Abstract (in Chinese)

# Diameters and Wide-Diameters of de Bruijn Graphs

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## Abstract

In graph theory, the study of fault tolerance and transmission delay of networks, the connectivity and diameter of a graph are two very important parameters. Since the de Bruijn graphs and generalized de Bruijn graphs are known to have small diameters, and simple routing strategies, they have been widely used as models for communication networks and multiprocessor systems.

The directed de Bruijn graph $B(d, n)$ has vertex-set $V = \{x_1x_2 \cdots x_n : x_i \in Z_d, i = 1, 2, \cdots, n\}$ and directed edge-set $E$, where for $\mathbf{x} = x_1x_2 \ldots x_n$, $\mathbf{y} = y_1y_2 \ldots y_n \in V$, $\mathbf{xy} \in E$ if and only if $y_i = x_{i+1}$ for $i = 1, 2, \ldots, n-1$. Clearly, $B(d, n)$ has $d^n$ vertices thus there is a restriction on the number of vertices. To conquer this disadvantage, a modification, generalized de Bruijn graphs, was obtained later by Imase and Itoh, and independently by Reddy, Pradhadn and Kuhl.

The generalized directed de Bruijn graph $G_B(n, m)$ is a directed graph whose vertices are $0, 1, 2, \cdots, m-1$ and the directed edges are of the form

$$i \to in + \alpha \pmod m, \forall i \in \{0, 1, \ldots, m-1\} \text{ and } \forall \alpha \in \{0, 1, \ldots, n-1\}.$$

Then, by replacing directed edges with undirected edges and omitting the loops and multi-edges of the directed de Bruijn graphs and generalized directed de Bruijn graphs, we have the undirected de Bruijn graphs and generalized undirected de Bruijn graphs respectively. In this thesis, we focus on these de Bruijn graphs.

First, we have "introduction and preliminaries" in Chapter 1. Then, in Chapter 2, we have a survey of several important known properties and applications. After survey, we study the wide-diameters of undirected de Bruijn graphs in Chapter 3, and study the diameters of generalized undirected de Bruijn graphs in Chapter 4. Finally, in Chapter 5, we conclude this thesis with several concluding remarks.

# Acknowledgments

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# Chapter 1

# Introduction and Preliminaries

In this chapter, we first list the basic notations, terminologies, and definitions of graph theory [53], which will be used throughout this thesis. Then, following the introduction of the *de Bruijn graphs* and *generalized de Bruijn graphs*, we present several fundamental and important properties of these graphs.

## 1.1 Motivation

Toward the $21^{st}$ century, communication (both wired and wireless) plays the most important role in human society. Therefore, it is essential to have reliable and efficient communication networks. It is well known that a network can be modelled by a directed graph in which the vertices represent switching elements or processors while the directed edges (arcs) represent communication links.

An essential factor for realizing a highly reliable and efficient network with a limited number of links consists of finding a directed graph with a minimal diameter and a maximal connectivity for a given number of vertices and degrees. This is due to the diameter is related to the transmission delay efficiency and the connectivity is directly related to the fault-tolerance capacity reliability of networks.

The best known general family of directed graphs which satisfy the above constraints was first defined by Imase and Itoh [30], the *directed de Bruijn graphs*. It was verified later by Xu et al. [54] that *directed de Bruijn networks* are suitable model for interconnection networks in parallel and distributed processing systems. Also, networks are regarded as good competitors for the hypercube and might constitute the next generation of parallel architectures. Therefore, our goal is to study this class of *de Bruijn graphs*. Thus, we can obtain a better understanding of the topology of these networks. As a modification, we shall also study the *generalized directed and undirected de Bruijn graphs* respectively.

## 1.2 Graphs

A **graph** $G$ is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its **endpoints**. We **draw** a graph on paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints. In this section, we focus on the undirected graphs in which all the edges all have no directions.

A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints. A **simple graph** is a graph having no loops or multiple edges. In this case an edge is determined by its endpoints and can be viewed as an unordered pair of vertices. Thus a simple graph can be specified by its vertex set and edge set, through treating the edge set as a set of unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge $e$ with endpoints $u$ and $v$.

Figure 1.1 is a drawing of a finite simple graph. The vertex set is $\{u, v, w, x, y\}$, and the edge set is $\{uv, uw, ux, vx, vw, xw, xy\}$.

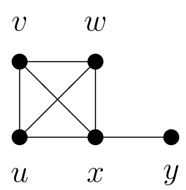


Figure 1.1: A drawing of a finite simple graph.

If vertex $v$ is an endpoint of edge $e$, then $v$ and $e$ are **incident**. The **degree** of vertex $v$ in a loopless graph $G$, namely $d_G(v)$, is the number of edges incident with $v$. The maximum degree of $G$ is denoted by $\Delta(G)$ and the minimum degree is $\delta(G)$. A vertex is **odd** (**even**) when its degree is odd (even). An **isolated vertex** is a vertex of degree 0. A graph $G$ is **bipartite** if $V(G)$ is the union of two disjoint (possibly empty) independent sets called **partite sets** of $G$. A **complete bipartite graph** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets.

When $u$ and $v$ are the endpoints of an edge, they are **adjacent** and are **neighbors**. The **neighborhood** of $v$ in $G$, written as $N(v)$, is the set of vertices adjacent to $v$. Furthermore, two edges are **incident** if they have one endpoint in common.

A **walk** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge $e_i$ has endpoints $v_{i-1}$ and $v_i$. A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A **trail** is a walk with no repeated edges. A $u, v$**-walk** or $u, v$**-trail** has first vertex $u$ and last vertex $v$ which are called **endpoints**. A $u, v$**-path** is a path whose vertices of degree 1 (its **endpoints**) are $u$ and $v$; the others are **internal vertices** [53]. The **length** of a walk, trail, path, or cycle is its number of edges. A walk or trail is **closed** if its endpoints are the same.

If $G$ has a $u, v$-path, then $u$ is **connected to** $v$ in $G$. A graph $G$ is **connected** if it has a $u, v$-path whenever $u, v \in V(G)$ (otherwise, $G$ is **disconnected**). A **maximal** connected subgraph of $G$ is a subgraph that is connected and is not contained in any other connected subgraph of $G$. The **components** of a graph $G$ are its maximal connected subgraphs. Components are pairwise disjoint that is for any two components have no vertices in sharing. A component (or a graph) is **trivial** if it has no edges; otherwise it is **nontrivial**.

## 1.3 Directed Graphs

A **directed graph** or **digraph** $D$ is a triple consisting of a vertex set $V(D)$, an edge set $E(D)$, and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the **tail** of the edge, and the second is the **head**; together, they are the **endpoints**. The terms "head" and "tail" come from the arrows used to draw directed graphs. As with graphs, we assign each vertex a point in the plane and each edge a curve joining its endpoints. When drawing a directed graph, the direction of a curve is from the tail to the head. Figure 1.2 shows a directed graph $D$ with vertex set $V(D) = \{000, 001, 010, 011, 100, 101, 110, 111\}$ and edge set $E(D) = \{(000, 000), (000, 001), (100, 000), (100, 001), (001, 011), (001, 010), (010, 100), (010, 101), (101, 010), (110, 100), (101, 011), (110, 101), (011, 110), (011, 111), (111, 110), (111, 111)\}$.

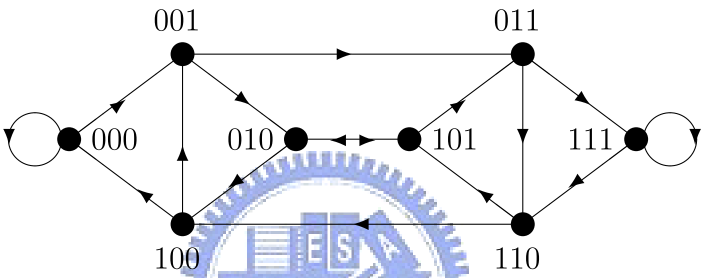


Figure 1.2: A *directed de Bruijn graph* $B(2, 3)$.

In a directed graph, a **loop** is an edge whose endpoints are equal, such as (000,000) and (111,111) in Figure 1.2. **Multiple edges** are edges having the same ordered pair of endpoints. A directed graph is **simple** if each ordered pair of vertices have at most one edge. And one loop may be present at each vertex. Therefore, Figure 1.2 is a simple directed graph.

In a simple directed graph, we write $uv$ for an edge with tail $u$ and head $v$. If there is an edge from $u$ to $v$, then $v$ is a **successor** of $u$, and $u$ is a **predecessor** of $v$. We write $u \to v$ for "there is an edge from $u$ to $v$".

A directed graph is a **path** if it is a simple directed graph whose vertices can be linearly ordered so that there is an edge with tail $u$ and head $v$ if and only if $v$ immediately follows $u$ in the vertex ordering. A **cycle** is defined similarly using an ordering of the vertices on a circle. The graph and directed graph models often share the same names of corresponding concepts. Also, a graph $G$ can be modelled using a directed graph $D$ in which each edge $uv \in E(G)$ is replaced with $uv, vu \in E(D)$. In this way, results from directed graphs can be applied to graphs. Since the notion of "edge" in directed graphs extends the notion of "edge" in graphs, using the same name makes sense.

The definitions of **trail** and **walk** are the same in graphs and directed graphs when we list edges as ordered pairs of vertices. In a directed graph, the successive edges in a walk must "follow the arrows". In a walk $v_0, e_1, v_1, \ldots, e_k, v_k$, the edge $e_i$ has tail $v_{i-1}$ and head $v_i$.

## 1.4 Notations

For the sake of brevity, we define $[a, b]$ as $\{a, a+1, \ldots, b\}$ for non-negative integers $a < b$. We use $Z_d$ for the representation of $\{0, 1, 2, \ldots, d-1\}$, $Z_d^*$ for $\{1, 2, \ldots, d-1\}$, and $N$ for $\{1, 2, 3, \ldots\}$.

For $x \in \mathbb{R}$, the **floor** $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. The **ceiling** $\lceil x \rceil$ is the smallest integer greater than or equal to $x$.

A graph $G$ is **regular** if $\Delta(G) = \delta(G)$. The graph is $k$**-regular** if the common degree is $k$. For instance, **cubic graph** is a graph that is 3-regular. An **even graph** is a graph with vertex degrees all even.

If $G$ has a $u, v$-path, then the **distance** from $u$ to $v$, written as $d_G(u, v)$ or simply $d(u, v)$, is the least length of the $u, v$-path. If $G$ has no $u, v$-path, then $d(u, v) = \infty$. The **diameter** of $G$ is $max_{u,v \in V(G)} d(u, v)$, written as $d(G)$. We use $< u, \ldots, v >$ to denote a path from $u$ to $v$ in $G$.

The concepts of container, wide-diameter, and fault diameter arose naturally from the study of routing, reliability, fault tolerance, and communication protocols in parallel architectures and distributed computer networks. Containers can be used to accelerate the transmission rate and to enhance the transmission reliability. The wide-diameter and fault diameter are two generalizations of the diameter. For all pairs of vertices, the diameter measures the length of shortest paths, while the wide-diameter measures the maximal length of the best containers. Practically, node faults may happen. The fault diameter estimates the maximal increment of the diameter when there are node faults. It is both practically and theoretically important to compute the wide-diameter and fault diameter of the networks. For the survey of these parameters, we refer to [27].

## 1.5 Directed and Undirected de Bruijn Graphs

Graphs are widely used in the design and analysis of parallel computer network systems. A vertex in the graph denotes a node (or processor) in the corresponding network, and an edge represents a communication link between two nodes. We will not discuss the difference between network and graph in this thesis.

The *de Bruijn interconnection network* is modelled by the *de Bruijn graph*, which is named after N. G. de Bruijn for his work in counting $d$-ary sequences of maximal period [6]. The *de Bruijn graph* was widely studied as a communication network model, and was proposed as a suitable processor interconnection network for VLSI implementation [50]. In this thesis, we use $B(d,n)$ [22] and $UB(d,n)$ to denote the *directed de Bruijn graph* and *undirected de Bruijn graph* respectively.

*De Bruijn networks* have many good features. In particular, they are among the best-known networks for a given degree and diameter [2]. They have good vulnerability properties and are able to tolerate up to $d-1$ faults in the directed case and $2d-3$ in the undirected case. They are adequate for various applications as one can embed in linear arrays, rings, and complete binary trees. They can also simulate without loss of time shuffle-exchanges or hypercubes for the class of ascend-descend algorithms.

The following shows the definition of *directed de Bruijn graph* $B(d,n)$ for $n \geq 1$ and $d \geq 2$. The *directed de Bruijn graph*$B(d,n)$ has a vertex-set

$$V = \{x_1x_2\dots x_n : x_i \in Z_d, i = 1,2,\dots,n\}$$

and a directed edge-set $E$, where $\mathbf{x} = x_1x_2\dots x_n$, $\mathbf{y} = y_1y_2\dots y_n \in V$, $\mathbf{xy} \in E$ if and only if $y_i = x_{i+1}$ for $i = 1,2,\dots,n-1$. It has been shown that the $B(d,n)$ is $d$-regular and has connectivity $\kappa = d-1$ [51]. Sridhar [51] also showed that for any two vertices $\mathbf{x}$ and $\mathbf{y}$ in $B(d,n)$, there are at least $d-1$ disjoint paths of length at most $n+1 = \lfloor \log N \rfloor + 1$, where $N = |V|$. We will use this concept again in the following chapters. The desirable structural properties of $B(d,n)$ may be found in an early survey by Bermond and Peyrat [4]. Figure 1.2 shows a *directed de Bruijn graph* $B(2,3)$.

The *undirected de Bruijn graph*, denoted $UB(d,n)$, is obtained from $B(d,n)$ by omitting the orientation of all directed edges, omitting multiple edges, and omitting loops. For instance $UB(2,3)$ in Figure 1.3 is obtained from $B(2,3)$ in Figure 1.2.

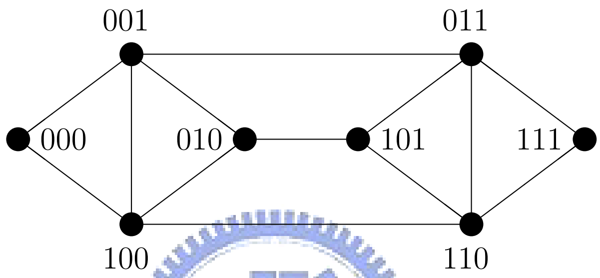


Figure 1.3: An *undirected de Bruijn graph* $UB(2,3)$.

In $UB(d,n)$, there are $d$ vertices with degree $2d-2$, $d(d-1)$ vertices with degree $2d-1$, and the others with degree $2d$. Moreover, $UB(d,n)$ has a diameter $n$, a vertex-connectivity $\kappa = 2d-2$, and an edge-connectivity $\lambda = 2d-2$ [20]. This implies that, for any two vertices $\mathbf{x}$ and $\mathbf{y}$ in $UB(d,n)$, there are at least $2d-2$ disjoint paths from $\mathbf{x}$ to $\mathbf{y}$. Yet, the length of these $2d-2$ paths may be very large compared to $n$. So it is important to know the $(2d-2)$-wide-diameter of $UB(d,n)$. The wide-diameter of *undirected de Bruijn graph* will be introduced in Chapter 3 again.

Interconnection networks with bidirectional communication links are usually modelled by undirected graphs. The *undirected de Bruijn graph* is one of the most important models. They have good properties, such as being able to tolerate up to $d-2$ (considering the vertices with loop) faults in the directed case and $2d-3$ in the undirected case while the diameter is still small. Several topics related to the fault-tolerant capabilities of *de Bruijn networks* have been studied, including the fault-tolerant routing, fault diagnoses, fault-tolerant VLSI designs, and the embedding of linear arrays, rings and complete binary trees [36, 50]. Therefore, studying *de Bruijn graphs* is an important work.

