{m_2}\}$ and $1 \leq r \leq m_1+1$. A lexicographic order for multiple indices $I_{m_1+1} = (\beta_1\beta_2\cdots\beta_{m_1}\beta_{m_1+1})$ is introduced, using

$$\mathcal{K}(I_{m_1+1}) = 1 + \sum_{r=2}^{m_1} 2^{m_2(m_1-r)}(\beta_r - 1). \tag{3.44}$$

Now, $A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$ can be represented by

$$A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}A_{\hat{x};2\times m_2\times m_3;\beta_2\beta_3}\cdots A_{\hat{x};2\times m_2\times m_3;\beta_{m_1}\beta_{m_1+1}}, \tag{3.45}$$

where

$$\alpha = \alpha(\beta_1, \beta_{m_1+1}) = 2^{m_2}(\beta_1 - 1) + \beta_{m_1+1}$$

and

$$k = \mathcal{K}(I_{m_1+1})$$

is as in Eq. (3.44). Accordingly, $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ can be expressed as

$$[A_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}}, \tag{3.46}$$

where $1 \leq \alpha \leq 2^{2m_2}$ and

$$A_{\hat{x};m_1,m_2,m_3;\alpha} = \sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}. \tag{3.47}$$

Moreover,

$$V_{\hat{x};m_1,m_2,m_3;\alpha} = (A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha})^t, \tag{3.48}$$

where $1 \leq k \leq 2^{m_2(m_1-1)}$, $V_{\hat{x};m_1,m_2,m_3;\alpha}$ is a $2^{m_2(m_1-1)}$ column vector that comprises all elementary patterns in $A_{\hat{x};m_1,m_2,m_3;\alpha}$. The ordering matrix $\mathbb{V}_{\hat{x};m_1,m_2,m_3}$ of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ is now defined as

$$[V_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}}, \tag{3.49}$$

where $1 \leq \alpha \leq 2^{2m_2}$. The ordering matrix $\mathbb{V}_{\hat{x};m_1,m_2,m_3}$ allows the elementary patterns to be tracked during the reduction from $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+1)}$ to $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$. This careful book-keeping constitutes a systematic way to generate the admissible patterns, and as in Sec. 3.2.2, lower-bound estimates of spatial entropy.

This simplest example is considered first to illustrate this concept.

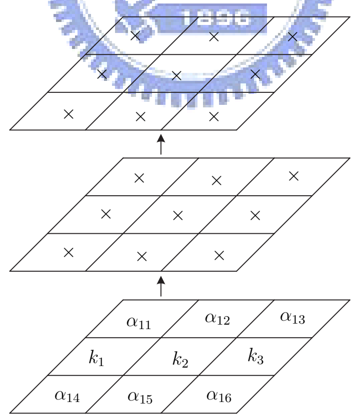
Example 3.2.1. *For $m_1 = 2$, $m_2 = 3$, $m_3 = 3$, the following can be easily verified;*

$$\mathbb{A}^2_{\hat{x};2\times 3\times 3} = [A_{\hat{x};2,3,3;\alpha_1}]_{2^3\times 2^3},$$

where $1 \leq \alpha_1 \leq 2^6$ and

$$A_{\hat{x};2,3,3;\alpha_1} = \sum_{k=1}^{2^3} A^{(k)}_{\hat{x};2,3,3;\alpha_1},$$

and for fixed α_1 and k the represented pattern of $A^{(k)}_{\hat{x};2,3,3;\alpha_1}$ are in the following form.



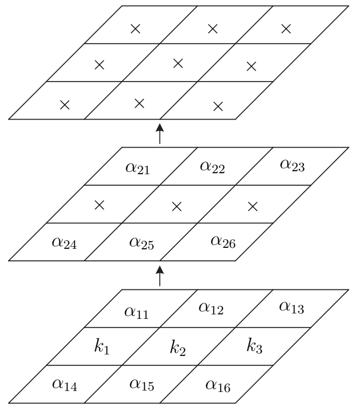
If the red symbol is defined equal to 1, and white symbol equals 0, then $\alpha_1 = 2^5\alpha_{11} + 2^4\alpha_{12} + 2^3\alpha_{13} + 2^2\alpha_{14} + 2\alpha_{15} + \alpha_{16} + 1$ and $k = 2^2k_1 + 2k_2 + k_3 + 1$. Hence

$$V_{\hat{x};2,3,3;\alpha_1} = (A^{(k)}_{\hat{x};2,3,3;\alpha_1})^t,$$

where $1 \leq k \leq 2^3$ and $1 \leq \alpha_1 \leq 2^6$. Define

$$V_{\hat{x};2,3,3;\alpha_1;\alpha_2} = (A^{(k)}_{\hat{x};2,3,3;\alpha_1;\alpha_2})^t,$$

where $1 \le k \le 2^3$ *and* $1 \le \alpha_1, \alpha_2 \le 2^6$ *and the represented pattern of* $A^{(k)}_{\hat{x};2,3,3;\alpha_1;\alpha_2}$ *is*

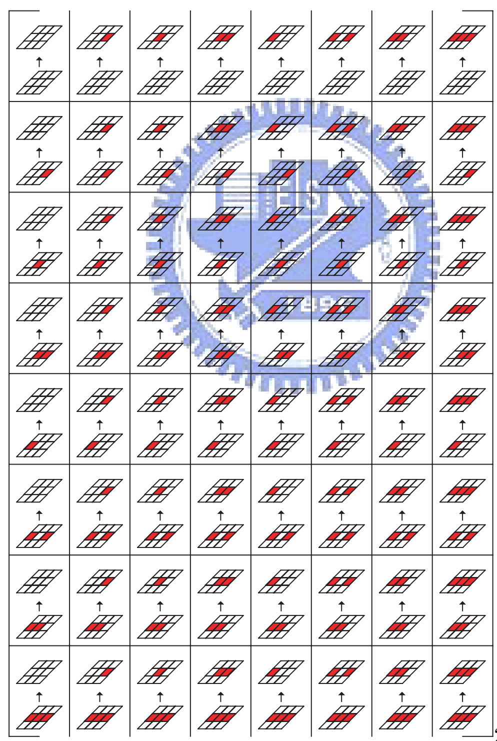


.

