# Unitary part of a contraction

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## ARTICLE INFO

## ABSTRACT

For a contraction $A$ on a Hilbert space $H$, we define the index $j(A)$ (resp., $k(A)$) as the smallest nonnegative integer $j$ (resp., $k$) such that $\ker(I - A^{j*}A^j)$ (resp., $\ker(I - A^{k*}A^k) \cap \ker(I - A^k A^{k*})$) equals the subspace of $H$ on which the unitary part of $A$ acts. We show that if $n = \dim H < \infty$, then $j(A) \leqslant n$ (resp., $k(A) \leqslant \lceil n/2 \rceil$), and the equality holds if and only if $A$ is of class $\mathcal{S}_n$ (resp., one of the three conditions is true: (1) $A$ is of class $\mathcal{S}_n$, (2) $n$ is even and $A$ is completely nonunitary with $\|A^{n-2}\| = 1$ and $\|A^{n-1}\| < 1$, and (3) $n$ is even and $A = U \oplus A'$, where $U$ is unitary on a one-dimensional space and $A'$ is of class $\mathcal{S}_{n-1}$).

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## 1. Introduction

Let $H$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $A$ be a contraction ($\|A\| \leqslant 1$) on $H$. There is a canonical decomposition of $H$ as the direct sum $H_1 \oplus H_2$ of reducing subspaces $H_1$ and $H_2$ for $A$ so that $A|H_1$ is unitary and $A|H_2$ is completely nonunitary. (Recall that an operator $B$ on $K$ is *completely nonunitary* (*c.n.u.*) if there is no (nonzero) reducing subspace $L$ of $B$ such that $B|L$ is unitary.) In this case, $H_1$ and $H_2$ are uniquely determined by $A$: $H_1 = \{x \in H: \|A^j x\| = \|x\| = \|A^{j*} x\| \text{ for all } j \geqslant 0\}$ and $H_2 = H \ominus H_1$, and $U = A|H_1$ and $A' = A|H_2$ are called the *unitary part* and *completely nonunitary part* of $A$, respectively (cf. [3, Theorem I.3.2]). Another expression for the subspace $H_1$ is $\bigcap_{j=0}^{\infty}(\ker(I - A^{j*}A^j) \cap \ker(I - A^j A^{j*}))$. Thus it is natural to consider the smallest integer $j$ (resp., $k$) for which $\ker(I - A^{j*}A^j) = H_1$ (resp., $\ker(I - A^{k*}A^k) \cap \ker(I - A^k A^{k*}) = H_1$). For this purpose, we give the following definitions:

**Definition 1.1.** For a contraction $A$ on $H$, let

$$H_j(A) = \begin{cases} \ker(I - A^{j*}A^j) & \text{if } 0 \leqslant j < \infty, \\ \bigcap_{j=0}^{\infty} \ker(I - A^{j*}A^j) & \text{if } j = \infty, \end{cases}$$

$$j(A) = \min\{j: 0 \leqslant j \leqslant \infty, \ H_j(A) = H_\infty(A) \cap H_\infty(A^*)\},$$

and

$$k(A) = \min\{k: 0 \leqslant k \leqslant \infty, \ H_k(A) \cap H_k(A^*) = H_\infty(A) \cap H_\infty(A^*)\}.$$

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The next example gives values of such indices for the *n*-by-*n* *Jordan block*

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

**Example 1.2.** If $A = J_n$, then $j(A) = n$ and $k(A) = \lceil n/2 \rceil$, the ceiling of $n/2$ (that is, the smallest integer which is larger than or equal to $n/2$). This is because simple computations yield that

$$H_j(A) = \begin{cases} \{0\} \oplus \mathbb{C}^{n-j} & \text{if } 0 \leqslant j \leqslant n, \\ \{0\} & \text{if } n < j \leqslant \infty, \end{cases}$$

$$H_j(A^*) = \begin{cases} \mathbb{C}^{n-j} \oplus \{0\} & \text{if } 0 \leqslant j \leqslant n, \\ \{0\} & \text{if } n < j \leqslant \infty, \end{cases}$$

and $H_\infty(A) \cap H_\infty(A^*) = \{0\}$.