## 1.6 Generalized Directed and Undirected de Bruijn Graphs

Since graphs $B(d,n)$ have short diameter, and simple routing strategies, they have been widely used as models for communication networks and multiprocessor systems. However, one of the disadvantage of $B(d,n)$ is the restriction on the number of vertices [16]. From $B(d,n)$ to $B(d,n+1)$, the number of vertices will increase from $d^n$ to $d^{n+1}$. As $d$ or $n$ increased, the gap between $d^n$ and $d^{n+1}$ becomes larger and larger, which also poses the problem of smooth expansion [16]. Therefore, this increase the difficulty for its applications.

In 1981, Imase and Itoh [30] modified the generating function of the *directed de Bruijn graph* to include any number of vertices. Reddy, Pradhadn and Kuhl [45] also proposed the same graph independently in 1980. In this thesis, we use $G_B(n,m)$ and $UG_B(n,m)$ to denote the *generalized directed de Bruijn graph* and *generalized undirected de Bruijn graph*.

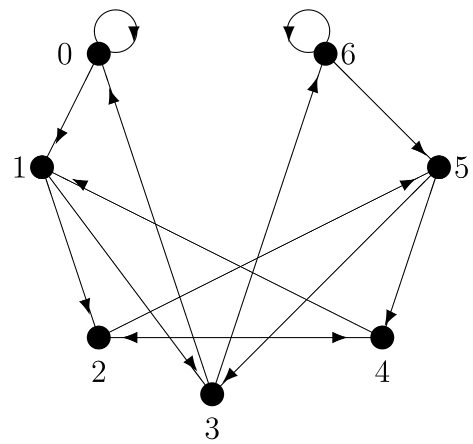


Figure 1.4: A *generalized directed de Bruijn graph* $G_B(2,7)$.

The $G_B(n,m)$ is the directed graph where the vertices are $0,1,\ldots,m-1$, and the directed edges (arcs) are expressed as:

$$i \rightarrow in + \alpha \pmod{m}, \forall i \in \{0, 1, \ldots, m-1\} \text{ and } \forall \alpha \in \{0, 1, \ldots, n-1\}.$$

Figure 1.4 demonstrates a $G_B(2,7)$ graph.

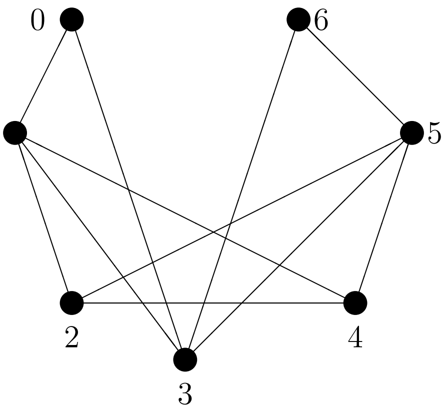


Figure 1.5: $UG_B(2,7)$.

The *generalized undirected de Bruijn graph* $UG_B(n,m)$ is the undirected graph, which is derived from the *generalized directed de Bruijn graph* $G_B(n,m)$ by replacing directed edges with undirected edges and omitting the loops and multi-edges. Figure 1.5, $UG_B(2,7)$ is derived from $G_B(2,7)$ in Figure 1.4. The set of neighbors of any vertex $i$ in $UG_B(n,m)$ can be expressed as:

$$N(i) = R(i) \cup L(i) \tag{1.1}$$

where

$$R(i) = \{in + \alpha \pmod{m} : \alpha \in [0, n-1]\}, \text{ and} \tag{1.2}$$

$$L(i) = \{j : jn + \beta \equiv i \pmod{m}, \text{ where } \beta \in [0, n-1], j \in [0, m-1]\}. \tag{1.3}$$

Therefore, if $j \in R(i)$ then $i \in L(j)$ in $UG_B(n,m)$. From the definitions, we obtained that the *directed de Bruijn graph* is a special case of $G_B(n,m)$ when $m$ is a power of $d$, so is the *undirected de Bruijn graph*.

From the $UG_B(2,7)$ in Figure 1.5, we can find the degree of vertex 0 is 2 less than the degree of vertex 1. But some researchers may rewrite the form to make

$UG_B(n,m)$ regular.

$$i \to in + \alpha \pmod{m}, \forall i \in [0, m-1] \text{ and } \forall \alpha \in [0, n-1], \alpha \not\equiv i \pmod{n}.$$

Figure 1.6 shows the regular graph $UG_B(3,12)$. Note that the *undirected de Bruijn graphs* are **almost regular**.

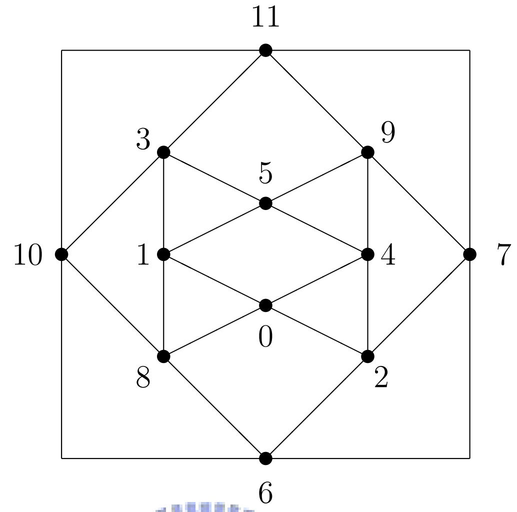


Figure 1.6: $UG_B(3,12)$.

Imase, Soneoka and Okada [31] proved that the *generalized directed de Bruijn graph* $G_B(n,m)$ is $(n-1)$-connected and its diameter is bounded from above by $\lceil \log_n m \rceil$. Therefore, $UG_B(n,m)$ is also $(n-1)$-connected and its diameter is also bounded above by the same value. Then the result is listed below for further references:

**Theorem 1.6.1.** *The diameter of* $UG_B(n,m)$ *is at most* $\lceil \log_n m \rceil$.

Due to their small diameters and simple routing strategies, *de Bruijn graphs* have been widely used as models for communication networks and multiprocessor systems [16]. However, $B(d,n)$ has the restriction on the number of vertices $d^n$, which limits its practical applications. The *generalized directed and undirected de Bruijn graphs* retain all of the properties of the de Bruijn graphs, but have no restrictions on the number of vertices [16]. Thus, determining the connectivity and diameter of $UG_B(n,m)$ is relevantly interesting and important. Detail properties will be described in Chapter 2.

# Chapter 2

# Known Results

From the literatures, there are many studies focus on the *directed and undirected de Bruijn graphs* [17, 18, 26, 31, 43, 45, 51], which shows the importance of these graphs in communication network models. In this chapter, we will give an overview of these networks.

## 2.1 Directed de Bruijn Graphs

*Directed de Bruijn graph* has a recursive structure which can define the $B(d, n)$ recursively. $B(d, n)$ is eulerian and Hamiltonian. All of these properties will be introduced in this section.

### 2.1.1 Recursive Structure of Directed de Bruijn Graphs

The *directed de Bruijn graph* $B(d, n)$ has the vertex-set

$$V = \{x_1 x_2 \dots x_n : x_i \in Z_d, i = 1, 2, \dots, n\}$$

and the directed edge-set $E$, where for $\mathbf{x} = x_1 x_2 \dots x_n$, $\mathbf{y} = y_1 y_2 \dots y_n \in V$, $\mathbf{xy} \in E$ if and only if $y_i = x_{i+1}$ for $i = 1, 2, \dots, n-1$. See Figure 1.2 as an example.

A more precise way of defining $B(d, n)$ can be obtained as follows. First, let $K_d^+$ denotes the directed graph obtained from the complete symmetric directed graph with $d \geq 2$ vertices by attaching a loop to each vertex. Let $V(K_d^+) = \{0, 1, 2, \cdots, d-1\}$. Then, $B(d, 1)$ is exactly isomorphic to $K_d^+$. Now, if we take the line graph of $K_d^+$ where a loop is considered as an edge (arc), then each arc of $K_d^+$ becomes a vertex of $L(K_d^+)$. Moreover, $(x_1, x_2)$ is incident to $(y_1, y_2)$ provided $x_2 = y_1$ and the new arc is $(x_1x_2, y_1y_2)$. This phenomenon shows that $B(d, 2) \cong L(K_d^+)$. Following the similar argument, we get $B(d, n) \cong L^{n-1}(K_d^+)$ for $n \geq 2$. Sridhar [51] showed the connectivity of $B(d, n)$ is $d - 1$. By Menget's Theorem, there are $d - 1$ internally disjoint paths between any two vertices. Then, by using the structure of line graphs, it is easier to find good properties of $B(d, n)$. The Theorem 2.1.1 obtained by Xu et al. [54] is an independent proof of an earlier result from Sridhar [51] which uses the above idea. For completeness, we also include the proof of this result.

**Theorem 2.1.1.** *[54] For any two distinct vertices $x$ and $y$ of $B(d, n)$, there are $d-1$ internally disjoint $(x, y)$-paths of length at most $n + 1$.*

**Proof.** We proceed by induction on $n \geq 1$. Since $B(d, 1) \cong K_d^+$, the theorem is true for $n = 1$. Suppose $n \geq 2$ and the theorem holds for any two vertices of $B(d, n - 1)$. Let $x$ and $y$ correspond to two vertices of $B(d, n)$. Since $B(d, n) = L(B(d, n-1))$, let such two vertices be $x = (w, w')$ and $y = (v, v')$, where $w$, $w'$, $v$, and $v'$ are vertices of $B(d, n - 1)$.

If $w' \neq v$, then by the induction hypothesis, there are $d - 1$ internally disjoint $(w', v)$-paths of length at most $n$ in $B(d, n-1)$, from which we can easily induce $d-1$ internally disjoint $(x, y)$-paths of length at most $n + 1$ in $B(d, n)$.

If $w' = v$, then $(x, y)$ is an edge of $B(d, n)$, and the two vertices can be written as

$$x = x_1x_2 \ldots x_n \text{ and } y = x_2x_3 \ldots x_nx_{n+1},$$

hence $(x_1, x_2, \ldots, x_n, x_{n+1})$ is a walk of length $n$ in $K_d^+$. Now, we construct the $d - 1$ internally disjoint $(x, y)$-walks $W_1, W_2, \ldots, W_{d-1}$ of length at most $n + 1$ in $B(d, n)$ as follows:

$$W_1 = (x_1, x_2, \ldots, x_n, x_{n+1}),$$

$$W_j = (x_1, x_2, \ldots, x_n, u_j, x_2, x_3, \ldots, x_n, x_{n+1}),\ j = 2, 3, \ldots, d - 1$$

where $u_2, \ldots, u_{d-1}$ are $d - 2$ distinct elements in $\{0, 1, \ldots, d - 1\} \setminus \{x_k, x_{k+1}\}$. It is clear that $|W_1| = 1$ and the others $|W_j| = n + 1$ for $j = 2, 3, \ldots, d - 1$. Thus we need prove that $W_2, \ldots, W_{d-1}$ are internally vertex disjoint in $B(d, n)$.

Suppose not. Then there exist $W_i$ and $W_j$, $2 \leq i \neq j \leq d-1$, which have common vertices. Let $u(\neq x, y)$ be the first common vertex of $W_i$ and $W_j$. Assume the section $W_i(x, u)$ is of length $a$ and the section $W_j(x, u)$ is of length $b$. Then we get $2 \leq a, b \leq n-1$. Let $u'$ and $u''$ be in-neighbors of $u$ on $W_i$ and $W_j$, respectively. Therefore we have $u' \neq u''$. Since $u$ can be reached in $a$ steps from $x$ along $W_i$ and in $b$ steps from $x$ along $W_j$, then it can be written as

$$u = x_{a+1}x_{a+2}\ldots x_n u_i x_2 \ldots x_a = x_{b+1}x_{b+2}\ldots x_n u_j x_2 \ldots x_b.$$

From this expression, we have $x_a = x_b$ since $2 \leq a, b \leq n-1$, namely

$$u' = x_a x_{a+1}x_{a+2}\ldots x_n u_i x_2 \ldots x_{a-1} = x_b x_{b+1}x_{b+2}\ldots x_n u_j x_2 \ldots x_{b-1} = u'',$$

which is a contradiction. This concludes the proof. □

Theorem 2.1.1 combines connectivity and diameter, which is called wide-diameter and will be mentioned again in Chapter 3.

### 2.1.2 de Bruijn Sequences and Hamiltonian Cycles

In this section, two main properties obtained from $B(d, n)$ will be presented. We start with the idea of encoding a sequence of $n$ digits or in the terminology of Coding Theory a codeword of length $n$. Fox example, let $n = 3$. Then there are $2^3$ distinct codewords of length 3: (0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0), and (1,1,1). One of the best ways to encode these words is to use the so-called rotating drum, as shown in Figure 2.1. By rotating the drum counterclockwise, we can encode these eight words as: 000,001,011,111,110,101,010,100. Since any two consecutive words have two digits in common, the above encoding can be represented as (00011101) in a cyclic order. It is not difficult to see this cyclic arrangement can be obtained from $B(2, 2)$. Therefore, the sequence obtained above is known as a de Bruijn sequence with $n = 2$ and $d = 2$. For convenience, this sequence is referred a de Bruijn $n$-sequence. In fact, for $d = 2$, a de Bruijn $n$-sequence with $n \geq 2$ exists.

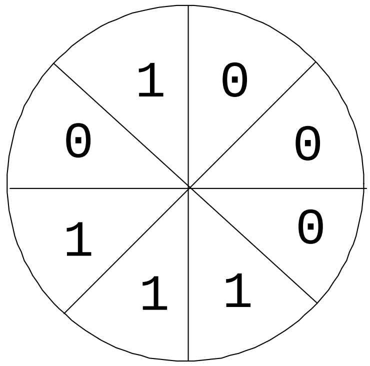


Figure 2.1: Rotating drum.

**Theorem 2.1.2.** *[53] The directed graph $B(2, n)$ is eulerian, and the edge labels on the edges in any eulerian circuit of $B(2, n)$ form a cyclic arrangement in which the $2^n$ consecutive segments of length $n$ are distinct.*

It is worth noting here that a de Bruijn sequence with $d = 2$ and $n = 3$ can be obtained from a *directed de Bruijn graph* $B(2, 2)$. Therefore, $B(2, n-1)$ gives a de Bruijn sequence which produces $2^n$ distinct codewords of length $n$. The idea comes from finding an eulerian circuit of $B(2, n-1)$.

A de Bruijn $n$-sequence can also be applied to find a Hamiltonian cycle in a *directed de Bruijn graph* $B(2, n)$, from the structure of $B(2, n)$. In fact, for $d \geq 2$, a *directed de Bruijn graph* $B(d, n)$ does contain a Hamiltonian cycle.

**Theorem 2.1.3.** *[23] The directed de Bruijn graph $B(d, n)$ contains a Hamiltonian cycle for $d \geq 2$ and $n \geq 2$.*

## 2.2 Undirected de Bruijn Graphs

The shortest path between any two distinct vertices in $B(d, n)$ is easy to determine. But for the $UB(d, n)$ the shortest path is difficult to solve. In this section, we focus on finding the shortest path of $UB(d, n)$. First, let us recall the definition of $B(d, n)$. The $B(d, n)$ has vertex-set $V = \{x_1 x_2 \cdots x_n : x_i \in Z_d, i = 1, 2, \cdots, n\}$ and directed edge-set $E$, where for $\mathbf{x} = x_1 x_2 \ldots x_n$, $\mathbf{y} = y_1 y_2 \ldots y_n \in V$, $\mathbf{xy} \in E$ if and only if

$y_i = x_{i+1}$ for $i = 1, 2, \ldots, n-1$. Define $R(x_1x_2\ldots x_n) = \{x_2\ldots x_n\alpha : \alpha \in Z_d\}$ and $L(x_1x_2\ldots x_n) = \{\alpha x_1x_2\ldots x_{n-1} : \alpha \in Z_d\}$.