Therefore, for instance,

$$V_{\hat{x};2,3,3;1;1} = S_{\hat{x};m_3;2,3;11} V_{\hat{x};2,3,2;1},$$

and the represented patterns of $S_{\hat{x};m_3;2,3;11}$



$_{2^3 \times 2^3}$.

The above derivation reveals that $V_{\hat{x};2,3,3;\alpha_1;\alpha_2}$ can be reduced to $V_{\hat{x};2,3,2;\alpha_2}$ by multiplication using connecting operator $S_{\hat{x};m_3;2,3;\alpha_1\alpha_2}$. This procedure can be extended to introduce the connecting operator $\mathbb{S}_{\hat{x};m_3;m_1m_2} = [S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$[4]$]$, where $1 \le \alpha_1, \alpha_2 \le 2^{2m_2}$, for all $m_1 \ge 2$, $m_2 \ge 2$.

Definition 3.2.2. *For* $m_1 \ge 2$, $m_2 \ge 2$, *define*

$$(\mathbb{C}_{\hat{x};m_3;m_1m_2})_{2^{2m_2} \times 2^{2m_2}} = (\mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2})_{2^{2m_2} \times 2^{2m_2}}, \tag{3.50}$$

[4]Use $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ to substitute $S_{\hat{x};m_3;m_1m_2;\alpha_1,\alpha_2}$ for simplicity afterward.

where the row matrix $\mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2}$ *of* $\mathbb{S}_{\hat{x};m_3;m_1m_2}$ *is defined in Eq. (3.6) and Eq. (3.7). And*

$$\begin{aligned} & C_{\hat{x};m_3;m_1m_2;i_1i_2} \\ = \; & [(A_{z;2\times m_2\times 2;i_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{A}_{z;(m_1-1)\times m_2\times 2})_{2^{m_2}\times 2^{m_2}}]_{2^{(m_1-1)m_2}\times 2^{(m_1-1)m_2}} \\ & \circ (E_{2^{(m_1-2)m_2}} \otimes ((\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2})_{2^{m_2}\times 2^{m_2}})_{2^{(m_1-1)m_2}\times 2^{(m_1-1)m_2}} \end{aligned} \tag{3.51}$$

where $(\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2}$ *is the* i_2*-th block of the matrix* $(\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}$, $(\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}$ *is the column matrix of* $\mathbb{A}^{(r)}_{z;2\times m_2\times 2}$, $\mathbb{A}^{(r)}_{z;2\times m_2\times 2}$ *is the row matrix of* $\mathbb{A}_{z;2\times m_2\times 2}$ *and* E_k *is the* $2^k\times 2^k$ *matrix with 1 as its entries.*

Remark 3.2.3. *By a similar method, the following connecting operators can also be defined.*

$$\begin{aligned} & C_{x;m_2;m_1m_3;i_1i_2} \\ = \; & [(A_{y;2\times 2\times m_3;i_1})_{2^{m_3}\times 2^{m_3}} \circ (\mathbb{A}_{y;(m_1-1)\times 2\times m_3})_{2^{m_3}\times 2^{m_3}}]_{2^{(m_1-1)m_3}\times 2^{(m_1-1)m_3}} \\ & \circ (E_{2^{(m_1-2)m_3}} \otimes ((\mathbb{A}^{(r)}_{y;2\times 2\times m_3})^{(c)}_{;i_2})_{2^{m_3}\times 2^{m_3}})_{2^{(m_1-1)m_3}\times 2^{(m_1-1)m_3}} \end{aligned}$$

$$\begin{aligned} & C_{\hat{y};m_3;m_1m_2;i_1i_2} \\ = \; & [(A_{\hat{z};m_1\times 2\times 2;i_1})_{2^{m_1}\times 2^{m_1}} \circ (\mathbb{A}_{\hat{z};m_1\times (m_2-1)\times 2})_{2^{m_1}\times 2^{m_1}}]_{2^{(m_2-1)m_1}\times 2^{(m_2-1)m_1}} \\ & \circ (E_{2^{(m_2-2)m_1}} \otimes ((\mathbb{A}^{(r)}_{\hat{z};m_1\times 2\times 2})^{(c)}_{;i_2})_{2^{m_1}\times 2^{m_1}})_{2^{(m_2-1)m_1}\times 2^{(m_2-1)m_1}} \end{aligned}$$

$$\begin{aligned} & C_{y;m_1;m_2m_3;i_1i_2} \\ = \; & [(A_{x;2\times 2\times m_3;i_1})_{2^{m_3}\times 2^{m_3}} \circ (\mathbb{A}_{x;2\times (m_2-1)\times m_3})_{2^{m_3}\times 2^{m_3}}]_{2^{(m_2-1)m_3}\times 2^{(m_2-1)m_3}} \\ & \circ (E_{2^{(m_2-2)m_3}} \otimes ((\mathbb{A}^{(r)}_{x;2\times 2\times m_3})^{(c)}_{;i_2})_{2^{m_3}\times 2^{m_3}})_{2^{(m_2-1)m_3}\times 2^{(m_2-1)m_3}} \end{aligned}$$

$$\begin{aligned} & C_{\hat{z};m_2;m_1m_3;i_1i_2} \\ = \; & [(A_{\hat{y};m_1\times 2\times 2;i_1})_{2^{m_1}\times 2^{m_1}} \circ (\mathbb{A}_{\hat{y};m_1\times 2\times (m_3-1)})_{2^{m_1}\times 2^{m_1}}]_{2^{(m_3-1)m_1}\times 2^{(m_3-1)m_1}} \\ & \circ (E_{2^{(m_3-2)m_1}} \otimes ((\mathbb{A}^{(r)}_{\hat{y};m_1\times 2\times 2})^{(c)}_{;i_2})_{2^{m_1}\times 2^{m_1}})_{2^{(m_3-1)m_1}\times 2^{(m_3-1)m_1}} \end{aligned}$$

$$\begin{aligned} & C_{z;m_1;m_2m_3;i_1i_2} \\ = \; & [(A_{\hat{x};2\times m_2\times 2;i_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{A}_{\hat{x};2\times m_2\times (m_3-1)})_{2^{m_2}\times 2^{m_2}}]_{2^{(m_3-1)m_2}\times 2^{(m_3-1)m_2}} \\ & \circ (E_{2^{(m_3-2)m_2}} \otimes ((\mathbb{A}^{(r)}_{\hat{x};2\times m_2\times 2})^{(c)}_{;i_2})_{2^{m_2}\times 2^{m_2}})_{2^{(m_3-1)m_2}\times 2^{(m_3-1)m_2}} \end{aligned}$$