In the following, we will show that if $A$ is a contraction on an *n*-dimensional space, then $j(A)$ and $k(A)$ can be at most $n$ and $\lceil n/2 \rceil$, respectively. The extremal cases can also be completely characterized. Recall that an *n*-dimensional operator $A$ is said to be *of class* $\mathcal{S}_n$ if it is a contraction, its eigenvalues are all in the open unit disc $\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$ and it satisfies $\text{rank}(I_n - A^*A) = 1$. One example of such operators is $J_n$. More generally, it is known [1, Corollary 1.3] that an *n*-by-*n* matrix is of class $\mathcal{S}_n$ if and only if it is unitarily equivalent to a matrix of the form $[a_{ij}]_{i,j=1}^n$, where $|a_{ii}| < 1$ for all $i$ and

$$a_{ij} = \begin{cases} (1 - |a_{ii}|^2)^{1/2}(1 - |a_{jj}|^2)^{1/2}(\prod_{k=i+1}^{j-1}(-\bar{a}_{kk})) & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

We show that a contraction $A$ on an *n*-dimensional space is such that $j(A) = n$ (resp., $k(A) = \lceil n/2 \rceil$) if and only if it is of class $\mathcal{S}_n$ (resp., one of the following holds: (1) $A$ is of class $\mathcal{S}_n$, (2) $n$ is even and $A$ is completely nonunitary with $\|A^{n-2}\| = 1$ and $\|A^{n-1}\| < 1$, and (3) $n$ is even and $A = U \oplus A'$, where $U$ is unitary on a one-dimensional space and $A'$ is of class $\mathcal{S}_{n-1}$). These will be given in Sections 2 and 3, respectively.

Throughout this paper, we will frequently use the fact that, for a contraction $A$ and an integer $j \geqslant 0$, $x$ is in $H_j(A)$ if and only if $\|A^j x\| = \|x\|$.

We end this section with some basic properties of the $H_j(A)$'s, $j(A)$ and $k(A)$.

**Proposition 1.3.** *Let $A$ be a contraction on $H$. Then*

(1) $H_0(A) = H_0(A^*) = H$,
(2) $H_j(A) \downarrow H_\infty(A)$ *as $j \to \infty$,*
(3) $\dim H_j(A) = \dim H_j(A^*)$ *for all $j$, $0 \leqslant j \leqslant \infty$,*
(4) $0 \leqslant k(A) \leqslant j(A) \leqslant \infty$,
(5) *the following conditions are equivalent*: (a) $j(A) = 0$, (b) $k(A) = 0$, *and* (c) $A$ *is unitary,*
(6) $j(A) = j$ $(1 \leqslant j < \infty)$ *if and only if $A = U \oplus A'$ on $H = H_1 \oplus H_2$, where $U$ is unitary, $H_j(A') = \{0\}$ and $H_{j-1}(A') \neq \{0\}$,*
(7) $j(A) = j(A^*)$ *and $k(A) = k(A^*)$,*
(8) $j(A \oplus B) = \max\{j(A), j(B)\}$ *and $k(A \oplus B) = \max\{k(A), k(B)\}$ for any other contraction $B$, and*
(9) *for $A$ normal, $H_j(A) = H_\infty(A)$ for all $j \geqslant 1$ and thus $j(A) = k(A) \leqslant 1$.*

For the proof of (3), we need the next lemma.

**Lemma 1.4.** *For any operator $A$ on $H$, let $A^*A = A_1 \oplus 0$ (resp., $AA^* = A_2 \oplus 0$) on $H = \overline{\text{ran } A^*} \oplus \ker A$ (resp., $H = \overline{\text{ran } A} \oplus \ker A^*$). Then $A_1$ and $A_2$ are unitarily equivalent.*

**Proof.** Since $\overline{\text{ran } A^*} = \overline{\text{ran}(A^*A)^{1/2}}$ (resp., $\overline{\text{ran } A} = \overline{\text{ran}(AA^*)^{1/2}}$), we may define $V : \overline{\text{ran } A^*} \to \overline{\text{ran } A}$ (resp., $W : \overline{\text{ran } A} \to \overline{\text{ran } A^*}$) by $V((A^*A)^{1/2}x) = Ax$ for $x$ in $H$ (resp., $W((AA^*)^{1/2}y) = A^*y$ for $y$ in $H$). Then $V$ and $W$ are surjective isometries satisfying