A $(v^{(0)}, v^{(k)})$-path denotes a path from $v^{(0)}$ to $v^{(k)}$. A $(v^{(0)}, v^{(k)})$-path:

$$Q =< v^{(0)}, v^{(1)}, \ldots, v^{(k)} >$$

is called an $R$-path if $v^{(i)} \in R(v^{(i-1)})$ for $1 \le i \le k$, and an $L$-path if $v^{(i)} \in L(v^{(i-1)})$ for $1 \le i \le k$. For convenience, we use $R$ and $L$ to denote an $R$-path and an $L$-path, respectively. The segment $< v^{(p)}, v^{(p+1)}, \ldots, v^{(q)} >$ in $Q$ for $0 \le p < q \le k$ is called a subpath of $Q$. We write $P = R_1L_1R_2L_2\ldots$ if the path $P$ consists of an $R$-path $R_1$, followed by an $L$-path $L_1$, an $R$-path $R_2$, an $L$-path $L_2$, and so on, where subscripts are used to distinguish different subpaths if necessary. Subscripts of these subpaths can be omitted if no ambiguity will occur, for example $P = R_1LR_2$ or $P = RL$.

We use $|P|$ to denote the length of the path $P$, and for two vertices $x$ and $y$, we use $P_{xy}$ to denote the shortest path between $x$ and $y$ in $UB(d, n)$. So, $|P_{xy}|$ denote the distance between $x$ and $y$. Liu and Sung [38] the presented properties of the shortest paths between any two vertices of $UB(d, n)$ and proposed two shortest-path routing algorithms, one of which has linear time complexity. First, we introduce a lemma about the properties.

**Lemma 2.2.1.** *[38] $P_{xy}$ cannot contain either $R_1L_1R_2L_2$ or $L_1R_1L_2R_2$ as subpaths.*

Liu and Sung derived the following corollary from the result of the above lemma.

**Corollary 2.2.2.** *[38] $P_{xy}$ must be one of the following types.*

- *type 1: $P_{xy} = R_1LR_2$ with $|L| > \max\{|R_1|, |R_2|\}$ or $L_1RL_2$ with $|R| > \max\{|L_1|, |L_2|\}$,*
- *type 2: $P_{xy} = RL$ or $LR$,*
- *type 3: $P_{xy} = R$ or $L$.*

By the Corollary 2.2.2, we denote $P^1_{xy}$, $P^2_{xy}$ and $P^3_{xy}$ for the different paths according to type 1, 2 and 3 respectively. Based on the above Lemma 2.2.1 and Corollary 2.2.2, we have the following results.

**Theorem 2.2.3.** *[38]* $|P_{xy}| = \min\{|P^1_{xy}|, |P^2_{xy}|, |P^3_{xy}|\} \leq n$.

**Corollary 2.2.4.** *The diameter of* $UB(d, n)$ *is* $n$.

**Proof.** The proof is directly from Theorem 2.2.3. Here we present another proof. The diameter of $UB(d, n)$ is at most $n$ since the distance between any two distinct vertices $x = x_1, x_2, \cdots, x_n$ and $y = y_1, y_2, \cdots, y_n$ is at most $n$ for a path

$$< x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n > .$$

On the other hand, the diameter of $UB(d, n)$ is at least $n$ since the distance between $00 \cdots 0$ and $11 \cdots 1$ is $n$. □

There are four different types of shortest paths and the corresponding *maximal matched substrings* between representations of $x$ and $y$. The key of finding the shortest paths between $x$ and $y$ is to find all the maximal matched substrings between representations of $x$ and $y$. According this, Liu and Sung provided two shortest-path routing algorithms, one of which has linear time complexity.

Mao and Yang [40] also proposed a shortest path routing algorithm, whose time complexity in $UB(2, n)$ is $O(n^2)$. Then, based on their shortest path routing algorithm, they proposed two fault-tolerant routing schemes. It is assumed that at most one node fails in the network. In their schemes, two node-disjoint paths are found; one is the shortest and the other one is very short [40].

## 2.3 Generalized Directed de Bruijn Graphs

In 1985, Imase, Soneoka, and Okada [31] showed that the vertex-connectivity $\kappa$ of $G_B(n, m)$ is at least $n - 1$ if the diameter greater than 3. Since $G_B(n, m)$ always contains self-loops, its connectivity cannot exceed $n - 1$, that is exactly $n - 1$.

For the communications in interconnection networks, it is always convenient to have the ability to use several paths between two vertices. The paths multiplicity has a property that might be considered because it allows us to select between several different combinations of paths to achieve a more well-balanced communication of a fault-tolerant communication [13, 44]

There is an arc from $x$ to $y$ in $G_B(n,m)$ if there exists an integer $\alpha \in [0, n-1]$ such that $y \equiv (nx + \alpha) \pmod m$. A path of length $k$ from a vertex $u$ to another vertex $v$ of $G_B(n,m)$ is a sequence of vertices: $u = x_0 \rightarrow x_1 \rightarrow \ldots \rightarrow x_k = v$. Suppose $x_{i+1} \equiv (nx_i + \alpha_i) \pmod m$, we have

$$x_k \equiv (n^k x_0 + \alpha_{k-1} + n\alpha_{k-2} + \ldots + n^{k-1}\alpha_0) \pmod m \tag{2.1}$$

The sum $R \equiv (\alpha_{k-1} + n\alpha_{k-2} + \ldots + n^{k-1}\alpha_0) \pmod m$ is called a *residue*, which is an $n$-ary number coding the $k$ arcs used to go from $u$ to $v$ in $k$ steps. So a path of length $k$ from $u$ to $v$ can be founded by calculating the residue:

$$R \equiv (v - n^k u) \pmod m \tag{2.2}$$

If such a path exists, then the residue can be written with $k$ letters in base $n$, such that $R < n^k$. On the contrary, the fact that there is no path of length $k$ between $u$ and $v$ can be detected by $R \geq n^k$.

Indeed, the distance between two vertices is the shortest path length connecting these vertices. In other words, it is the least $k$ such that there is a path of length $k$ between these two vertices. And the definition can be formulated by the following residues:

$$d(u,v) = min\{k : (v - n^k u) \pmod m < n^k\} \tag{2.3}$$

There are several shortest paths (of length $k = d(u,v)$) from $u$ to $v$ when several valid residues modulo $m$ can be found. Thus when it exists $\gamma \in N$, we have

$$R + (\gamma - 1)m < n^k < R + \gamma m \tag{2.4}$$

Then, there are $\gamma$ paths corresponding to the different residues $R$, $R+m, \ldots$, $R+(\gamma-1)m$, respectively.

For example, consider the graph $G_B(4,6)$. There is no path of length 1 from 1 to 3 since the residue $R \equiv 3 - 4 \times 1 \equiv 5 \geq 4 = n$. When looking for a path of length 2, we find the residue $R \equiv 3 - 4^2 \times 1 \equiv 5 < n^2$. Thus, vertex 1 is at distance 2 from 3. If other residue exists, they should be written with 2 letters in base 4, i.e. $R_i < 16$. We get $R_0 \equiv 3 - 4^2 \times 1 \equiv 5_{10} = 11_4$, $R_1 \equiv R_0 + m \equiv 11_{10} = 23_4$, and

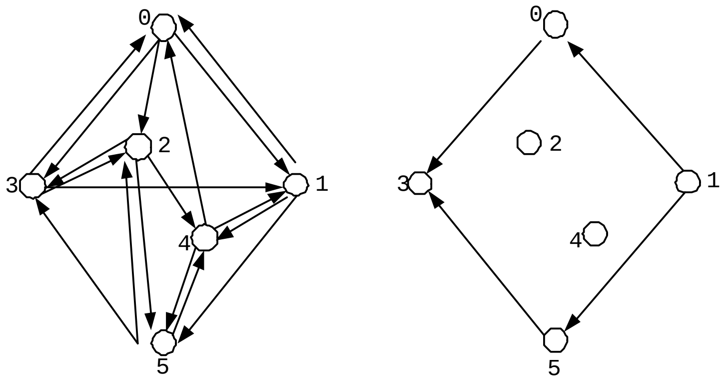


Figure 2.2: $G_B(4,6)$ and the shortest path from 1 to 3

$R_2 = R_1 + m = 17 > 16$. So, the two shortest paths are $1 \to 5 \to 3$ and $1 \to 0 \to 3$ (see the Figure 2.2).

## 2.4 Generalized Undirected de Bruijn Graphs

Since the study of the diameter of an interconnection network is to investigate the fault tolerance and transmission delay, it is interesting to determine the diameter of $UG_B(n,m)$.

In what follows we introduce some known results of $d(UG_B(n,m))$. First, Escuadro and Muga II obtained:

**Theorem 2.4.1.** *[21] For $n \geq 3$, $d(UG_B(n,n^2)) = 2$.*

Then, Nochefranca and Sy proved the following:

**Theorem 2.4.2.** *[41] $d(UG_B(n,n(n^2+1))) = 4$ for odd $n \geq 3$.*

Caro, Nochefranca and Sy proved:

**Theorem 2.4.3.** *[8] $d(UG_B(n,n^2+1)) \leq 4$ for odd $n \geq 5$.*

All of the above three theorems are for a fixed $m$. In 2002, Caro and Zeratsion obtained more values for $m$ corresponding to $n$ where $m \leq n^3$.

**Theorem 2.4.4.** *[10]* $d(UG_B(n,m)) = 2$ *for* $m \in [n+1, n^2]$ *and* $n|m$.

**Theorem 2.4.5.** *[10]* $d(UG_B(n,m)) = 3$ *for* $m \in [n^2+1, n^3]$ *and* $n|m$.

Although the diameter of $UG_B(n,m)$ is an important parameter, only some special cases have been detected. There are a lot of diameters unsolved yet.

# Chapter 3

# Wide-Diameters of Undirected De Bruijn Graphs

The wide-diameter of a graph, which combines the connectivity with the diameter, is a parameter that measures simultaneously the fault-tolerance and the efficiency of a parallel processing computer network. In this chapter, we first introduce the wide-diameters. In the second section, we will show a result of finding wide-diameters of $UB(d,n)$.

## 3.1 Introduction of Wide-Diameters

Diameter with width $w$ of a graph $G$ is defined as the minimum integer $l$ for which between any two distinct vertices in $G$ there exist at least $w$ internally vertex disjoint paths of length at most $l$. The notation of $w$-wide-diameter was introduced by Hsu [27] to unify the concept of diameter and connectivity [37]. Throughout this chapter, we use "disjoint paths" for the notations of "internally vertex disjoint paths".

A **container** $C(u,v)$ is a set of disjoint paths between two distinct vertices $u$ and $v$ in $G$. The **width** of $C(u,v)$, written as $w(C(u,v))$, is its cardinality. The **length** of $C(u,v)$, written as $l(C(u,v))$, is the length of the longest path in $C(u,v)$. The $w$-**wide distance** between $u$ and $v$ is $l(C(u,v))$, where $C(u,v)$ is the minimum length container between $u$ and $v$ with width $w$. Let $\kappa$ be the connectivity of $G$. The **wide-diameter** of $G$ is the maximum of $\kappa$ wide distances among all pairs of vertices $u$, $v$ in $G$, where $u \neq v$. The **fault diameter** of a connected graph $G$ is the maximum diameter of any subgraph of $G$ obtained by removing at most $\kappa - 1$ vertices.

We use $P[x_1x_2\dots x_m]$, $m > n$, to denote the following path by deleting the repeated vertices and having shortest path.

$$x_1\dots x_n \to x_2\dots x_{n+1} \to \dots \to x_{m-n}\dots x_{m-1} \to x_{m-n+1}\dots x_m.$$

For example: for $n = 5$ and $d = 3$, $P[00100200] = 00100 \to 01002 \to 10020 \to 00200$, and $P[00000101010] = 00000 \to 00001 \to 00010 \to 00101 \to 01010$.

The boldface of a character is a vertex of $UB(d, n)$, for example $\mathbf{x}, \mathbf{y}$. Suppose that $\mathbf{x} = x_1x_2\dots x_n$, $\mathbf{y} = y_1y_2\dots y_n$. In what follows, for each $a \in Z_d$, we use $P[\mathbf{x}a\mathbf{y}]$ to denote the path

$$\mathbf{x} \to x_2\dots x_na \to x_3\dots x_nay_1 \to x_4\dots x_nay_1y_2 \to \dots \to ay_1\dots y_{n-1} \to \mathbf{y}.$$

$P(\mathbf{x}a\mathbf{y}]$ is the path of vertices including $\mathbf{y}$ and excluding $\mathbf{x}$ (and similarly for $P[\mathbf{x}a\mathbf{y})$ and $P(\mathbf{x}a\mathbf{y}))$ [20]. So we can also use $\mathbf{x} \to P(\mathbf{x}a\mathbf{y}]$ or $P[\mathbf{x}a\mathbf{y}) \to \mathbf{y}$, or $\mathbf{x} \to P(\mathbf{x}a\mathbf{y}) \to \mathbf{y}$ to mean $P[\mathbf{x}a\mathbf{y}]$. If necessary,

$$x_1x_2\dots x_n \to P(x_1x_2\dots x_nay_1y_2\dots y_n) \to y_1y_2\dots y_n \text{ equals } P[\mathbf{x}a\mathbf{y}].$$

We add superscripts to mean coordinates for long string for reading easily. For example:

$$P[x_1x_2\dots x_nay_1y_2\dots y_n] = P[x_1^1x_2^2\dots x_n^na^{n+1}y_1^{n+2}y_2^{n+3}\dots y_n^{2n+1}]$$

## 3.2 Wide-Diameters of $UB(d, n)$

Now, we are ready to show our main results. We set $WD$ as the wide-diameters between $x$ and $y$. The following lemmas are essential for the proof of the main Theorem 3.2.4.

**Lemma 3.2.1.** *[33] For any vertex $\mathbf{y} = y_1y_2\dots y_n \in UB(d, n)$, there are exactly $2d - 2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x} = 00\dots 0$ of length at most $2n - 2$.*

**Proof.** Define $A_1 = \{00\dots0a : a \in {Z_d}^*\}$, $B_1 = \{b0\dots0 : b \in {Z_d}^*\}$,

$$A_i = \{0^1\dots0^{n-i}ay_1\dots y_{i-1} : a \in {Z_d}^*\}, i = 2, 3, \dots, n, \text{ and}$$

$$B_j = \{y_{n-j+2}\dots y_n b0^{j+1}\dots0^n : b \in {Z_d}^*\}, j = 2, 3, \dots, n.$$

For any $i, j = 1, 2, \dots, n$, and $i \neq j$, $A_i \cap A_j = \phi$ and $B_i \cap B_j = \phi$, since the numbers of 0 before $a$ or after $b$ are different. If $A_i \cap B_j = \phi$ for any $i, j = 1, 2, \dots, n$, then we have found $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x}$ with length $n+1$, namely $WD = \{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}] : a, b \in {Z_d}^*\}$.

Now, suppose $A_i \cap B_j = \{0\dots0a\dots b0\dots0\} \neq \phi$, for some $a, b \in {Z_d}^*$, and $t = i+j-n > 0$ is the length of overlap between $A_i$ and $B_j$ by deleting the consecutive 0's from head and from tail. Then the length of $a\dots b$ in $A_i \cap B_j$ is $t(\leq n)$. According to $i$ and $j$, we have the following cases.