Theorem 3.2.4. *For any* $m_2 \geq 2$, $m_3 \geq 2$ *and* $1 \leq i_1, i_2 \leq 2^{2m_2}$,

$$C_{\hat{x};m_3;(m_1+1)m_2;i_1i_2} = [a_{\hat{x};2\times m_2\times 2;i_1i}C_{\hat{x};m_3;m_1m_2;ii_2}], \tag{3.52}$$

where $1 \leq i \leq 2^{2m_2}$.

Proof. By Theorem 3.1.11 and Remark 3.1.12,

$$\mathbb{A}_{z;m_1\times m_2\times 2} = [A_{z;2\times m_2\times 2;i_1} \circ \mathbb{A}_{z;(m_1-1)\times m_2\times 2}],$$

where $1 \le i_1 \le 2^{2m_2}$. Hence, by

$$\begin{aligned}
& C_{\hat{x};m_3;(m_1+1)m_2;i_1i_2} \\
= & [(A_{z;2\times m_2\times 2;i_1}) \circ \mathbb{A}_{z;m_1\times m_2\times 2}] \circ [E_{2^{(m_1-1)m_2}} \otimes (A^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2}] \\
= & [a_{\hat{x};2\times m_2\times 2;i_1 i}(A_{z;2\times m_2\times 2;i} \circ \mathbb{A}_{z;(m_1-1)\times m_2\times 2})] \\
& \circ [E_{2^{m_2}} \otimes (E_{2^{(m_1-2)m_2}} \otimes (A^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2})] \\
= & [a_{\hat{x};2\times m_2\times 2;i_1 i} C_{\hat{x};m_3;m_1m_2;ii_2}]_{2^{m_2}\times 2^{m_2}}
\end{aligned}$$

where $1 \le i \le 2^{2m_2}$. The proof is complete. □

Notably, Eq. (3.52) implies $C_{\hat{x};m_3;m_1m_2;ij}$ is

$$a_{\hat{x};2\times m_2\times 2;i_1i_2} a_{\hat{x};2\times m_2\times 2;i_2i_3} \cdots a_{\hat{x};2\times m_2\times 2;i_{m_1}i_{m_1+1}}$$

with $i_1 = i$ and $i_{m_1+1} = j$. $C_{\hat{x};m_3;m_1m_2;ij}$ comprises all paths of length m_1+1, that start at i and end at j. Indeed, the entries of $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ and $\mathbb{A}_{z;(m_1+1)\times m_2\times 2}$ are the same. However, the arrangements differ.

Substituting m_3 for m_3+1 into Eq. (3.45) and using Eq. (3.32), $A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha}$ could be represented by

$$\begin{aligned}
& A_{\hat{x};2\times m_2\times(m_3+1);\beta_1\beta_2} A_{\hat{x};2\times m_2\times(m_3+1);\beta_2\beta_3} \cdots A_{\hat{x};2\times m_2\times(m_3+1);\beta_{m_1}\beta_{m_1+1}} \\
= & \prod_{j=1}^{m_1} [a_{\hat{x};2\times m_2\times 2;\alpha_j\hat{\alpha}} A_{\hat{x};2\times m_2\times m_3;\hat{\beta}_1\hat{\beta}_2}]_{2^{m_2}\times 2^{m_2}},
\end{aligned} \tag{3.53}$$

where $1 \le \hat{\beta}_1, \hat{\beta}_2 \le 2^{m_2}$ and $\alpha_j = \alpha(\beta_j, \beta_{j+1})$ and $\hat{\alpha} = \alpha(\hat{\beta}_1, \hat{\beta}_2)$ for $1 \le j \le m_1$.

After m_1 matrix multiplications have been performed as in Eq. (3.53),

$$A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1} = [A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}]_{2^{m_2}\times 2^{m_2}}, \tag{3.54}$$

where $1 \le \alpha_2 \le 2^{2m_2}$ and $A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}$ can be represented by

$$\sum_{\ell=1}^{2^{m_2(m_1-1)}} K(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, \ell) A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2} \tag{3.55}$$

which is a linear combination of $A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ with the coefficients $K(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, \ell)$ which are products of $a_{\hat{x};2\times m_2\times 2;\alpha_j\hat{\alpha}}$, $1 \le j \le m_1$. $K(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, \ell)$ must be studied in more details. Notably,

$$\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+1)} = [A_{\hat{x};m_1,m_2,m_3+1;\alpha_1}]_{2^{m_2}\times 2^{m_2}} \tag{3.56}$$

where $1 \le \alpha_1 \le 2^{2m_2}$,

$$A_{\hat{x};m_1,m_2,m_3+1;\alpha_1} = \sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1}$$

and

$$\sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1} = [\sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \le \alpha_2 \le 2^{2m_2}$. Now, $V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}$ is defined as

$$V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2} = (A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2})^t. \tag{3.57}$$

From Eq. (3.55) and Eq. (3.57),

$$V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2} = \mathbb{K}(\hat{x}, m_1m_2; \alpha_1\alpha_2) V_{\hat{x};m_1,m_2,m_3;\alpha_2} \tag{3.58}$$

where

$$\mathbb{K}(\hat{x}, m_1m_2; \alpha_1\alpha_2) = (K(\hat{x}, m_1m_2; \alpha_1\alpha_2; k, \ell)),$$

$1 \le k, \ell \le 2^{m_2(m_1-1)}$ is a $2^{m_2(m_1-1)} \times 2^{m_2(m_1-1)}$ matrix. Now

$$\mathbb{K}(\hat{x}, m_1m_2; \alpha_1\alpha_2) = S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$$

must be shown as follows.