$$\langle V(A^*A)^{1/2}x, (AA^*)^{1/2}y \rangle = \langle Ax, (AA^*)^{1/2}y \rangle = \langle x, A^*(AA^*)^{1/2}y \rangle$$
$$= \langle x, (A^*A)^{1/2}A^*y \rangle = \langle (A^*A)^{1/2}x, W(AA^*)^{1/2}y \rangle$$
$$= \langle W^*(A^*A)^{1/2}x, (AA^*)^{1/2}y \rangle$$

for all $x$ and $y$. Thus $V = W^*$. We have

$$
\begin{aligned}
\left(V^* A_2 V\right)\left(A^* A\right)^{1/2} x &= W A_2 A x = W\left(A A^*\right) A x \\
&= W\left(A A^*\right)^{1/2}\left(A A^*\right)^{1/2} A x = A^*\left(A A^*\right)^{1/2} A x = \left(A^* A\right)\left(A^* A\right)^{1/2} x \\
&= A_1\left(A^* A\right)^{1/2} x,
\end{aligned}
$$

which shows that $V^* A_2 V = A_1$, completing the proof. □

**Proof of Proposition 1.3.** (1), (2), (4), (5), (6) and (8) are trivial. (3) is an easy consequence of Lemma 1.4 and (7) follows from (6) and (3). We now prove (9). Indeed, if $A$ is normal, then from $I - A^{j*} A^j = (\sum_{k=0}^{j-1} A^{k*} A^k)(I - A^* A)$ we obtain $H_1(A) \subseteq H_j(A)$ for any $j \geqslant 1$. Since $H_j(A) \subseteq H_1(A)$ by (2), the equality $H_j(A) = H_1(A)$ holds for all $j \geqslant 1$. Our assertions follow immediately. □

## 2. The index $j(A)$

The main result of this section is the following theorem.

**Theorem 2.1.** *If $A$ is a contraction on an $n$-dimensional space, then $j(A) \leqslant n$. The equality holds if and only if $A$ is of class $\mathcal{S}_n$.*

For its proof, we need the next two lemmas.

**Lemma 2.2.** *If $A$ is a contraction on an $n$-dimensional space with $\dim H_j(A) > n - j$ for some $j$, $1 \leqslant j \leqslant n$, then $j(A) < j$.*

**Proof.** Assume that $A$ acts on the space $H$ with $\dim H = n$. Since there are $j + 1$ subspaces in the sequence $H_j(A) \subseteq H_{j-1}(A) \subseteq \cdots \subseteq H_1(A) \subseteq H_0(A) (= H)$, our assumption on $H_j(A)$ implies, by the pigeonhole principle, the equality of $H_k(A)$ and $H_{k+1}(A)$ for some $k$, $0 \leqslant k < j$. If $x$ is in $H_{k+1}(A)$, then

$$
\|x\| = \left\|A^{k+1} x\right\| \leqslant \|A x\| \leqslant \|x\|,
$$

which yields that $\|A^k(Ax)\| = \|Ax\| = \|x\|$. Hence $Ax$ is in $H_k(A) = H_{k+1}(A)$ and thus $A$ maps $H_k(A)$ into itself isometrically. Therefore, $A = A_1 \oplus A_2$ on $H = H_k(A) \oplus H_k(A)^\perp$ with $A_1$ unitary. This shows that $H_k(A) \subseteq H_\infty(A) \cap H_\infty(A^*)$. Since the converse containment is trivial, we have $H_k(A) = H_\infty(A) \cap H_\infty(A^*)$ and therefore $j(A) \leqslant k < j$ as asserted. □

We note that the above arguments are essentially contained in the proof of [2, Lemma 2.3].

**Lemma 2.3.** *A contraction $A$ on an $n$-dimensional space is of class $\mathcal{S}_n$ if and only if $\dim H_j(A) = n - j$ for all $j$, $1 \leqslant j \leqslant n$.*

This characterization of $\mathcal{S}_n$-operators is from [2, Theorem 3.1].

**Proof of Theorem 2.1.** If $H_n(A) = \{0\}$, then obviously $H_\infty(A) \cap H_\infty(A^*) = \{0\}$ and $j(A) \leqslant n$; otherwise, the assertion $j(A) < n$ follows from Lemma 2.2.