**Case 1:** $n-j+2 > i-1 \Rightarrow t < 3$

**case 1-1:** $t = 1 \Rightarrow \mathbf{y} = 0^1\dots0^{i-1}y_i0^{n-i+2}\dots0^n$

Note that $y_i > 0$ and $y_j = 0$, for $j \neq i$.

- $i = 1$:

  Let $y_1 = w > 0$. Then,

  $WD = \{\mathbf{x} \to 0\dots0a \to \mathbf{y} : a \in {Z_d}^*\} \cup \{\mathbf{x} \to \mathbf{y}\} \cup$

  $\{\mathbf{y} \to bw0\dots0 \to P[w^10\dots0b^n0^{n+1}\dots0^{2n}) \to \mathbf{x} : b \in {Z_d}^*\backslash\{w\}\}$.

- $i = n$ :

  Let $y_n = w > 0$. Then,

  $WD = \{\mathbf{y} \to a0\dots0 \to \mathbf{x} : a \in {Z_d}^*\} \cup \{\mathbf{x} \to \mathbf{y}\} \cup$

  $\{\mathbf{x} \to P(0^1\dots0^nb^{n+1}0^{n+2}\dots0^{2n-1}w^{2n}b^{2n+1}] \to \mathbf{y} : b \in {Z_d}^*\backslash\{w\}\}$.

- $1 < i < n$:

  Let $y_i = w > 0$. Then,

  $WD = \{\mathbf{x} \to P(0^1\dots0^na^{n+1}\dots a^{n+i}) \to 0^1\dots0^{n-i}a^{n-i+1}\dots a^n$

  connect $\mathbf{y} \to P(0^1\dots w^i\dots0^na^{n+1}\dots a^{n+i}) \to 0^1\dots0^{n-i}a^{n-i+1}\dots a^n : a \in {Z_d}^*\} \cup \{b^1\dots b^i0^{i+1}\dots0^n \to P(b^1\dots b^i0^{i+1}\dots0^{n+i}) \to \mathbf{x}$

connect $b^1 \dots b^i 0^{i+1} \dots 0^n \rightarrow P(b^1 \dots b^i 0^{i+1} \dots w^{2i} \dots 0^{n+i}) \rightarrow \mathbf{y} : b \in Z_d{}^*\}$.

This case has the longest path of length $2n-2$.

**case 1-2:** $t = 2 \Rightarrow \mathbf{y} = b0^2 \dots 0^{i-1}0^{n-j+2} \dots 0^{n-1}a$

Let $\mathbf{y} = w0 \dots 0u, w, u > 0$. Then,

$WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d{}^* \backslash \{w\}\} \cup \{\mathbf{x} \rightarrow w0 \dots 0 \rightarrow 0w0 \dots 0 \rightarrow w0 \dots 0u = \mathbf{y}\}$.

**Case 2:** $n - j + 2 \leq i - 1 \Rightarrow t \geq 3$

**case 2-1:** $n - t + 2 \leq t - 1 \Rightarrow n + 3 \leq 2t \Rightarrow \frac{n+3}{2} \leq t \leq n$

$$\mathbf{y} = y_1 \dots y_{n-j+1} 0^{n-j+2} \dots 0^{n-t+1} a^{n-t+2} y_{n-t+3} \dots y_{t-2} b^{t-1} 0^t \dots 0^{i-1} y_i \dots y_n,$$

with $y_1 \dots y_{t-2} = y_{n-t+3} \dots y_n$. Let $y_{t-1} = b = w > 0$. Then,

$WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d{}^* \setminus \{w\}\} \cup \{P[\mathbf{y}ww\mathbf{x}]\}$.

example 1: $n + 3 = 2t$.

$y = 0^1 \dots 0^{n-j} y_{n-j+1} 0^{n-j+2} \dots 0^{n-t+1} a^{n-t+2=t-1} 0^t \dots 0^{i-1} y_{n-j+1} 0^{i+1} \dots 0^n$.

example 2: $n + 4 = 2t$.

$\mathbf{y} = b^1 0^2 \dots 0^{n-t+1=t-3} a^{t-2} b^{t-1} 0^t \dots 0^{n-1} a^n$.

**case 2-2:** $n - t + 2 > t - 1 \Rightarrow n + 3 > 2t$

Define Min $= \min\{t, n-j+2\}$ and Max $= \max\{i-1, n-t+1\}$. Then $\mathbf{y} = y_1 \dots y_{Min-1} 0^{Min} \dots 0^{Max} y_{Max+1} \dots y_n$ with $y_1 \dots y_{t-2} = y_{n-t+3} \dots y_n$.

**case 2-2-1:** Min $= t$ and Max $= n - t + 1$

$t \leq n - j + 2$ and $i - 1 \leq n - t + 1 \Rightarrow 3 \leq t \leq \frac{n+4}{3}$.

$\mathbf{y} = y_1 \dots y_{t-2} b^{t-1} 0^t \dots 0^{n-t+1} a^{n-t+2} y_{n-t+3} \dots y_n$. Let $y_{t-1} = b = w > 0$. Then,

$WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d{}^* \setminus \{w\}\} \cup \{P[\mathbf{y}ww\mathbf{x}]\}$.

**case 2-2-2:** Min $= t$ and Max $= i - 1$

$\mathbf{y} = 0^1 \dots 0^{i+t-n-1} y_{i+t-n} \dots y_{t-2} b^{t-1} 0^t \dots 0^{i-1} y_i \dots y_n \Rightarrow a = y_{n-t+2} = 0$, this is impossible.

**case 2-2-3:** Min $= n - j + 2$ and Max $= n - t + 1$

$\mathbf{y} = y_1 \dots y_{n-j+1} 0^{n-j+2} \dots 0^{n-t+1} a^{n-t+2} y_{n-t+3} \dots y_{2n-t-j+3} 0^{2n-t-j+4} \dots 0^n$, and $b = y_{t-1} = 0$, this is impossible.

**case 2-2-4:** Min $= n - j + 2$ and Max $= i - 1$

$\mathbf{y} = 0^1 \ldots 0^{i+t-n-1} y_{i+t-n} \ldots y_{n-j+1} 0^{n-j+2} \ldots 0^{i-1} y_i \ldots y_{2n-t-j+3} 0^{2n-t-j+4} \ldots 0^n$, $a = y_{n-t+2} = 0$, and $b = y_{t-1} = 0$. This is impossible.

Since cases 2-2-2, 2-2-3, and 2-2-4 are impossible, we get exactly $2d-2$ disjoint paths. This completes the Lemma. □

Except Lemma 3.2.1, we still need another lemma very similar to Lemma 3.2.1.

**Lemma 3.2.2.** *[33] For any vertex* $\mathbf{y} = y_1 y_2 \ldots y_n \in UB(d,n)$, *there are at least* $2d-2$ *disjoint paths from* $\mathbf{y}$ *to* $\mathbf{x} = 100\ldots0$ *of length at most* $2n$.

**Proof.** First, we consider the case when $\mathbf{y} = u0\ldots0$, $u \neq 1$. If $u = 0$, then by Lemma 3.2.1, we are done. Now, suppose $u > 1$. Then, the $2d-2$ disjoint paths are
$WD = \{\mathbf{x} \rightarrow 0\ldots0a \rightarrow \mathbf{y} : a \in {Z_d}^*\} \cup$
$\{\mathbf{y} \rightarrow P[b^1 u^2 0^3 \ldots 0^n b^{n+1} 1^{n+2} 0^{n+3} \ldots 0^{2n}] \rightarrow \mathbf{x} : b \in {Z_d}^*\}$.

In what follows, we suppose $\mathbf{y} \neq u0\ldots0$.

Define $A_1 = \{00\ldots0a : a \in {Z_d}^*\}$, $B_1 = \{b10\ldots0 : b \in {Z_d}^*\}$,

$$A_i = \{0^1 \ldots 0^{n-i} a y_1 \ldots y_{i-1} : a \in {Z_d}^*\}, i = 2,3,\ldots,n, \text{ and}$$

$$B_j = \{y_{n-j+2} \ldots y_n b 1^{j+1} 0^{j+2} \ldots 0^n : b \in {Z_d}^*\}, j = 2,3,\ldots,n.$$

For $i,j = 1,2,\ldots,n$, $i \neq j$, $A_i \cap A_j = \phi$ and $B_i \cap B_j = \phi$, since the numbers of consecutive 0's are different. If $A_i \cap B_j = \phi$ for any $i,j = 1,2,\ldots,n$, then we have found $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x}$ with length $n+1$, namely $WD = \{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}] : a,b \in {Z_d}^*\}$.

Now, suppose $A_i \cap B_j = \{0\ldots0a\ldots b10\ldots0\} \neq \phi$, for some $a,b \in {Z_d}^*$, where $t = i+j-n > 0$ is the length of overlap between $A_i$ and $B_j$ by deleting the consecutive 0's from head and from tail and the special 1 following after $b$. In other words, $a\ldots b1$ has length $t$. According to $i$ and $j$, we have the following cases.

**Case 1:** $n-j+2 > i-1 \Rightarrow t < 3$

**case 1-1:** $t = 1 \Rightarrow \mathbf{y} = 10\ldots0y_i0\ldots0$

Note that $y_i > 0$ and $i > 1$.

- $i = n$

  $WD = \{\mathbf{x} \to a10\ldots0 \to \mathbf{y} : a \in Z_d{}^*\} \cup \{\mathbf{x} \to P(\mathbf{x}b^{n+1}\ldots b^{2n}) \to b\ldots b$ connect $\mathbf{y} \to P(\mathbf{y}b^{n+1}\ldots b^{2n}) \to b\ldots b : b \in Z_d{}^*\}$. This case has the longest length $2n$.

- $1 < i < n$

  $WD = \{\mathbf{x} \to P(\mathbf{x}a^{n+1}\ldots a^{n+i+1}) \to 0^1\ldots 0^{n-i-1}a^{n-i}\ldots a^n$ connect $\mathbf{y} \to P(\mathbf{y}a^{n+1}\ldots a^{n+i+1}) \to 0^1\ldots 0^{n-i-1}a^{n-i}\ldots a^n, : a \in Z_d{}^*\} \cup \{b^1\ldots b^n \to P(b^1\ldots b^n\mathbf{x}) \to \mathbf{x}$ connect $b^1\ldots b^n \to P(b^1\ldots b^n\mathbf{y}) \to \mathbf{y} : b \in Z_d{}^*\}$. This case also has the longest length $2n$.

**case 1-2:** $t = 2 \Rightarrow \mathbf{y} = b\delta 0^3\ldots 0^{i-1}0^{n-j+2}\ldots 0^{n-1}a$, where $\delta = 0$ or 1.

Let $y = w\delta 0\ldots 0u$, $\delta = 0$ or $1, w, u > 0$.

- $w = 1$ and $\delta = 0$

  $WD = \{\mathbf{x} \to a10\ldots0 \to \mathbf{y} : a \in Z_d{}^*\} \cup \{\mathbf{x} \to P(\mathbf{x}b^{n+1}\ldots b^{2n}) \to b\ldots b$ connect $\mathbf{y} \to P(\mathbf{y}b^{n+1}\ldots b^{2n}) \to b\ldots b : b \in Z_d{}^*\}$.

- otherwise

  $WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d \setminus \{w\}\}$.

**Case 2:** $n - j + 2 \le i - 1 \Rightarrow t \ge 3$

**case 2-1:** $n - t + 2 \le t - 1 \Rightarrow n + 3 \le 2t \Rightarrow \frac{n+3}{2} \le t \le n$

$$\mathbf{y} = y_1\ldots y_{n-j+1}0^{n-j+2}\ldots 0^{n-t+1}a^{n-t+2}y_{n-t+3}\ldots y_{t-2}b^{t-1}\delta^t 0^{t+1}\ldots 0^{i-1}y_i\ldots y_n,$$

with $y_1\ldots y_{t-2} = y_{n-t+3}\ldots y_n$, and $\delta = 0$ or 1. Let $y_{t-1} = b = w > 0$. Then, $WD = \{P[\mathbf{x}a\mathbf{y}] : a \in Z_d{}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in Z_d{}^* \setminus \{w\}\} \cup \{P[\mathbf{y}ww\mathbf{x}]\}$.

**case 2-2:** $n - t + 2 > t - 1 \Rightarrow n + 3 > 2t$

Define Min $= \min\{t, n - j + 2\}$ and Max $= \max\{i - 1, n - t + 1\}$. Then $\mathbf{y} = y_1\ldots y_{Min-1}0^{Min}\ldots 0^{Max}y_{Max+1}\ldots y_n$ with $y_1\ldots y_{t-2} = y_{n-t+3}\ldots y_n$, and $y_t = \delta = 0$ or 1.

**case 2-2-1:** Min $= t$ and Max $= n - t + 1$

$t \le n - j + 2$ and $i - 1 \le n - t + 1 \Rightarrow 3 \le t \le \frac{n+4}{3}$.

$\mathbf{y} = y_1\ldots y_{t-2}b^{t-1}\delta^t 0^{t+1}\ldots 0^{n-t+1}a^{n-t+2}y_{n-t+3}\ldots y_n$. Let $y_{t-1} = b = w > 0$. Then,

$WD = \{P[\mathbf{x}a\mathbf{y}] : a \in {Z_d}^*\} \cup \{P[\mathbf{y}b\mathbf{x}] : b \in {Z_d}^* \setminus \{w\}\} \cup \{P[\mathbf{y}ww\mathbf{x}]\}$

**case 2-2-2:** Min $= t$ and Max $= i-1$

$\mathbf{y} = 0^1 \ldots 0^{i+t-n-1} y_{i+t-n} \cdots y_{t-2} b^{t-1} \delta^t 0^{t+1} \ldots 0^{i-1} y_i \ldots y_n$, and $a = y_{n-t+2} = 0$, this is impossible.

**case 2-2-3:** Min $= n-j+2$ and Max $= n-t+1$

$\mathbf{y} = y_1 \ldots y_{n-j+1} 0^{n-j+2} \ldots 0^{n-t+1} a^{n-t+2} y_{n-t+3} \cdots y_{2n-t-j+3} 0^{2n-t-j+4} \ldots 0^n$, and $b = y_{t-1} = 0$, this is impossible.

**case 2-2-4:** Min $= n-j+2$ and Max $= i-1$

$\mathbf{y} = 0^1 \ldots 0^{i+t-n-1} y_{i+t-n} \cdots y_{n-j+1} 0^{n-j+2} \ldots 0^{i-1} y_i \ldots y_{2n-t-j+3} 0^{2n-t-j+4} \ldots 0^n$, and $a = b = 0$, this is impossible.

Combining all the cases, we have at least $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x} = 100\ldots0$ of length at most $2n$. □

**Lemma 3.2.3.** *[33] For any vertex $\mathbf{y} \in UB(d,n)$, there are at least $2d-2$ disjoint paths from $\mathbf{y}$ to distinct vertex $\mathbf{x}$ of length at most $2n$ when $\mathbf{x} = cc\cdots c$, or $ce\cdots e$, or $c\ldots ce$, where $c, e \in Z_d$, and $c \neq e$.*

**Proof.** For the case of $x = cc\cdots c$, we can prove by replacing 0 and $c$ with $c$ and 0 respectively in Lemma 3.2.1. For the case of $x = ce\cdots e$, we can prove by replacing 0 and 1 with $e$ and $c$ respectively in Lemma 3.2.2. As for the case of $x = c\cdots ce$, we can prove by Lemma 3.2.2. □

Now, we are ready to prove the main result of this section.