Theorem 3.2.5. *For any $m_1 \ge 2$, $m_2 \ge 2$ and $m_3 \ge 2$, let $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ be given as in Eq. (3.50) and Eq. (3.51). Then,*

$$V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2} = S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} V_{\hat{x};m_1,m_2,m_3;\alpha_2}, \tag{3.59}$$

or equivalently, the recursive formula

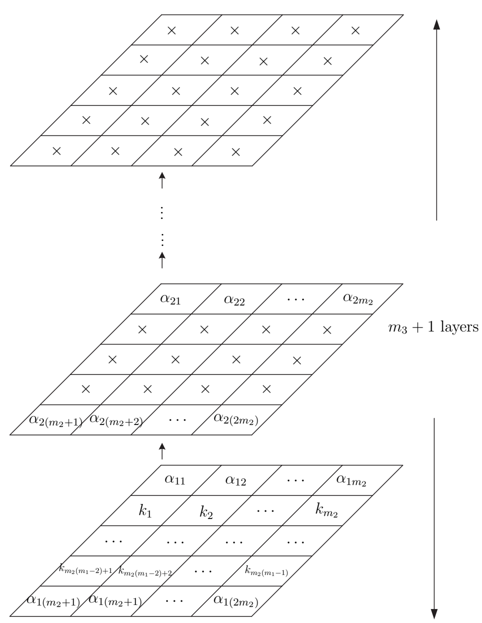
$$\begin{aligned} & A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1} \\ = & \; [\sum_{\ell=1}^{2^{m_2(m_1-1)}} (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{k\ell} A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}]_{2^{m_2}\times 2^{m_2}}, \end{aligned} \tag{3.60}$$

where $1 \le \alpha_2 \le 2^{2m_2}$. Moreover, for $m_3 = 1$,

$$A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1} = [\sum_{\ell=1}^{2^{m_2(m_1-1)}} (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{k\ell}]_{2^{m_2}\times 2^{m_2}}, \tag{3.61}$$

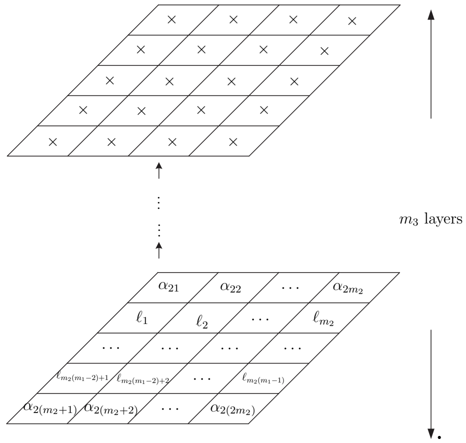
where $1 \le \alpha_2 \le 2^{2m_2}$ for any $1 \le k \le 2^{m_2(m_1-1)}$ and $\alpha_1 \in \{1, 2, \ldots, 2^{2m_2}\}$.

Proof. From Eq. (3.54), $A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}$ can be represented as the pattern



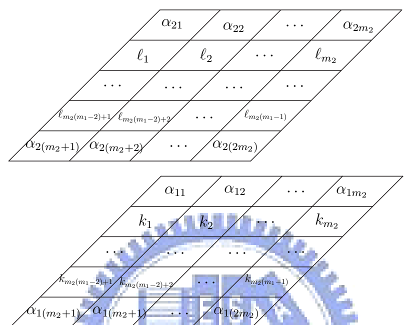
(3.62)

and $A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ as the pattern,



. (3.63)

From Definition 3.2.2, $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ represents the following pattern



. (3.64)

Therefore, Eq. (3.60) follows from Eqs. (3.62), (3.63) and (3.64). Also, from Eq. (3.58), equation (3.59) follows.

Next, equation (3.61) follows simply from Eqs. (3.62) and (3.64). □

For any positive integer $p \geq 2$, applying Theorem 3.2.5 p times allows the elementary patterns of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ to be expressed as products of a sequence of $S_{\hat{x};m_3;m_1m_2;\alpha_i\alpha_{i+1}}$ and the elementary patterns in $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$. The elementary pattern in $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ is first considered. For any $p \geq 2$ and $1 \leq q \leq p-1$, define

$$\begin{aligned} & A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_q} \\ = \; & [A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_q;\alpha_{q+1}}]_{2^{m_2}\times 2^{m_2}}, \end{aligned}$$

where $1 \leq \alpha_{q+1} \leq 2^{2m_2}$. Then $A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}$ can be represented as

$$\sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} (\prod_{i=2}^{p+1} K(\hat{x};m_1m_2;\alpha_{i-1}\alpha_i;\ell_{i-1},\ell_i))A^{(\ell_{p+1})}_{\hat{x};m_1;m_2;m_3;\alpha_{p+1}} \quad (3.65)$$

where and $\ell_1 = k$ can be easily verified.

Hence, for any $p \geq 2$, equation (3.56) can be generalized for $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ as a $(2^{m_2})^{p+1} \times (2^{m_2})^{p+1}$ matrix

$$\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)} = [A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}], \quad (3.66)$$

where

$$A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}} = \sum_{k=1}^{2^{(m_1-1)m_2}} A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}.$$

In particular, if $\alpha_1, \alpha_2, \ldots, \alpha_{p+1} \in \{2^{m_2}(s-1)+s|1 \leq s \leq 2^{m_2}\}$ then $A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}$ lies on the diagonal of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ in Eq. (3.66). Now, define

$$V_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}} = (A^{(k)}_{m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}})^t.$$

Therefore, Theorem 3.2.5 can be generalized to the following Theorem.