**Theorem 3.2.4.** *[33] For any two distinct vertices $\mathbf{y} = y_1y_2\ldots y_n$, $\mathbf{z} = z_1z_2\ldots z_n$ $\in UB(d,n)$, there are $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{z}$ with length at most $2n+1$.*

**Proof.** If $\mathbf{y}$ or $\mathbf{z}$ is in the forms of $\mathbf{x}$ in Lemma 3.2.3 then we are done. Therefore, we suppose that both of $\mathbf{y}$ and $\mathbf{z}$ are not in the forms of $\mathbf{x}$ in Lemma 3.2.3.

By Lemma 3.2.1, we can get $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x} = 0\ldots0$ and from $\mathbf{z}$ to $\mathbf{x}$, respectively. Now, let the sets $A_i$ and $B_j$ are defined as in Lemma 3.2.1. Similar to the same Lemma 3.2.1, define $C_1 = A_1$, $D_1 = B_1$,

$$C_i = \{0^1 \dots 0^{n-i} a z_1 \dots z_{i-1} : a \in Z_d{}^*\}, i = 2, 3, \dots, n, \text{ and}$$

$$D_j = \{z_{n-j+2} \dots z_n b 0^{j+1} \dots 0^n : b \in Z_d{}^*\}, j = 2, 3, \dots, n.$$

We use $A_i(a')(\in A_i)$ to denote the vertex $0^1 \dots 0^{n-i} a' y_1 \dots y_{i-1}$ in the set of $A_i$. Then $A_i \cap A_j = B_i \cap B_j = C_i \cap C_j = D_i \cap D_j = \phi$, for $i \neq j$, and $i, j = 1, 2, \dots, n$. This immediately implies $C_i(a') = B_i(b')$ and $B_i(b') = C_j(a'')$ can not happen at the same time by $C_i \cap C_j = \emptyset$.

Since $\mathbf{y} \neq \mathbf{z}$, we let $p = min\{i : y_i \neq z_i\}$ and $q = max\{i : y_i \neq z_i\}$, then $1 \leq p \leq q \leq n$. Now, $A_i \cap C_j = \phi$ for $i, j > p$, and $B_i \cap D_j = \phi$, for $i, j > n - q + 1$.

If $A_i(a') = B_j(b')$ ( or $C_{i'}(a") = D_{j'}(b")$), then we choose an arbitrary $\beta$ from $Z_d{}^* \setminus \{b', b"\}$, and replace $\mathbf{x}$ with $\beta\beta \dots \beta$. This implies that there are $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x} = \beta\beta \dots \beta$, $\{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}], a, b \in Z_d \setminus \{\beta\}\}$, with $A_i \cap B_j = \phi$ (respectively from $\mathbf{z}$ to $\mathbf{x}$ with $C_i \cap D_j = \phi$).

If $\mathbf{y} = a'b'a'b' \dots a'b'$ (or $a'b'a'b' \dots a'b'a'$), then $\mathbf{y}$ has degree $2d-1$. The case $a = b'$ and $b = a'$ (or $a = b'$, $b = b'$) may cause the vertex $b'a'b'a' \dots b'a'$ (or $b'a'b' \dots b'a'b'$) repeated. Then we replace $\mathbf{x}$ with $\beta\beta \dots \beta$, where $\beta = a'$. Now we have $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{x}$ and $WD = \{P[\mathbf{x}a\mathbf{y}], P[\mathbf{y}b\mathbf{x}], a, b \in Z_d \setminus \{a'\}\}$ with $A_i \cap B_j = \phi$. As to $\mathbf{z}$, the process is similar, we omit the details.

Now we have case $A_i \cap D_j \neq \phi$ or $B_i \cap C_j \neq \phi$ left to consider. Suppose not. Then, $A_i \cap D_j = B_i \cap C_j = \phi$, we can find $2d-2$ disjoint paths from $\mathbf{y}$ to $\mathbf{z}$ as follows: $WD = \{0 \dots 0 a y_1 \dots y_{p-1} \to P(0^1 \dots 0^{n-p} a^{n-p+1} y_1{}^{n-p+2} \dots y_n{}^{2n-p}) \to \mathbf{y}$ connect $0 \dots 0 a z_1 \dots z_{p-1} \to P(0^1 \dots 0^{n-p} a^{n-p+1} z_1{}^{n-p+2} \dots z_n{}^{2n-p}) \to \mathbf{z} : a \in Z_d{}^*\} \cup \{\mathbf{y} \to P(y_1{}^1 \dots y_n{}^n b^{n+1} 0^{n+2} \dots 0^{n+q}) \to y_{q+1} \dots y_n b 0 \dots 0$ connect $\mathbf{z} \to P(z_1{}^1 \dots z_n{}^n b^{n+1} 0^{n+2} \dots 0^{n+q}) \to z_{q+1} \dots z_n b 0 \dots 0 : b \in Z_d{}^*\}$.

Note that $0 \dots 0 a y_1 \dots y_{p-1} = 0 \dots 0 a z_1 \dots z_{p-1}$ and $y_{q+1} \dots y_n b 0 \dots 0 = z_{q+1} \dots z_n b 0 \dots 0$. Moreover, 0 may be replaced with $\beta$ if it is necessary in the previous cases. We call this "normal process".

Suppose $A_{j'} \cap D_{i'} \neq \phi$ or $B_i \cap C_j \neq \phi$:

- $A_{j'} \cap D_{i'} = \phi$ and $B_i(b') = v = C_j(a')$, for some $i > n-q+1$, $j > p$, $a'$ and $b'$. We fix the two disjoint paths as:

$$\mathbf{y} \to B_n(b') \to B_{n-1}(b') \to \cdots \to v \to C_{j+1}(a') \to \cdots \to C_n(a') \to \mathbf{z}$$

$$\mathbf{z} \to P(z_1{}^1 \ldots z_n{}^n b' 0^{n+2} \ldots 0^{2n}] \to P[0^1 \ldots 0^{n-1} a' y_1{}^{n+1} \ldots y_n{}^{2n}) \to \mathbf{y}.$$

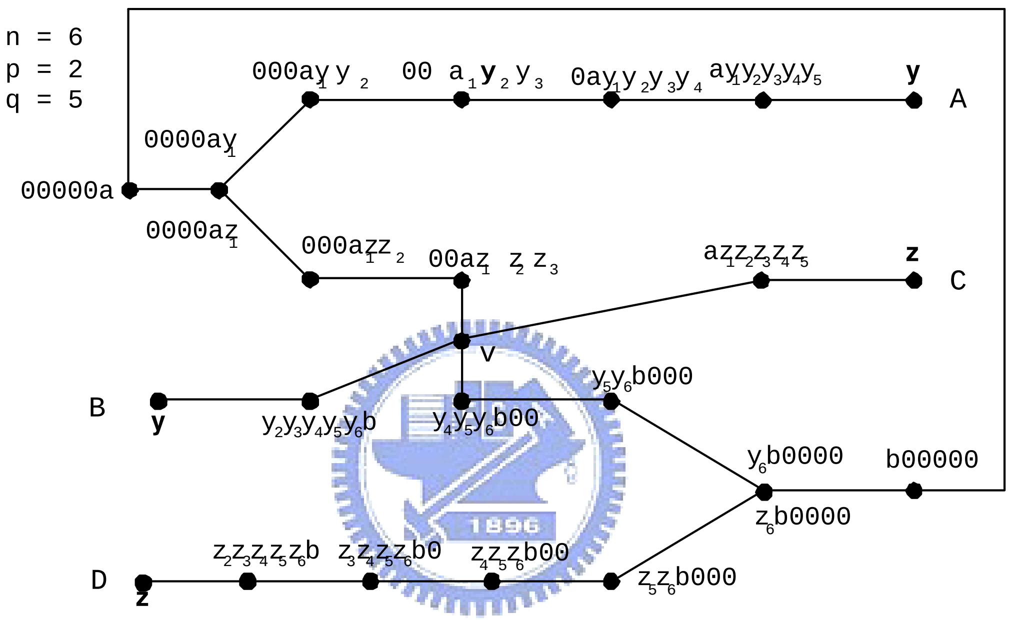


Figure 3.1: Main idea.

- $A_{j'}(a') = w = D_{i'}(b')$ and $B_i \cap C_j = \phi$, for some $i', j', a'$ and $b'$.

$$\mathbf{z} \to D_n(b') \to D_{n-1}(b') \to \cdots \to w \to A_{j'+1}(a') \to \cdots \to A_n(a') \to \mathbf{y}$$

$$\mathbf{y} \to P(y_1{}^1 \ldots y_n{}^n b' 0^{n+2} \ldots 0^{2n}] \to P[0^1 \ldots 0^{n-1} a' z_1{}^{n+1} \ldots z_n{}^{2n}) \to \mathbf{z}.$$

- $A_{j'}(a') = w = D_{i'}(b')$ and $B_i(b') = v = C_j(a')$ for some $i, j, i', j', a'$ and $b'$.

$$\mathbf{y} \to B_n(b') \to B_{n-1}(b') \to \cdots \to v \to C_{j+1}(a') \to \cdots \to C_n(a') \to \mathbf{z},$$

$$\mathbf{z} \to D_n(b') \to D_{n-1}(b') \to \cdots \to w \to A_{j'+1}(a') \to \cdots \to A_n(a') \to \mathbf{y}.$$

We call the previous three cases "main process". Finally, $B_i(b') = v = C_j(a')$ and $B_{i'}(b') = w = C_k(a")$ happen at the same time for $i$, $i'$, $j$, $k$, $a'$, $a"$, $b'$ and $i > i'$. In this case, we have $\mathbf{y} \to B_n(b') \cdots \to B_i(b') = v = C_j(a') \to \cdots \to C_n(a') \to \mathbf{z}$ first, and the others are decided by Figure 3.2. We call this "special process". Note here

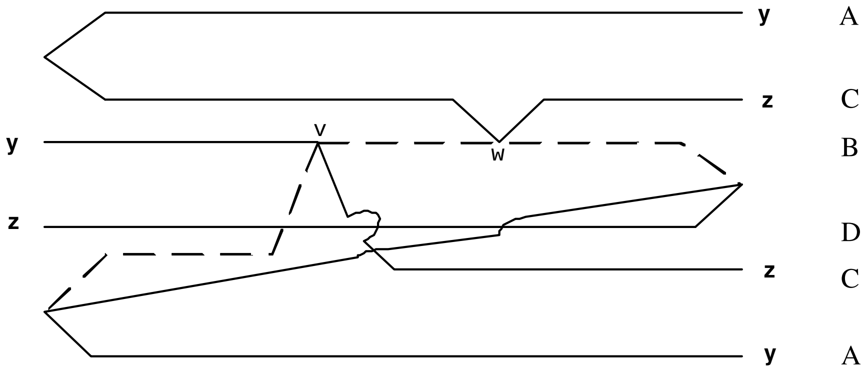


Figure 3.2: Main and special process of wide-diameter of $UB(d, n)$.

that $\mathbf{y}$ and $\mathbf{z}$ are not equal to either $ce \ldots e$ or $c \ldots ce$ (in Lemma 3.2.3), so we always have $a0 \ldots 0$ and $0 \ldots 0b$ (complete bipartite graph) to use. Since the main process and the special process do not occur usually, the "normal process" always work. □

## Remark

We give an algorithm for the reader to find the wide-diameter quickly.

Algorithm:

step 1: According to $\mathbf{y}$ and $\mathbf{z}$, define $A_i$, $B_j$, $C_i$, and $D_j$. If $A_i(a') = B_j(b')$ or $C_i(a") = D_j(b")$, then choose an arbitrary $\beta$ from $Z_d * \setminus \{b', b"\}$, and let $\mathbf{x} = \beta \ldots \beta$.

step 2: Routing by "normal process".

step 3: repeat if $B_i(b') = C_j(a')$ and $B_{i'}(b') = C_k(a")$

routing "special process"

end-repeat.

step 4: repeat if $A_i \cap D_j \neq \emptyset$ or $B_i \cap C_j \neq \emptyset$

routing "main process"

end-repeat.

end.

# Chapter 4

# Diameters of Generalized Undirected de Bruijn Graphs

In this chapter, we mainly study the diameter of *generalized undirected de Bruijn graphs* $UG_B(n,m)$ whenever $n^2 < m \le n^3$.

## 4.1 $d(UG_B(n,m))$ for $n^2 < m \le n^2 + n$

By the definition of $UG_B(n,m)$, we conclude that the diameter of a generalized undirected de Bruijn graph $G$ is at most the diameter of its corresponding *generalized directed de Bruijn graph* $\widetilde{G}$, since the distance of two distinct vertices in $G$ is not greater than that in $\widetilde{G}$. Therefore, the diameter of $UG_B(n,m)$ is at most $\lceil \log_n m \rceil$ by Theorem 1.6.1. Then for $m \le n^3$, $d(UG_B(n,m)) \le 3$. If we want to show a graph $G = UG_B(n,m)$ such that $d(G) \le 3$, we must find a pair of $(x,y)$ such that $d(x,y) \le 3$. On the other hand, if we want to show $d(G) \le 2$, we must show that for any two distinct vertices $x$ and $y$, $d_G(x,y) \le 2$. We start with the smallest case $m = n^2 + 1$.

**Proposition 4.1.1.** *[34] $d(UG_B(n,n^2+1)) = 3$ for $n \ge 4$.*

**Proof.** Let $m = n^2 + 1$ and $n \ge 4$. Consider $x = n - 2$ and $y = n^2 - n + 2$ in $G = UG_B(n,m)$. We shall claim $d_G(x,y) \ge 3$. Since $(n^2 - n + 2)n + \alpha \equiv n + 1 + \alpha > n - 2 = x$ and $(n-2)n + \alpha \le n^2 - n - 1 < n^2 - n + 2 = y$, $x \notin R(y)$ and $y \notin R(x)$ follow. Hence, it is left to show that $[R(x) \cup L(x)] \cap [R(y) \cup L(y)] = \emptyset$. We can check the Figure 4.1.

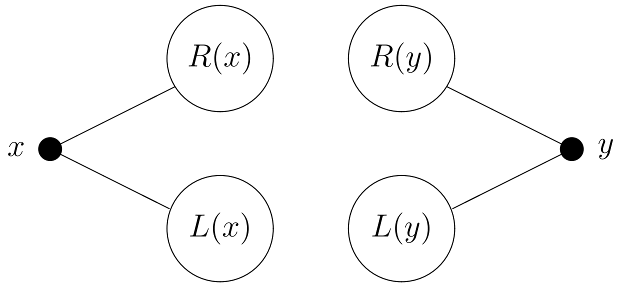


Figure 4.1: The main idea of $N(x) \cap N(y) = \emptyset$

Thus, we have four cases to consider.

- $R(x) \cap R(y) = \emptyset$

  Since $(n-2)n + \alpha \equiv (n^2 - n + 2)n + \beta \pmod{m}$, $\alpha - \beta \equiv 3n + 2 \pmod{m}$. Clearly, there are no solutions for $\alpha$ and $\beta$ when $n \geq 4$, the proof follows.

- $R(x) \cap L(y) = \emptyset$

  Since $(nx + \alpha)n + \beta \equiv y \Rightarrow \alpha n + \beta \equiv x + y = n^2 \pmod{m}$. By the fact $|\alpha n + \beta| \leq n^2 - 1$, no solutions either.

- $L(x) \cap R(y) = \emptyset$

  Since $(ny + \alpha)n + \beta \equiv x \Rightarrow \alpha n + \beta \equiv n^2 \pmod{m}$, we are not able to find solutions for $\alpha$ and $\beta$.

- $L(x) \cap L(y) = \emptyset$

  Suppose not, then there exists a $k \in [0, n^2]$ such that $kn + \alpha \equiv x \pmod{m}$ and $kn + \beta \equiv y \pmod{m}$. Therefore, $|\alpha - \beta| = |x - y| = |n^2 - 2n + 4| > n - 1$. Again, this is not possible.

We note that $d(UG_B(n, n^2 + 1)) = 2$ for $n = 2, 3$. Surprisingly, if $m = n^2 + 2$, then the diameter of $UG_B(n, m)$ is equal to 2, which will shown immediately. $\square$

**Proposition 4.1.2.** *[34] $d(UG_B(n, n^2+2)) = 2$ for $n \geq 3$.*

**Proof.** Let $m = n^2 + 2$. For any two distinct vertices $x$ and $y$ in $UG_B(n, m)$, we claim that $d_G(x, y) \leq 2$. It suffices to show that $N(x) \cap N(y) \neq \emptyset$. Since $N(x) = R(x) \cup L(x)$ and $N(y) = R(y) \cup L(y)$, we have to prove that one of the following four conditions holds: (1)$R(x) \cap L(y) \neq \emptyset$, (2)$R(y) \cap L(x) \neq \emptyset$, (3) $R(x) \cap R(y) \neq \emptyset$ and (4)$L(x) \cap L(y) \neq \emptyset$.

Observe that if $R(x) \cap L(y) \neq \emptyset$, then $(nx + \alpha)n + \beta \equiv y \pmod m$ for some $0 \leq \alpha, \beta \leq n-1$. Therefore, $y + 2x \equiv \alpha n + \beta \in [0, n^2 - 1] \pmod m$. This implies that if $y + 2x \in [0, n^2 - 1] \pmod m$, then $d(x, y) \leq 2$. On the other hand, by considering $R(y) \cap L(x) \neq \emptyset$, we have that if $x + 2y \in [0, n^2 - 1] \pmod m$, then $d(x, y) \leq 2$.

So, it is left to check when both $x + 2y$ and $2x + y$ are equal to either $n^2$ or $n^2 + 1$ $\pmod m$ since $[0, m-1] \setminus [0, n^2 - 1] = \{n^2, n^2 + 1\}$. Since $0 \leq x \neq y \leq n^2 + 1$, there are only six possible cases to consider. But, if $2x + y = n^2$ and $2y + x = 2n^2 + 2$, then $3n^2 + 2 \equiv 0 \pmod 3$ which is impossible. By the same reason, $2x + y = n^2 + 1$ and $2y + x = 2n^2 + 3$ is impossible. Furthermore, if $2x + y = n^2$ and $2y + x = 2n^2 + 3$, then $y - x = n^2 + 3$, this is impossible. Thus, it left three cases to check.

- $2x + y = n^2$ and $2y + x = n^2 + 1$

  In this case, since $2n^2 + 1 \equiv 0 \pmod 3$, we may let $n = 3p + 1$. Then $x = 3p^2 + 2p$ and $y = 3p^2 + 2p + 1$. Hence, we have a path $< 3p^2 + 2p, p, 3p^2 + 2p + 1 >$ from $x$ to $y$. This concludes the proof.

- $2x + y = n^2 + 1$ and $2y + x = 2n^2 + 2$

  We have $x = 0$ and $y = n^2 + 1$. Therefore, the path $< 0, n, n^2 + 1 >$ connects $x$ and $y$ for $n \geq 3$ gives the proof.

- $2x + y = 2n^2 + 2$ and $2y + x = 2n^2 + 3$

  Since $4n^2 + 5 \equiv 0 \pmod 3$, it suffices to consider the cases $n \equiv 1, 2 \pmod 3$. First, if $n = 3p + 1$, then let $x = 6p^2 + 4p + 2$ and $y = 6p^2 + 4p + 1$. It is easy to see that $< 6p^2+4p+1, 2p, 6p^2+4p+2 >$ is a path from $x$ to $y$. If $n = 3p+2$, the proof follows by letting $x = 6p^2+8p+4$ and $y = 6p^2+8p+3$.

$\square$

## 4.2 $d(UG_B(n,m))$ for $n^2+n < m \leq n^2 + (\frac{\sqrt{5}+1}{2})n$

**Proposition 4.2.1.** *$d(UG_B(n, n^2+n+1)) = 2$ for $n \geq 2$.*

**Proof.** Let $m = n^2+n+1$. For any two distinct vertices $x$ and $y$ in $UG_B(n,m)$, we claim that $d_G(x,y) \leq 2$. It suffices to show that $N(x) \cap N(y) \neq \emptyset$. Since $N(x) = R(x) \cup L(x)$ and $N(y) = R(y) \cup L(y)$, we have to prove that one of the following four conditions holds: (1)$R(x) \cap L(y) \neq \emptyset$, (2)$R(y) \cap L(x) \neq \emptyset$, (3) $R(x) \cap R(y) \neq \emptyset$ and (4)$L(x) \cap L(y) \neq \emptyset$.

By $R(x) \cap L(y) \neq \emptyset$, we get $(xn+\alpha)n+\beta \equiv y \pmod m$ for some $0 \leq \alpha, \beta \leq n-1$. Then we have

$$\alpha n + \beta \equiv y - xn^2 \equiv xn + x + y \in [0, n^2-1] \pmod m.$$

So, if $R(x) \cap L(y) = \emptyset$, then

$$xn + x + y \equiv n^2 + t \pmod m, \; 0 \leq t \leq n. \tag{4.1}$$

Through the same process, if $R(y) \cap L(x) = \emptyset$, then

$$yn + y + x \equiv n^2 + s \pmod m, \; 0 \leq s \leq n. \tag{4.2}$$

Therefore, when $R(x) \cap L(y) = \emptyset$ and $R(y) \cap L(x) = \emptyset$, we get $-n \leq (x-y)n \leq n$. On the other hand, if $R(x) \cap R(y) \neq \emptyset$ then $xn + \gamma \equiv yn + \delta$, where $\gamma, \delta \in [0, n-1]$

$$(x-y)n \in [0, n-1] \cup [m-n+1, m-1] \pmod m. \tag{4.3}$$

From the previous three equations, we prove that one of the three cases: $R(x) \cap L(y) \neq \emptyset$, $R(y) \cap L(x) \neq \emptyset$, and $R(x) \cap R(y) \neq \emptyset$ will hold when $(x-y)n \not\equiv n \pmod m$ or $(y-x)n \not\equiv n \pmod m$. Thus, we complete the proof.

Now, suppose $(x-y)n \equiv n \pmod{m}$. The case of $(y-x)n \not\equiv n \pmod{m}$ has the same process. Then $(x-y)n = mk + n = (n^2+n+1)k + n$ for some $k$.

$$(x-y)n = n^2k + nk + k + n = n(nk+k+1) + k$$

$$x - y = (nk+k+1) + \frac{k}{n} \tag{4.4}$$

Only if $k = 0$ or $k = n$, the equation 4.4 has solutions. If $k = n$, then $x - y = n^2 + n + 2$, a contradiction. If $k = 0$, then $x - y = 1$.

The resulting case, we can suppose $x = y + 1$. If $\lfloor \frac{x}{n} \rfloor = \lfloor \frac{y}{n} \rfloor$, then we have the path $< x, \lfloor \frac{x}{n} \rfloor, y >$. Otherwise, we can suppose $x = an$ and $y = an - 1$, and we have $< an, n^2 + a - 1, an - 1 >$. □

**Proposition 4.2.2.** *$d(UG_B(n, n^2 + n + 2)) = 3$ for $n \geq 2$.*

**Proof.** Let $[0, m-1]$ be the vertex set of $G = UG_B(n, m)$. It suffices to show that there exists a pair of vertices $x$ and $y$ in $[0, m-1]$ such that $d_G(x, y) \geq 3$. Here, we let $x = 0$ and $y = n^2 + 2$ to satisfy the inequality $d_G(0, y) \geq 3$. Now, we show the following six statements are all true.

(1) $y \notin R(0)$.

This is a direct consequence of $R(0) = \{0n + \alpha | \alpha \in [0, n-1]\} = [1, n-1]$ and $n \geq 3$, since $y \notin [1, n-1]$.

(2) $0 \notin R(y)$.

This is a direct consequence of $R(y) = \{yn + \alpha | \alpha \in [0, n-1]\} = [n+2, 2n+1]$ and $n \geq 3$, since $0 \notin [n+2, 2n+1]$.

(3) $R(0) \cap R(y) = \emptyset$.

By (1) and (2), we have $[1, n-1] \cap [n+2, 2n+1] = \emptyset$

(4) $R(0) \cap L(y) = \emptyset$.

By (1), we have $\bigcup_{i\in R(0)} R(i) = \bigcup_{i\in[1,n-1]} R(i) = [n, n^2-1]$. So, $y \notin \bigcup_{i\in R(0)} R(i)$.

(5) $L(0) \cap R(y) = \emptyset$.

By (2), we have $\bigcup_{i\in R(y)} R(i) = \bigcup_{i\in[n+2,2n+1]} R(i) = [n-2, n^2+n-3]$. So, $0 \notin \bigcup_{i\in R(y)} R(i)$.

(6) $L(0) \cap L(y) = \emptyset$.

Suppose not. Then $L(0) \cap L(y) \neq \emptyset$. Therefore, there exists a $k \in [0, m-1]$ such that both 0 and $y$ are in $R(k)$. This implies that there exist $\alpha$ and $\beta$, where $0 \leq \alpha, \beta \leq n-1$, satisfying

$$\begin{cases} kn+\alpha \equiv 0 \pmod{m}, \\ kn+\beta \equiv y \pmod{m}. \end{cases} \tag{4.5}$$

This implies that $y \equiv \beta - \alpha \pmod{m}$ and $\beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$. But $y = m - n \notin ([0, n-1] \cup [m-n+1, m-1])$. Thus, the system (4.5) has no solutions for $(\alpha, \beta)$. Hence, (6) is true.

Since we can find a pair of vertices $x = 0$ and $y = n^2+2$ that satisfy properties (1) to (6), $N[x] \cap N[y] = \emptyset$, we have the proof. □

**Proposition 4.2.3.** *$d(UG_B(n,m)) = 3$ for $n^2+n+3 \leq m \leq n^2+\frac{3}{2}n$ and $n \geq 3$.*

**Proof.** Let $[0, m-1]$ be the vertex set of $G = UG_B(n,m)$. It suffices to show that there exists a pair of vertices $x$ and $y$ in $[0, m-1]$ such that $d_G(x,y) \geq 3$. This time we shall let $x = 1$ and find an element $y \in Y = [n^2-n, n^2-1]$ to satisfy the above inequality, i.e., $d_G(1,y) \geq 3$. First, we claim the following six statements are true.

(1) For each $y \in Y$, $1 \notin L(y)$.

This is a direct consequence of $R(1) = \{n+\alpha | \alpha \in [0, n-1]\} = [n, 2n-1]$ and $n \geq 3$, since $Y \cap [n, 2n-1] = \emptyset$.

(2) For each $y \in Y$, $R(1) \cap L(y) = \emptyset$.

By (1), $R(1) = [n, 2n-1]$. Therefore

$$\bigcup_{i \in R(1)} R(i) = [n^2, 2n^2 - 1] = [n^2, m-1] \ \cup \ [0, 2n^2 - 1 - m]. \tag{4.6}$$

Now, the proof follows by the fact that

$$2n^2 - 1 - m \leq 2n^2 - 1 - (n^2 + n + 3) = n^2 - n - 4 < n^2 - n,$$

since $Y \cap [n^2, 2n^2 - 1] = \emptyset$.

(3) For each $y \in Y$, $L(1) \cap L(y) = \emptyset$.

Suppose not. Then $L(1) \cap L(y) \neq \emptyset$. Therefore there exists $k \in [0, m-1]$ such that both 1 and $y$ are in $R(k)$. This implies that there exist $\alpha$ and $\beta$, where $0 \leq \alpha, \beta \leq n-1$, satisfying

$$\begin{cases} kn + \alpha \equiv 1 \pmod{m}, \\ kn + \beta \equiv y \pmod{m}. \end{cases} \tag{4.7}$$

This implies that $\beta - \alpha \equiv y - 1 \pmod{m}$ and $\beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$. But $y - 1 \in [n^2 - n - 1, n^2 - 2]$. Thus system (4.7) has no solutions for $(\alpha, \beta)$. Hence, we have (3).

(4) There exists a set $Y'$ of at most two elements, such that for each $y \in Y \setminus Y'$, $R(1) \cap R(y) = \emptyset$.

Observe that

$$\bigcup_{y \in Y} R(y) = \bigcup_{y \in Y} \{yn + \alpha : \alpha \in [0, n-1]\} = [n^3 - n^2, n^3 - 1]$$

and $R(1) = [n, 2n-1]$. Since $|\bigcup_{y \in Y} R(y)| = n^2 < m$, there are at most two elements in $Y$ satisfying $R(1) \cap R(y) \neq \emptyset$. Moreover, since $[n^3 - n^2, n^3 - 1] \pmod{m}$ is a set of consecutive $n^2$ integers, the possible two elements are $y_1, y_1 + 1$ for some $y_1$ in $Y$. Hence, by letting $Y'$ be $\{y_1, y_1 + 1\}$, we conclude the proof.

(5) There exist at least one elements $y$ in $Y$ such that $L(1) \cap R(y) = \emptyset$.

Observe that

$$\bigcup_{i\in R(y)} R(i) = \bigcup_{i\in[n^3-n^2,n^3-1]} R(i) = [n^4-n^3, n^4-1]$$

which has $n^3$ consecutive positive integers. Therefore, by taking modulo $m \geq n^2 + n + 3$, we have at most $n-1$ integers which are congruent to 1 modulo $m$, since $n^3 = (n^2+n+3)(n-2) + (n^2-n+6)$ and $n \geq 3$. This implies that there are at least one elements, say $y_2$, in $Y$, such that $L(1) \cap R(y_2) = \emptyset$.

(6) For each $y \in Y$ satisfying (4) and (5), $1 \notin R(y)$.

Suppose not. Then $1 \in R(y)$, i.e., $yn + \alpha \equiv 1 \pmod m$ for some $0 \leq \alpha \leq n-1$. First, if $\alpha = 0$, by the fact that $yn+(n-1) \equiv n \pmod m$, $n \in R(1)\cap R(y)$, it is a contradiction. On the other hand, if $\alpha \neq 0$ then $yn+(\alpha-1) \equiv 0 \pmod m$ which implies that $0 \in L(1)\cap R(y)$, it is also a contradiction. Together, we conclude the proof of (6).

Now, we are ready to find the pair $(1, y)$ satisfying $d_G(1, y) \geq 3$. Clearly, if there exists a $y \in \{y_2\} \setminus \{y_1, y_1+1\}$ in (4) and (5), then this $y$ satisfies the conditions from (1) to (6). This implies that $N[1]\cap N[y] = \emptyset$ and the proof follows. On the other hand, if $\{y_2\} \setminus Y' = \emptyset$, then $\{y_2\} \subseteq Y'$. Suppose $R(1) \cap R(y_2) \neq \emptyset$ and $L(1) \cap R(y_2) = \emptyset$. We will claim this assumption is a contradiction.