Theorem 3.2.6. *For any $m_1 \geq 2$, $m_2 \geq 2$, $m_3 \geq 2$ and $p \geq 1$, $V_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}$ could be represented as*

$$S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_p\alpha_{p+1}}V_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}}$$

where $1 \leq \alpha_i \leq 2^{2m_2}$ and $1 \leq i \leq p+1$.

Proof. From Eqs. (3.65), (3.58) and (3.60),

$$\begin{aligned}
& A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}} \\
= & \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} \Big(\prod_{i=2}^{p+1} K(\hat{x}; m_1m_2; \alpha_{i-1}\alpha_i; \ell_{i-1}, \ell_i)\Big) A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}} \\
= & \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} \Big(\prod_{i=2}^{p+1} (S_{\hat{x};m_3;m_1m_2;\alpha_{i-1}\alpha_i})_{\ell_{i-1}\ell_i}\Big) A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}} \\
= & \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} \Big((S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{\ell_1\ell_2} (S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3})_{\ell_2\ell_3} \\
& \cdots (S_{\hat{x};m_3;m_1m_2;\alpha_p\alpha_{p+1}})_{\ell_p\ell_{p+1}}\Big) A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}} \\
= & \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} \big(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_p\alpha_{p+1}}\big)_{k\ell_{p+1}} A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}}.
\end{aligned}$$

The proof is complete. □

3.2.2 Lower bound of entropy

In this subsection, the connecting operator $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ is adopted to estimate the lower bound of entropy and in particular, to confirm that is positive. The following notation is used.

Definition 3.2.7. *Let $V = (V_1, \cdots, V_M)^t$, where V_k are $N \times N$ matrices. Define the sum over V_k as*

$$|V| = \sum_{k=1}^{N} V_k. \tag{3.67}$$

If $\mathbb{M} = [M_{ij}]$ is a $M \times M$ matrix, then

$$|\mathbb{M}V| = \sum_{i=1}^{M}\sum_{j=1}^{M} M_{ij}V_j$$

Notably, (3.67) implies

$$|V_{\hat{x};m_1,m_2,m_3;\alpha}| = \sum_{k=1}^{2^{(m_1-1)m_2}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha} = A_{\hat{x};m_1,m_2,m_3;\alpha}.$$

As is typical, the set of all matrices with the same order can be partially ordered.

Definition 3.2.8. *Let $\mathbb{M} = [M_{ij}]$ and $\mathbb{N} = [N_{ij}]$ be two $M \times M$ matrices; $\mathbb{M} \geq \mathbb{N}$ if $M_{ij} \geq N_{ij}$ for all $1 \leq i, j \leq M$.*

Notably, if $\mathbb{A}_{x;2\times2\times2} \geq \mathbb{A}'_{x;2\times2\times2}$, then $\mathbb{A}_{\hat{x};2\times m_2\times m_3} \geq \mathbb{A}'_{\hat{x};2\times m_2\times m_3}$ for all m_2, $m_3 \geq 2$. Therefore, $h(\mathbb{A}_{x;2\times2\times2}) \geq h(\mathbb{A}'_{x;2\times2\times2})$. Hence, the spatial entropy as a function of $\mathbb{A}_{x;2\times2\times2}$ is monotonic with respect to the partial order $\geq$.

Definition 3.2.9. *A $P+1$ multiple index*

$$\mathcal{A}_P \equiv (\alpha_1\alpha_2\cdots\alpha_P\alpha_{P+1}) \tag{3.68}$$

is called a periodic cycle if

$$\alpha_{P+1} = \alpha_1, \tag{3.69}$$

where $1 \leq \alpha_i \leq 2^{2m_2}$ and $1 \leq i \leq P+1$. It is called diagonal cycle if Eq. (3.69) holds and

$$\alpha_i \in \{2^{m_2}(s-1) + s | 1 \leq s \leq 2^{m_2}\}$$

for each $1 \leq i \leq P+1$. For a diagonal cycle Eq. (3.68)

$$\bar{\alpha_P} = \alpha_1; \alpha_2; \cdots; \alpha_P$$

and

$$\bar{\alpha_P}^n = \bar{\alpha_P}; \bar{\alpha_P}; \cdots; \bar{\alpha_P}. \quad \textit{(n-times)}$$

First, prove the following Lemma.

Lemma 3.2.10. *Let $m_1 \geq 2$, $m_2 \geq 2$, $P \geq 1$, $\mathcal{A}_P$ be a diagonal cycle. Then, for any $m_3 \geq 1$,*

$$\begin{aligned} & \rho(\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3P+2)}) \\ \geq\ & \rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_{P+1}})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}|). \end{aligned} \tag{3.70}$$

Proof. Since $\mathcal{A}_P$ is a periodic cycle, Theorem 3.2.6 implies

$$\begin{aligned} & V_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1} \\ =\ & (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_{P+1}})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}. \end{aligned} \tag{3.71}$$

Furthermore, $\mathcal{A}_P$ is diagonal and $|V_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1}| = A_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1}$ lies in the diagonal part of Eq. (3.66), with $m_3 + p = m_3P + 2$. Accordingly,

$$\rho(\mathbb{A}^{m_1}_{\hat{x};m_1,m_2,m_3P+2}) \geq \rho(|V_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1}|). \tag{3.72}$$

Therefore, equation (3.70) follows from Eqs. (3.71) and (3.72). The proof is complete. □

The following Lemma is useful in evaluating maximum eigenvalue of Eq. (3.70).

Lemma 3.2.11. *For any $m_1 \geq 2$, $m_2 \geq 2$, $1 \leq k \leq 2^{(m_1-1)m_2}$ and $\alpha_1 \in \{(s-1)2^{m_2}+s | 1 \leq s \leq 2^{m_2}\}$, if*

$$tr(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) = 0,$$

then for all $1 \leq \ell \leq 2^{(m_1-1)m_2}$,

$$(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{k\ell} = 0, \tag{3.73}$$

for all $\alpha_2 \in \{(s-1)2^{m_2}+s | 1 \leq s \leq 2^{m_2}\}$, such that the k-th rows of matrices $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ are zeros. For any diagonal cycle $\mathcal{A}_P$, let $U = (u_1u_2\cdots u_{2^{m_2(m_1-1)}})$ be an eigenvector of $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}$. If $u_k \neq 0$ for some $1 \leq k \leq 2^{(m_1-1)m_2}$, then $tr(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) > 0$.