Since $L(1) \cap R(y_2) = \emptyset$, we have $y_2 n^2 \pmod m \in [2, m-n^2+1]$. Since $R(1) = [n, 2n-1]$, the corresponding $in_{i\in R(1)} = \{n^2, n^2+n, n^2+2n, \cdots, 2n^2-n\}$. Observation

$$\{n^2, n^2+n, n^2+2n, \cdots, 2n^2-n\} \cap [2, m-n^2+1] \pmod m = \{n^2+2n\} \pmod m,$$

we have $y_2 n = n+2$ and $y_2 n^2 = n^2+2n$. Let $m = n^2+n+t$ and $y_2 = n^2 - s$ where $3 \leq t \leq \frac{1}{2}n$ and $1 \leq s \leq n$. Then,

$$(n^2 - s)n \equiv n + 2 \pmod m$$

$$(-n - t - s)n \equiv n + 2 \pmod m$$

$$-n^2 - tn - sn \equiv n + 2 \pmod m$$

$$n + t - tn - sn \equiv n + 2 \pmod{m}$$

$$t - tn - sn \equiv 2 \pmod{m}$$

$$t - 2 \equiv (t + s)n \pmod{m}$$

$$t - 2 + n^2 + n + t = (t + s)n$$

$$n(n + 1 - t - s) = 2 - 2t \in \{-4, -6, \cdots, 2 - n\} \tag{4.8}$$

This equation 4.8 has no solutions which is a contradiction. Therefore, we complete the proof. □

**Proposition 4.2.4.** *$d(UG_B(n, m)) = 3$ for $n^2 + \frac{3}{2}n < m \leq n^2 + (\frac{\sqrt{5}+1}{2})n$ and $n \geq 12$.*

**Proof.** Let $[0, m-1]$ be the vertex set of $G = UG_B(n, m)$. It suffices to show that there exists a pair of vertices $x$ and $y$ in $[0, m-1]$ such that $d_G(x, y) \geq 3$. Here, we let $x = 0$ and try to find an element $y \in Y = [n^2, m - n]$ to satisfy the inequality $d_G(0, y) \geq 3$. Now, we show the following six statements are true.

(1) For each $y \in Y$, $0 \notin L(y)$.

This is a direct consequence of $R(0) = \{0n + \alpha | \alpha \in [0, n-1]\} = [1, n-1]$ and $n \geq 3$, since $Y \cap [1, n-1] = \emptyset$.

(2) For each $y \in Y$, $R(0) \cap L(y) = \emptyset$.

From (1), $R(0) = [1, n-1]$, and so we have

$$\bigcup_{i \in R(0)} R(i) = \bigcup_{i \in [1, n-1]} R(i) = [n, n^2 - 1]. \tag{4.9}$$

Therefore, $Y \cap [n, n^2 - 1] = \emptyset$ for $n \geq 3$ and (2) is true.

(3) For each $y \in Y$, $L(0) \cap L(y) = \emptyset$.

Suppose not. Then $L(0) \cap L(y) \neq \emptyset$. Therefore, there exists a $k \in [0, m-1]$ such that both 0 and $y$ are in $R(k)$. This implies that there exist $\alpha$ and $\beta$, where

$0 \leq \alpha, \beta \leq n-1$, satisfying

$$\begin{cases} kn + \alpha \equiv 0 & \pmod{m}, \\ kn + \beta \equiv y & \pmod{m}. \end{cases} \tag{4.10}$$

This implies that $y \equiv \beta - \alpha \pmod{m}$ and $\beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$. But $Y \cap ([0, n-1] \cup [m-n+1, m-1]) = \emptyset$. Thus, the system (4.10) has no solutions for $(\alpha, \beta)$. Hence, (3) is true.

(4) There exists an element $y \in Y$ such that $L(0) \cap R(y) = \emptyset$.

It suffices to show that there exists an element $y \in Y$ such that $0 \notin \bigcup_{i \in R(y)} R(i)$. We claim there exists a $\widetilde{y}$ satisfying $\widetilde{y}n^2 \in [1, m-n^2]$. Then

$$\bigcup_{i \in R(\widetilde{y})} R(i) = [n^2, m - n^2 + n^2 - 1] = [1, m-1]$$

and $0 \notin \bigcup_{i \in R(y)} R(i)$ holds.

Let $m = n^2 + n + t$ and we have $\frac{1}{2}n < t \leq (\frac{\sqrt{5}-1}{2})n$. Observe that $(n^2+t-1)n^2 \equiv nt \pmod{m}$, $(n^2+t-2)n^2 \equiv nt + (n+t) \pmod{m}$ and so on. Therefore, if there exists a smallest integer $s$, $2 \leq s \leq t$, satisfying $(n^2+t-s)n^2 \equiv nt + (s-1)(n+t) \in [1, m-1] \pmod{m}$, then let $\widetilde{y} = n^2 + t - s$ and we have the proof.

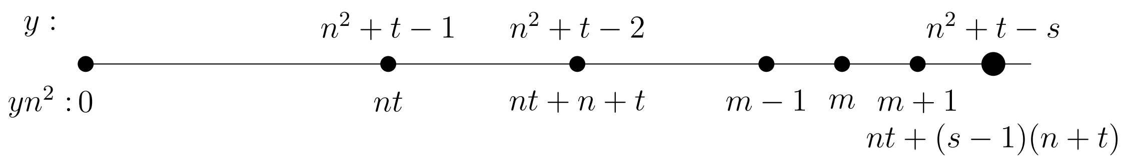


Figure 4.2: $L(0) \cap R(y) = \emptyset$

Suppose not. Then $nt + (t-1)(n+t) \leq m$.

$$nt + nt - n + t^2 - t \leq n^2 + n + t$$

$$n^2 - 2nt + 2n + 2t - t^2 \geq 0$$

$$n^2 + (2-2t)n + t(2-t) \geq 0$$

$$n \geq (t-1) + \sqrt{2t^2 - 4t + 1}$$

But,

$$n \geq (t-1) + \sqrt{2t^2 - 4t + 1} > \left(\frac{n}{2} - 1 + \sqrt{2(\frac{n}{2})^2 - 4(\frac{n}{2}) + 1}\right).$$

has solution for $0 < n < 12$. A contradiction to $n \geq 12$. So, we can find a $\widetilde{y} \in Y$ satisfying $\widetilde{y}n^2 \in [1, m - n^2]$.

This concludes the proof of (4).

(5) For $y \in Y$ satisfying (4), $R(0) \cap R(y) = \emptyset$.

Suppose not. Then $R(0) \cap R(y) \neq \emptyset$ and thus $yn + \alpha \equiv 0n + \beta \pmod{m}$ has solutions for some $\alpha$ and $\beta$ where $0 \leq \alpha, \beta \leq n-1$. Therefore,

$$yn \equiv \beta - \alpha \in [0, n-1] \cup [m-n+1, m-1], \text{ and} \tag{4.11}$$

$$yn^2 \in \{0, n, 2n, \ldots, (n-1)n\} \cup \{2n+t, 3n+t, \ldots, n^2+t\}. \tag{4.12}$$

But, since $y \in Y$ satisfies (4), $yn^2 \in [1, m-n^2] = [1, n+t]$. This implies that

$$yn^2 \in (\{0, n, 2n, \ldots, (n-1)n\} \cup \{2n+t, 3n+t, \ldots, n^2+t\}) \cap [1, n+t] = \{n\}.$$

From equations 4.11 and 4.12, $yn \equiv 0, 1, \ldots, n-1, m-n+1, \ldots, m-1 \pmod{m}$ and the corresponding $yn^2 \equiv 0, n, \ldots, n^2-n, 2n+t, 3n+t, \ldots, n^2+t \pmod{m}$, we get $yn \equiv 1 \pmod{m}$, since $yn^2 \equiv n \pmod{m}$. Now, by letting $y = n^2 + s$, $0 \leq s \leq t$, we have

$$yn \equiv (n^2+s)n \equiv (-n-t+s)n \equiv -tn + sn + n + t \equiv 1 \pmod{m}.$$

It is followed by $t - 1 \equiv (t-s-1)n \pmod{m}$. Clearly, this equation has no solutions for $\frac{1}{2}n < t \leq (\frac{\sqrt{5}-1}{2})n$. This concludes the proof of (5).

(6) For $y \in Y$ satisfying (4) and (5), $0 \notin R(y)$.

Suppose not. Then $yn + \alpha \equiv 0 \pmod{m}$. If $\alpha = 0$, then $yn^2 \equiv (yn)n \equiv 0 \pmod{m}$, a contradiction to (4), $L(0) \cap R(y) = \emptyset$. If $\alpha \neq 0$, then $yn^2 \equiv -\alpha n \notin [1, m-n^2]$, a contradiction to (4) again. Hence, $0 \notin R(y)$.

Since we can always find a pair of vertices $x = 0$ and $y \in [n^2, m-n]$ that satisfy properties (1) to (6), $N[x] \cap N[y] = \emptyset$. We have the proof. $\square$

## 4.3 $d(UG_B(n,m))$ for $n^2 + (\frac{\sqrt{5}+1}{2})n < m \leq 2n^2$

**Proposition 4.3.1.** *[35] $d(UG_B(n,m)) = 3$ for $n^2 + (\frac{\sqrt{5}+1}{2})n < m \leq n^2 + 2n$ and $n \geq 2$.*

**Proof.** Let $[0, m-1]$ be the vertex set of $G = UG_B(n,m)$. It suffices to show that there exists a pair of vertices $x$ and $y$ in $[0, m-1]$ such that $d_G(x,y) \geq 3$. Here, we let $x = 0$ and try to find an element $y \in Y = [n^2, m-n]$ to satisfy the inequality $d_G(0,y) \geq 3$. Now, we claim the following six statements are true. The proof uses a similar idea in the above proposition.

(1) For each $y \in Y$, $0 \notin L(y)$.

This is a direct consequence of $R(0) = \{0n + \alpha | \alpha \in [0, n-1]\} = [1, n-1]$ and $n \geq 2$, since $Y \cap [1, n-1] = \emptyset$.

(2) For each $y \in Y$, $R(0) \cap L(y) = \emptyset$.

By (1), $R(0) = [1, n-1]$, we have

$$\bigcup_{i \in R(0)} R(i) = \bigcup_{i \in [1,n-1]} R(i) = [n, n^2 - 1]. \tag{4.13}$$

Therefore, $Y \cap [n, n^2-1] = \emptyset$ for $n \geq 2$ and (2) is true.

(3) For each $y \in Y$, $L(0) \cap L(y) = \emptyset$.

Suppose not. Then $L(0) \cap L(y) \neq \emptyset$. Therefore, there exists a $k \in [0, m-1]$ such that both 0 and $y$ are in $R(k)$. This implies that there exist $\alpha$ and $\beta$, where $0 \leq \alpha, \beta \leq n-1$, satisfying

$$\begin{cases} kn + \alpha \equiv 0 \pmod{m}, \\ kn + \beta \equiv y \pmod{m}. \end{cases} \tag{4.14}$$

This implies that $y \equiv \beta - \alpha \pmod{m}$ and $\beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$. But $Y \cap ([0, n-1] \cup [m-n+1, m-1]) = \emptyset$. Thus, the system (4.14) has no solutions for $(\alpha, \beta)$. Hence, (3) is true.

(4) There exists an element $y \in Y$ such that $L(0) \cap R(y) = \emptyset$.

It suffices to claim that there exists an element $y \in Y$ such that $0 \notin \bigcup_{i \in R(y)} R(i)$. First, let $t = m-n^2-n$ and $A_i = [(i-1)(m-n^2)+1, i(m-n^2)]$, where $i = 1, 2, \ldots, t+1$. Therefore,

$$|\bigcup_{i=1}^{t+1} A_i| = (t+1)(m-n^2) = (t+1)(n+t).$$

Since $t \geq (\frac{\sqrt{5}-1}{2})n$,

$$(t+1)(n+t) = t^2 + t + nt + n \geq (\frac{\sqrt{5}-1}{2}n)^2 + (\frac{\sqrt{5}-1}{2})n^2 + t + n = m.$$

By the fact that $\{A_i\}_{i=1}^{t+1}$ is a collection of disjoint sets and $\bigcup_{i=1}^{t+1} A_i = [1, (t+1)(m-n^2)]$, there exists an $i_0$ such that $n^4 \pmod{m} \in A_{i_0}$.

Now, let $y = n^2 + i_0 - 1$. Then we have

$$yn^2 \equiv (n^2 + i_0 - 1)n^2 \equiv n^4 + (i_0 - 1)(n^2) \pmod{m}$$

$$\in [(i_0-1)(m-n^2) + 1 + (i_0-1)(n^2), i_0(m-n^2) + (i_0-1)(n^2)] \pmod{m}$$

$$= [1 + (i_0-1)m, m - n^2 + (i_0-1)m] \pmod{m} = [1, m-n^2] \pmod{m}.$$

This implies that

$$\bigcup_{i \in R(y)} R(i) = \{ni + \alpha | i \in R(y) \text{ and } \alpha \in [0, n-1]\}$$

$$= \{ni + \alpha | i \in [n(n^2 + i_0 - 1), n(n^2 + i_0 - 1) + (n-1)] \text{ and } \alpha \in [0, n-1]\}$$

$$\subseteq [1, m - n^2 + (n^2 - 1)] = [1, m-1].$$

This concludes the proof of (4).

(5) For $y \in Y$ satisfying (4), $R(0) \cap R(y) = \emptyset$.

Suppose not. Then $R(0) \cap R(y) \neq \emptyset$ and thus $yn + \alpha \equiv 0n + \beta \pmod{m}$ has solutions for some $\alpha$ and $\beta$ where $0 \leq \alpha, \beta \leq n-1$. Therefore,

$$yn \equiv \beta - \alpha \in [0, n-1] \cup [m-n+1, m-1], \text{ and}$$

$$yn^2 \in \{0, n, 2n, \ldots, (n-1)n\} \cup \{2n+t, 3n+t, \ldots, n^2+t\}.$$

But, since $y \in Y$ satisfying (4), $yn^2 \in [1, m-n^2] = [1, n+t]$. This implies that

$yn^2 \in (\{0, n, 2n, \ldots, (n-1)n\} \cup \{2n+t, 3n+t, \ldots, n^2+t\}) \cap [1, n+t] = \{n\}, i.e.$

$yn \equiv 1 \pmod{m}$. Now, by letting $y = n^2 + s$, $0 \le s \le t$, we have

$$yn \equiv (n^2+s)n \equiv (-n-t+s)n \equiv -tn + sn + n + t \equiv 1 \pmod{m}.$$

It is followed by $t-1 \equiv (t-s-1)n \pmod{m}$. Clearly, this equation has no solutions. This concludes the proof of (5).

(6) For $y \in Y$ satisfying (4) and (5), $0 \notin R(y)$.

Suppose not. Then $yn + \alpha \equiv 0 \pmod{m}$. If $\alpha = 0$, then $yn^2 \equiv (yn)n \equiv 0$, a contradiction to (4), $L(0) \cap R(y) = \emptyset$. If $\alpha \ne 0$, then $yn^2 \equiv -\alpha n \notin [1, m-n^2]$, a contradiction to (4) again. Hence, $0 \notin R(y)$.

Since we can always find a pair of vertices $x = 0$ and $y \in [n^2, m-n]$ to satisfy properties (1) to (6), $N[x] \cap N[y] = \emptyset$, we have the proof. □

**Proposition 4.3.2.** *[35] $d(UG_B(n,m)) = 3$ for $n^2 + 2n < m \le 2n^2$ and $n \ge 7$.*

**Proof.** Let $[0, m-1]$ be the vertex set of $G = UG_B(n,m)$. Again, it suffices to show that there exists a pair of vertices $x$ and $y$ in $[0, m-1]$ such that $d_G(x,y) \ge 3$. Note that we shall let $x = 1$ and find an element $y \in Y = [n^2-2n, n^2-1]$ to satisfy the above inequality, i.e., $d_G(1,y) \ge 3$. First, we claim the following six statements are true.

(1) For each $y \in Y$, $1 \notin L(y)$.