Proof. Since $A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}$ can be expressed as Eq. (3.61). $tr(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) = 0$ if and only if Eq. (3.73) holds for all $1 \leq \ell \leq 2^{(m_1-1)m_2}$. The second part of the Lemma 3.2.11 follows easily from the first part. The proof is complete. □

By Lemma 3.2.10 and Lemma 3.2.11, the lower bound of entropy can be determined as follows.

Theorem 3.2.12. *Let $\alpha_1\alpha_2\cdots\alpha_P\alpha_1$ be a diagonal cycle. Then, for any $m_1 \geq 2$, $m_2 \geq 2$,*

$$\begin{aligned} & h(\mathbb{A}_{x;2\times2\times2}) \\ \geq\ & \lim_{m_2\to\infty}\frac{1}{m_1m_2P}\log\rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}). \end{aligned} \tag{3.74}$$

Proof. First, by the methods used to prove Lemma 2.1.10, Lemma 2.1.11 and Theorem 2.1.12 in Sec. 2.1.2,

$$\limsup_{m_3\to\infty}\frac{1}{m_3}(\log\rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}|))$$

$$=\log\rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}) \tag{3.75}$$

is obtained. The detailed proofs are omitted here for brevity. Now,

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_2\to\infty}\frac{1}{m_1m_2P}\limsup_{m_3\to\infty}\frac{1}{m_3}(\log\rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}|))$$

is established. Indeed, from Eqs. (3.33) and (3.70),

$$\begin{aligned} h(\mathbb{A}_{x;2\times2\times2}) &= \lim_{m_2m_3\to\infty}\frac{1}{(m_3P+2)m_2}\log\rho(\mathbb{A}_{\hat{x};2\times m_2\times(m_3P+2)}) \\ &= \lim_{m_2m_3\to\infty}\frac{1}{m_1(m_3P+2)m_2}\log\rho(\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3P+2)}) \\ &\geq \lim_{m_2\to\infty}\frac{1}{m_1m_2P}\limsup_{m_3\to\infty}\frac{1}{m_3}(\log\rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\cdots \\ &\qquad S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}|)). \end{aligned}$$

Applying Eq. (3.75) which completes the proof. □

Remark 3.2.13. *By the similar method, the following lower bounds of entropy can also be estimated.*

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_3\to\infty}\frac{1}{m_1m_3P}\log\rho(S_{x;m_2;m_1m_3;\alpha_1\alpha_2}S_{x;m_2;m_1m_3;\alpha_2\alpha_3}\cdots S_{x;m_2;m_1m_3;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_1\to\infty}\frac{1}{m_1m_2P}\log\rho(S_{\hat{y};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{y};m_3;m_1m_2;\alpha_2\alpha_3}\cdots S_{\hat{y};m_3;m_1m_2;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_3\to\infty}\frac{1}{m_2m_3P}\log\rho(S_{y;m_1;m_2m_3;\alpha_1\alpha_2}S_{y;m_1;m_2m_3;\alpha_2\alpha_3}\cdots S_{\hat{x};m_1;m_2m_3;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_1\to\infty}\frac{1}{m_1m_3P}\log\rho(S_{\hat{z};m_2;m_1m_3;\alpha_1\alpha_2}S_{\hat{z};m_2;m_1m_3;\alpha_2\alpha_3}\cdots S_{\hat{z};m_2;m_1m_3;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_2\to\infty}\frac{1}{m_2m_3P}\log\rho(S_{z;m_1;m_2m_3;\alpha_1\alpha_2}S_{z;m_1;m_2m_3;\alpha_2\alpha_3}\cdots S_{z;m_1;m_2m_3;\alpha_P\alpha_1}).$$

Remark 3.2.14. *The results in last three sections can be generated to p-symbols on* $\mathbf{Z}_{2\ell\times2\ell\times2\ell}$ *such as in two dimensional case [4] and [5] and the details are omitted here for brevity.*

3.3 Applications to 3DCNN

This section elucidates an interesting model in 3DCNN of the application of the method. The method is elucidated by considering $a_{0,0,0} = a$, $a_{1,0,0} = a_x$, $a_{0,1,0} = a_y$ and $a_{0,0,1} = a_z$, which are nonzero; in other cases, $a_{\alpha,\beta,\gamma}$ and $b_{\alpha,\beta,\gamma}$ are zero. Then, the 3DCNN is of the form as Eq. (1.41)

$$\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + af(u_{i,j,k}) + a_x f(u_{i+1,j,k}) + a_y f(u_{i,j+1,k}) + a_z f(u_{i,j,k+1}).$$

The stationary solution to Eq. (1.41) satisfies

$$u_{i,j,k} = w + av_{i,j,k} + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1},$$

for $(i, j, k) \in \mathbf{Z}^3$ as in Eq. (1.42).

Firstly, consider the mosaic solution $u = (u_{i,j,k})$ to Eq. (1.42). If $u_{i,j,k} \geq 1$, i.e., $v_{i,j,k} = 1$, then

$$(a-1) + w + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1} \geq 0. \tag{3.76}$$

If $u_{i,j,k} \leq -1$, i.e., $v_{i,j,k} = -1$, then

$$(a-1) - w - (a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1}) \geq 0. \tag{3.77}$$

Equation (1.42) has five parameters w, a, a_x, a_y and a_z. Three procedures are adopted to partition these parameters:

Procedure (I): The parameters a_x, a_y and a_z are initially expressed into three-dimensional coordinates, to solve Eqs. (3.76) and (3.77), as in Fig. 3.

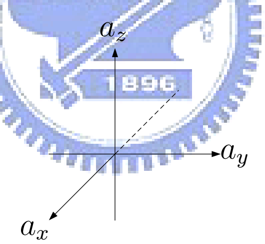


Figure 3: Primary partition of (a_x, a_y, a_z).

Clearly 2^3 octants (I)∼(VIII) exist in (a_x, a_y, a_z) three-dimensional coordinates.

Procedure (II): In each octant are 3! relations

$$\begin{array}{llllll}
\text{(i)} & : & |a_x| & > & |a_y| & > & |a_z|, \\
\text{(ii)} & : & |a_x| & > & |a_z| & > & |a_y|, \\
\text{(iii)} & : & |a_y| & > & |a_x| & > & |a_z|, \\
\text{(iv)} & : & |a_y| & > & |a_z| & > & |a_x|, \\
\text{(v)} & : & |a_z| & > & |a_x| & > & |a_y|, \\
\text{(vi)} & : & |a_z| & > & |a_y| & > & |a_x|.
\end{array} \tag{3.78}$$

Procedure (III): Each relations, denoted it by $|a_1| > |a_2| > |a_3|$, two situations apply

$$\begin{array}{llll}
(1) & |a_1| & > & |a_2| + |a_3| \\
(2) & |a_1| & < & |a_2| + |a_3|.
\end{array} \tag{3.79}$$

However, in the (a, w)-planes, two sets of 2^3 straight lines are important. The first set is

$$\ell_r^+ \ : \ (a-1) + w + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1} = 0.$$

which is related to Eq. (3.76). The second set is

$$\ell_r^- \ : \ (a-1) - w - (a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1}) = 0.$$

which is related to Eq. (3.77), where $v_{i+1,j,k}, v_{i,j+1,k}, v_{i,j,k+1} \in \{-1, 1\}$ and $1 \le r \le 8$. When (a_x, a_y, a_z) lines in the open region (I)$\sim$(VIII), (i)$\sim$(vi) and (1)$\sim$(2) as in Fig. 3, Eqs. (3.78) and (3.79) are used to partition the $(w, a-1)$-plane, as in Fig. 4.

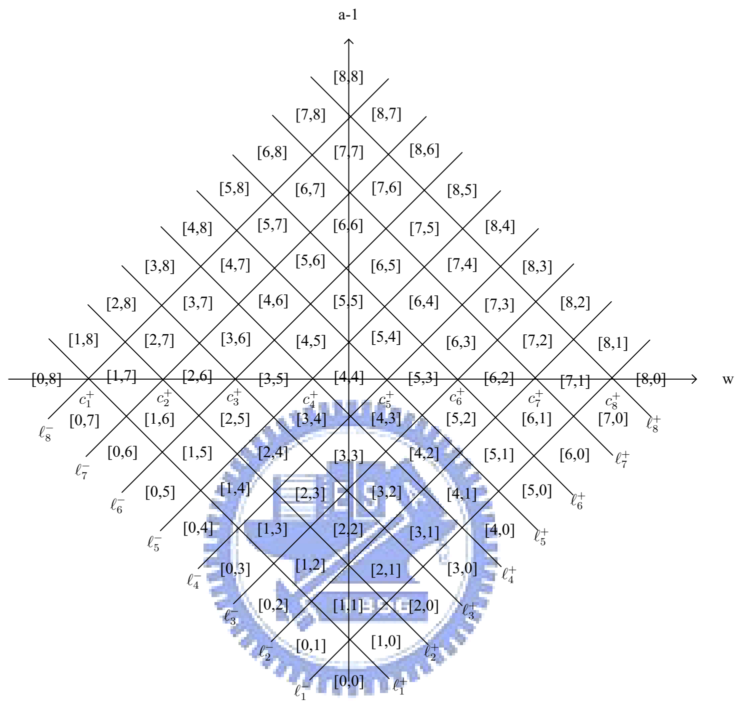


Figure 4: Partition of $(w, a-1)$-plane.

The symbols [m,n] in Fig. 4 have the following meanings. Consider, for example, (a_x, a_y, a_z) lies in regions (VIII), (i) and (1) as in Fig. 3, Eq. (3.78) and Eq. (3.79). This situation is expressed as (VIII)-(i)-(1), and considered $a_x < a_y < a_z < 0$ and $|a_x| > |a_y| + |a_z|$. Denoted by

	$(v_{i+1,j,k}, v_{i,j+1,k}, v_{i,j,k+1})$	$-(a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1})$
$c_1^+ = c_8^-$	$(-1,-1,-1)$	$a_x + a_y + a_z$
$c_2^+ = c_7^-$	$(-1,-1,1)$	$a_x + a_y - a_z$
$c_3^+ = c_6^-$	$(-1,1,-1)$	$a_x - a_y + a_z$
$c_4^+ = c_5^-$	$(-1,1,1)$	$a_x - a_y - a_z$
$c_5^+ = c_4^-$	$(1,-1,-1)$	$-a_x + a_y + a_z$
$c_6^+ = c_3^-$	$(1,-1,1)$	$-a_x + a_y - a_z$
$c_7^+ = c_2^-$	$(1,1,-1)$	$-a_x - a_y + a_z$
$c_8^+ = c_1^-$	$(1,1,1)$	$-a_x - a_y - a_z$

Table 1: The intersects of ℓ_i^+ and ℓ_j^-.

Then, $c_8^+ > c_7^+ > c_6^+ > c_5^+ > 0 > c_4^+ > c_3^+ > c_2^+ > c_1^+ >$ are the intersects of ℓ_i^+ and ℓ_j^- on the w-axis displayed in Fig. 4.

With reference to the local patterns on cube-cells, +1 is represented by the symbol + and −1 is represented by the symbol −. The 2^4 local patterns can be listed and ordered, as in Fig. 5.