This is a direct consequence of $R(1) = \{n+\alpha | \alpha \in [0, n-1]\} = [n, 2n-1]$ and $n \ge 4$, since $Y \cap [n, 2n-1] = \emptyset$.

(2) For each $y \in Y$, $R(1) \cap L(y) = \emptyset$.

By (1), $R(1) = [n, 2n-1]$. Therefore

$$\bigcup_{i \in R(1)} R(i) = [n^2, 2n^2-1] = [n^2, m-1] \ \cup \ [0, 2n^2-1-m]. \tag{4.15}$$

Now, the proof is followed by the fact that $2n^2-1-m \leq 2n^2-1-n^2-2n < n^2-2n$, since $Y \cap [n^2, 2n^2-1] = \emptyset$.

(3) For each $y \in Y$, $L(1) \cap L(y) = \emptyset$.

Suppose not. Then $L(1) \cap L(y) \neq \emptyset$. Therefore there exists a $k \in [0, m-1]$ such that both 1 and $y$ are in $R(k)$. This implies that there exist $\alpha$ and $\beta$, where $0 \leq \alpha, \beta \leq n-1$, satisfying

$$\begin{cases} kn+\alpha \equiv 1 \pmod{m}, \\ kn+\beta \equiv y \pmod{m}. \end{cases} \tag{4.16}$$

This implies that $\beta - \alpha \equiv y-1 \pmod{m}$ and $\beta - \alpha \in [0, n-1] \cup [m-n+1, m-1]$. But $y-1 \in [n^2-2n-1, n^2-2]$. Thus system (4.16) has no solutions for $(\alpha, \beta)$. Hence, we have (3).

(4) There exists a set $Y'$ of at most four elements, such that for each $y \in Y \setminus Y'$, $R(1) \cap R(y) = \emptyset$.

Observe that

$$\bigcup_{y \in Y} R(y) = \bigcup_{y \in Y} \{yn+\alpha | \alpha \in [0, n-1]\} = [n^3-2n^2, n^3-1]$$

and $R(1) = [n, 2n-1]$. Since $m \leq |\bigcup_{y \in Y} R(y)| = 2n^2 < 2m$, there are at most four elements of $Y$ satisfying $R(1) \cap R(y) \neq \emptyset$. Moreover, since $[n^3-2n^2, n^3-1] \pmod{m}$ is a set of consecutive $2n^2$ integers, the possible four elements are $y_1, y_1+1, y_2, y_2+1$ for some $y_1$ and $y_2$ in $Y$. Hence, by letting $Y'$ be $\{y_1, y_1+1, y_2, y_2+1\}$ and checking the Figure 4.3, we conclude the proof.

(5) There exist at least three elements $y$ in $Y$ such that $L(1) \cap R(y) = \emptyset$.

Observe that

$$\bigcup_{i \in R(y)} R(i) = \bigcup_{i \in [n^3-2n^2, n^3-1]} R(i) = [n^4-2n^3, n^4-1]$$

which has $2n^3$ consecutive positive integers. Therefore, by taking modulo $m \geq n^2+2n$, we have at most $2n-3$ integers which are congruent to 1 modulo $m$, since $2n^3 =$

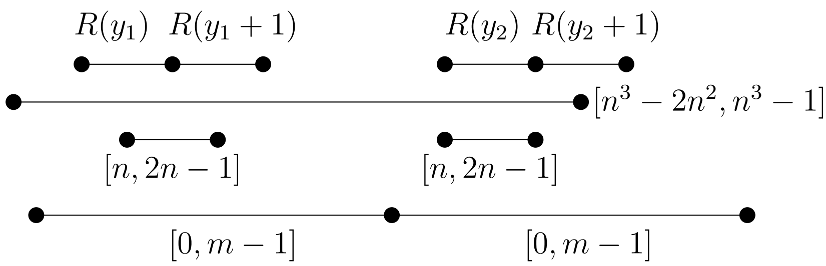


Figure 4.3: $R(y_1)$, $R(y_1+1)$, $R(y_2)$, $R(y_2+1)$, and $[n^3-2n^2, n^3-1]$.

$(n^2+2n)(2n-4)+8n$ and $n \geq 7$. This implies that there are at least three elements, say $y_3, y_4, y_5$, in $Y$, such that $L(1) \cap R(y_i) = \emptyset$, for $i = 3, 4, 5$.

(6) For each $y \in Y$ satisfying (4) and (5), $1 \notin R(y)$.

Suppose not. Then $1 \in R(y)$, i.e., $yn + \alpha \equiv 1 \pmod{m}$ for some $0 \leq \alpha \leq n-1$. First, if $\alpha = 0$, by the fact that $yn + (n-1) \equiv n \pmod{m}$, $n \in R(1) \cap R(y)$, a contradiction. On the other hand, if $\alpha \neq 0$ then $yn + (\alpha - 1) \equiv 0 \pmod{m}$ which implies that $0 \in L(1) \cap R(y)$, a contradiction. This concludes the proof of (6).

Now, we are ready to find the pair $(1, y)$ satisfying $d_G(1, y) \geq 3$. Clearly, if there exists a $y \in \{y_3, y_4, y_5\} \setminus \{y_1, y_1+1, y_2, y_2+1\}$ in (4) and (5), then this $y$ satisfies the conditions from (1) to (6). This implies that $N[1] \cap N[y] = \emptyset$ and the proof follows. On the other hand, if $\{y_3, y_4, y_5\} \setminus Y' = \emptyset$, then $\{y_3, y_4, y_5\} \subseteq Y'$. Without loss of generality, let $y_3 = y_1$ and $y_4 = y_3 + 1$. Then by (5), $L(1) \cap R(y_1) = \emptyset$ and $L(1) \cap R(y_1+1) = \emptyset$. This implies that $L(1) \cap [R(y_1) \cup R(y_1+1)] = \emptyset$. But,

$$\bigcup_{i \in R(y_1) \cup R(y_1+1)} R(i) = \bigcup_{i \in [y_1 n, y_1 n + 2n - 1]} R(i) = [y_1 n^2, y_1 n^2 + 2n^2 - 1]$$

which is a set of $2n^2$ consecutive integers and thus $[y_1 n^2, y_1 n^2 + 2n^2 - 1] \pmod{m} \supseteq [0, m-1] \supseteq L(1)$, a contradiction. So there exists a $y \in Y$ such that $d_G(1, y) \geq 3$. This concludes the proof. $\square$

Combining the above two propositions, we have the following theorem.

**Theorem 4.3.3.** *[35] $d(UG_B(n,m)) = 3$ for $n^2 + (\frac{\sqrt{5}+1}{2})n < m \leq 2n^2$ and $n \geq 2$.*

**Proof.** By previous two propositions and the following results.

$d_G(0,y) \geq 3$ for $G = UG_B(n,m)$ in the following table where giving $m$, $n$, $y$ and letting $x = 0$ we have $d_G(0,y) \geq 3$. (The table is obtained by the aid of computer and for clearness, we include a program following the table).

| n | m | y |
|---|---|---|
| 2 | 8 | 5 |
| 3 | 15,16,18 | 9 |
| 3 | 17 | 10 |
| 4 | 25 | 16 |
| 4 | 24,27,29,30 | 17 |
| 4 | 31 | 18 |
| 4 | 26,28,32 | 23 |
| 5 | 39,44,47,48,50 | 25 |
| 5 | 36,38,40,45,46,49 | 26 |
| 5 | 35,37,42 | 27 |
| 5 | 41,43 | 28 |
| 6 | 56,58,61,64,67,68 | 36 |
| 6 | 49,51,53,57,60,63,66,69,70,72 | 37 |
| 6 | 52,54,59,62,65,71 | 38 |
| 6 | 48,50 | 39 |
| 6 | 55 | 40 |

## Program A

This program generates the results of $d_G(x,y) \geq 3$.

For $n = 2$ to $100$

For $m = n^2 + 2n$ to $2n^2$

For $y = n^2 - 2n$ to $n^2 - 1$

If $distance(x = 0, y) \geq 3$

print $m$, $n$, $x$, $y$

Endif
Endfor $y$
Endfor $m$
Endfor $n$
function *distance*
for $\alpha = 1$ to $n - 1$
test=$congm(x * n + \alpha)$
select xnei.dbf
append blank
replace element with test
endfor

for $\beta = 0$ to $m - 1$
if $congm(x - \beta * n) < n$
select xnei.dbf
append blank
replace element with $\beta$
endif
endfor

for $\alpha = 1$ to $n - 1$
test=$congm(y * n + \alpha)$
select ynei.dbf
append blank
replace element with test
endfor

for $\beta = 0$ to $m - 1$
if $congm(y - \beta * n) < n$
select ynei.dbf

append blank

replace element with $\beta$

endif

endfor

If (x-Neighbor $\cap$ y-Neighbor $= \emptyset$) return 3 else return 2

function *congm*

if $t > (m-1)$ then t=t-m endif

if $t < 0$ then t=t+m endif

return $t$

## 4.4 $d(UG_B(n,m))$ for $2n^2 < m \leq n^3$

**Proposition 4.4.1.** *[34] For positive integers $n \geq 2$ and $2n^2 < m \leq n^3$, the diameter of $UG_B(n,m)$ is* 3.

**Proof.** Let $[0, m-1]$ be the vertex set of $G = UG_B(n,m)$. We claim that either $d_G(0, m-n) = 3$ or $d_G(0, m-n-1) = 3$. For convenience, let $j_1 = m-n$ and $j_2 = m-n-1$. By observation, we have $j_1 \notin N(0)$ and $j_2 \notin N(0)$. Therefore, it suffices to prove that either $N(0) \cap N(j_1) = \emptyset$ or $N(0) \cap N(j_2) = \emptyset$ which implies that $d(G) \geq 3$.

By definition, $N(0) = R(0) \cup L(0)$ and $N(j) = R(j) \cup L(j)$ where $j = j_1$ or $j_2$ as the case may be. Therefore, it is equivalent to show that $[R(0) \cup L(0)] \cap [R(j) \cup L(j)] = \emptyset$. We split the proof into four cases and the first three cases deal with $j = j_1$ or $j_2$.

Case 1. $R(0) \cap L(j) = \emptyset$. Since $\bigcup_{i \in R(0)} R(i) = \bigcup_{i \in [1,n-1]} R(i) = [n, n^2-1]$, neither $j_1$ nor $j_2$ are in $\bigcup_{i \in R(0)} R(i)$. This implies that $R(0) \cap L(j) = \emptyset$.

Case 2. $R(0) \cap R(j) = \emptyset$. By definition of $R(j)$, $R(j) = \{jn + \alpha \pmod{m} : \alpha \in$

$[0, n-1]\}$. Hence, it is clear that $R(0) \cap R(j) = \emptyset$.

Case 3. $L(0) \cap L(j) = \emptyset$. Assume that $L(0) \cap L(j) \neq \emptyset$. Then there exists a $k$ such that $0 \in R(k)$ and $j \in R(k)$. This implies that there exist $\alpha$ and $\beta$ where $0 \leq \alpha, \beta \leq n-1$ satisfying

$$\begin{cases} kn + \alpha \equiv 0 \pmod{m}, \\ kn + \beta \equiv j \pmod{m}. \end{cases} \tag{4.17}$$

Therefore, $\beta - \alpha \equiv j \pmod{m}$ and $-(n-1) \leq \beta - \alpha \leq n-1$. Since $\beta - \alpha \neq j$ if $\beta - \alpha \geq 0$ and $(-\beta + \alpha) + m - n < m$ or $(-\beta + \alpha) + m - n - 1 < m$, we conclude that no solution $(\alpha, \beta)$ for (2.1). Hence the case is proved.

Case 4. $L(0) \cap R(j) = \emptyset$, $j = j_1$ or $j_2$. First, we define $\delta(j_1) = 0$ and $\delta(j_2) = 1$. We shall claim that either $0 \notin \bigcup_{i \in R(j_1)} R(i)$ or $0 \notin \bigcup_{i \in R(j_2)} R(i)$. Assume that the above assertion is not true. Then, there exist $0 \leq \alpha, \beta, \gamma, \epsilon \leq n-1$ such that

$$\begin{cases} ((m - n - \delta(j_1))n + \alpha)n + \beta \equiv 0, & \pmod{m}; \\ ((m - n - \delta(j_2))n + \gamma)n + \epsilon \equiv 0, & \pmod{m}. \end{cases}$$

Thus,

$$\begin{cases} -n^3 + \alpha n + \beta \equiv 0, & \pmod{m}; \\ -n^3 - n^2 + \gamma n + \epsilon \equiv 0, & \pmod{m}. \end{cases}$$

This implies that $n^2 + (\alpha - \gamma)n + (\beta - \epsilon) \equiv 0 \pmod{m}$. Since both $\alpha - \gamma$ and $\beta - \epsilon$ are integers between $-(n-1)$ and $(n-1)$, we have $2n^2 > n^2 + (\alpha - \gamma)n + (\beta - \epsilon) > 0$. Therefore, we are not able to find $(\alpha, \beta, \gamma, \epsilon)$ to satisfy $n^2 + (\alpha - \gamma)n + (\beta - \epsilon) \equiv 0 \pmod{m}$. Hence, we conclude that either $0 \notin \bigcup_{i \in R(j_1)} R(i)$ or $0 \notin \bigcup_{i \in R(j_2)} R(i)$ and thus either $L(0) \cap R(j_1) = \emptyset$ or $L(0) \cap R(j_2) = \emptyset$.

Now, combining the above four cases and $j \notin N(0)$, we have either $d_G(0, j_1) = 3$ or $d_G(0, j_2) = 3$. This concludes the proof. □

To summarize, we are able to determine the diameter of $UG_B(n, m)$ for almost all $n^2 < m \leq n^3$ except certain $m$'s which are very close to $n^2$. From the facts that the diameter of $UG_B(n, n^2 + 1)$ is 3 and the diameter of $UG_B(n, n^2 + 2)$ is 2, we notice that $d(UG_B(n, m_1))$ may not be larger than $d(UG_B(n, m_2))$ whenever

$m_1 > m_2$. Nevertheless, we do prove that for $n^2+n+2 \leq m \leq n^3$, $d(UG_B(n,m)) = 3$. Hopefully, we can settle the rest of cases for $n^2 < m \leq n^3$ in the near future.

# Chapter 5

# Conclusion

With the growing popularity and usage of communication networks, it is expected that an efficient design of networks will make a tremendous impact on our daily life. Since *de Bruijn networks* meet the requirements of being a good one, knowing how to apply these networks deserves more attention. In this thesis, we have made a substantial effort in finding wide-diameters of $UB(d, n)$ and diameters of $UG_B(n, m)$ for $n^2 < m \leq n^3$ except certain $m$'s. But, there are many more problems left to be answered. So, for our future study, we would like to solve the following two problems first.

Problem 1. Determine $d(UG_B(n, m))$ for all $n^2 \leq m \leq n^3$.

Problem 2. Is $d(UG_B(n, m)) = k$ whenever $n^{k-1} < m \leq n^k$, $k \geq 4$?

There also remains something to do in our study of wide-diameters. So far, we can find $2d - 2$ disjoint paths with length at most $2n + 1$ for any two vertices in the network $UB(d, n)$. It seems that the lengths of paths can be shorten to $n + 2$. So, we have the next problem to try.

Problem 3. For any two vertices in $UB(d, n)$, find $2d - 2$ vertex-disjoint paths with length as short as possible.

Finally, we would like to put our long-term research goal of the *de Bruijn networks* in solving the following problems.

Problem 4. Find the wide-diameters of $G_B(n,m)$ and $UG_B(n,m)$ respectively.

Problem 5. Determine the connectivity of $UG_B(n,m)$.

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