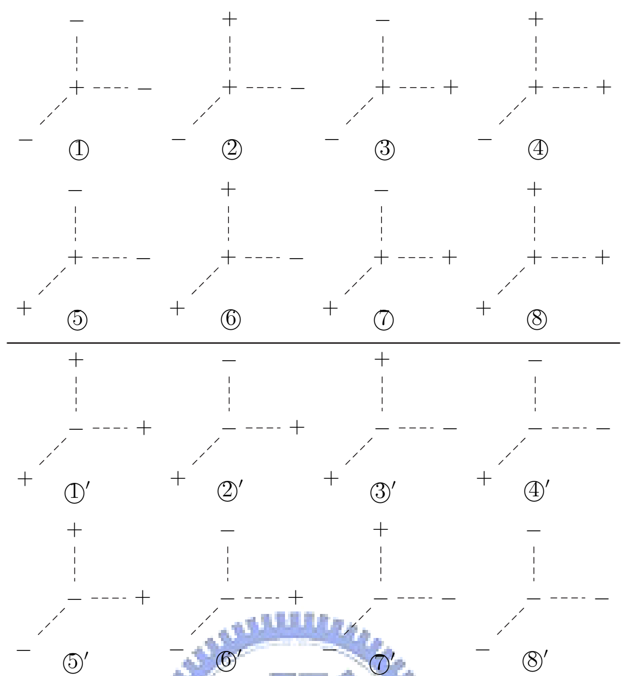


Figure 5: Ordering of local patterns in partition (VIII)-(i)-(1).

Now, when $(a-1, w)$ lies in region [m,n] in Fig. 4, the only admissible patterns are exactly ①, ②, $\cdots$, ⓜ and ①′, ②′, $\cdots$, ⓝ′. For instance, in region (VIII)-(i)-(1) and $(w, a-1) \in [4, 8]$ only ①, ②, ③, ④ and ①′, ②′, ③′, ④′⑤′, ⑥′⑦′, ⑧′ can be produced. This fact is equivalent to the holding of inequalities in Eqs. (3.76) and (3.77) if and only if $v_{i,j,k}$, $v_{i+1,j,k}$, $v_{i,j+1,k}$ and $v_{i,j,k+1}$ are of the form ①, ②, ③, ④ and ①′, ②′, ③′, ④′⑤′, ⑥′⑦′, ⑧′.

Next, the transition matrix of local patterns in region (VIII)-(i)-(1)-[4,8] can be derived as

$$\mathbb{A}_{x;2\times2\times2} = G \otimes E \otimes E \otimes E.$$

Then, according to Proposition 3.1.15, the admissible local patterns in $\Sigma_{2\times m_2\times m_3}$ and its corresponding transition matrices are

$$\begin{aligned} \mathbb{A}_{x;2\times m_2\times 2} &= \otimes(G \otimes E)^{m_2-1} \otimes (\otimes E^2), \\ \mathbb{A}_{\hat{x};2\times m_2\times 2} &= (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1}), \\ \mathbb{A}_{\hat{x};2\times m_2\times m_3} &= \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2}), \end{aligned}$$

as in Eqs. (3.36), (3.37) and (3.38).

Finally, the connecting operator is adopted to examine the complexity of the set of mosaic patterns in 3DCNN. That is, the lower bound of spatial entropy in the region (VIII)-(i)-(1)-[4,8] can be estimated.

Proposition 3.3.1. *Consider* $\mathbb{A}_{x;2\times 2\times 2} = G\otimes E\otimes E\otimes E$, *then*

$$S_{z;m_1;m_2 2;11} = C_{z;m_1;m_2 2;11} = (\otimes G^{m_2-1})\otimes E,$$

$$\rho(S_{z;m_1;m_2 2;11}) = 2g^{m_2-1}$$

and

$$h(\mathbb{A}_{x;2\times 2\times 2}) \geq \frac{1}{2}\log g,$$

where $g = \frac{1+\sqrt{5}}{2}$ *is the golden-mean. Moreover, since*

$$\mathbb{A}_{\hat{x};2\times m_2\times m_3} = \otimes((\otimes G^{m_2-1})\otimes E)^{m_3-1}\otimes(\otimes E^{m_2})$$

and

$$\rho(\mathbb{A}_{\hat{x};2\times m_2\times m_3}) = 2^{m_2+m_3-1}g^{(m_2-1)(m_3-1)},$$

the spatial entropy can be exactly computed as

$$h(\mathbb{A}_{x;2\times 2\times 2}) = \log g$$

as in Proposition 3.1.15.

Proof. According to Eq. (3.37),

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = (\otimes G^{m_2-1})\otimes(\otimes E^{m_2+1})$$

is obtained. Evidently,

$$A_{\hat{x};2\times m_2\times 2;1} = \otimes E^{m_2}$$

and

$$(\mathbb{A}^{(r)}_{\hat{x};2\times m_2\times 2})^{(c)}_{;1} = (\otimes G^{m_2-1})\otimes E.$$

By Remark 3.2.3, the connecting operator

$$\begin{aligned} C_{z;m_1;m_2 2;11} &= A_{\hat{x};2\times m_2\times 2;1}\circ(\mathbb{A}^{(r)}_{\hat{x};2\times m_2\times 2})^{(c)}_{;1} \\ &= (\otimes G^{m_2-1})\otimes E. \end{aligned}$$

Therefore, based on Remark 3.2.13, the lower bound of spatial entropy is estimated as

$$\begin{aligned} h(\mathbb{A}_{x;2\times 2\times 2}) &\geq \lim_{m_2\to\infty}\frac{1}{2m_2}\log\rho(S_{z;m_1;m_2 2;11}) \\ &= \lim_{m_2\to\infty}\frac{\log 2g^{m_2-1}}{2m_2} \\ &= \frac{1}{2}\log g. \end{aligned}$$

□

Remark 3.3.2. *For the general template* $A = (a_{\alpha,\beta,\gamma})$ *where* $a_{\alpha,\beta,\gamma}\neq 0$, *the basic set in* $\Sigma_{3\times 3\times 3}$ *must be extend to the basic set in* $\Sigma_{4\times 4\times 4}$. *Then, the method described above can be applied, as stated in Remark 3.2.14. The details are omitted here for brevity.*

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