For the remaining part, we need only check, in view of Lemma 2.3, that $j(A) = n$ implies $\dim H_j(A) = n - j$ for all $j$, $1 \leqslant j \leqslant n$. Indeed, if $\dim H_j(A) < n - j$ for some $j$, $1 \leqslant j < n$, then, as in the proof of Lemma 2.2, the pigeonhole principle yields the equality of $H_k(A)$ and $H_{k+1}(A)$ for some $k$, $j \leqslant k < n$. We then proceed as before to infer that $A$ maps $H_k(A)$ into itself isometrically and that $H_k(A) = H_\infty(A) \cap H_\infty(A^*)$. This leads to $n = j(A) \leqslant k < n$, a contradiction. Thus we must have $\dim H_j(A) \geqslant n - j$ for all $j$. If the strict inequality ">" holds for any $j$, $1 \leqslant j \leqslant n$, then Lemma 2.2 says that $j(A) < j \leqslant n$, again a contradiction. Therefore, we have $\dim H_j(A) = n - j$ for all $j$. Thus $A$ is of class $\mathcal{S}_n$ by Lemma 2.3. This completes the proof. □

To conclude this section, we give two remarks. Firstly, the assertion $j(A) \leqslant n$ in Theorem 2.1 can be slightly improved. Namely, *if $A$ is a finite-dimensional contraction, then $j(A)$ is at most the degree of the minimal polynomial of $A$.* To prove this, let $m$ denote this degree. Then, for any $x$ in $H_m(A)$, we have $\|A^m x\| = \|x\|$. Thus $\|A^{m-1} x\| = \cdots = \|A x\| = \|x\|$, which shows that $A^{m-1} x, \ldots, A x, x$ are all in $H_1(A)$. Since $A^m x$ is a linear combination of these vectors, it is also in $H_1(A)$. We can then deduce successively that all the $A^j x$'s, $j \geqslant m$, are in $H_1(A)$. If $K$ denotes the subspace generated by $x, Ax, A^2 x, \ldots$, then $K$ is an invariant subspace of $A$ contained in $H_1(A)$. Hence $A_1 \equiv A|K$ is isometric on $K$. On a finite-dimensional space, $A_1$ is unitary. Thus $K$ is contained in $H_\infty(A) \cap H_\infty(A^*)$. This shows that $x \in H_\infty(A) \cap H_\infty(A^*)$ for any $x$ in $H_m(A)$ or $H_m(A) \subseteq H_\infty(A) \cap H_\infty(A^*)$. Since the converse containment is trivial, we obtain $H_m(A) = H_\infty(A) \cap H_\infty(A^*)$. Thus $j(A) \leqslant m$ follows.

Secondly, *for any finite-dimensional contraction A, the inequality*

$$\dim H_{j+1}(A) \leqslant \max\{\dim H_j(A) - 1, \dim(H_\infty(A) \cap H_\infty(A^*))\} \tag{a}$$

*holds for all* $j \geqslant 0$. This is a consequence of [2, Lemma 2.3]. Indeed, if $\dim H_{j+1}(A) \leqslant \dim H_j(A) - 1$, then we are done. Otherwise, since $H_{j+1}(A) \subseteq H_j(A)$, we must have $H_{j+1}(A) = H_j(A)$. By [2, Lemma 2.3] or the arguments in the proof of Lemma 2.2, $A|H_{j+1}(A)$ is unitary. This implies that $H_{j+1}(A) \subseteq H_\infty(A) \cap H_\infty(A^*)$. Hence these two subspaces are equal and thus (a) also holds. Note that this can be used to give an alternative proof of $j(A) \leqslant n$ in Theorem 2.1.

## 3. The index $k(A)$

Our first result relates $k(A)$ to the norm-one index $k_A$ for a contraction $A$. Recall that if $A$ is a contraction, then its *norm-one index* $k_A$ is defined as $\sup\{k \geqslant 0: \|A^k\| = 1\}$ (cf. [2, p. 364]). It is easily seen that (1) $0 \leqslant k_A \leqslant \infty$, (2) $k_A = 0$ if and only if $\|A\| < 1$, and (3) $k_A = \infty$ if and only if its spectral radius $r(A)$ ($\equiv \max\{|\lambda|: \lambda \in \sigma(A)\}$) equals 1. In fact, it was shown in [2, Proposition 2.1(c) or Theorem 2.2] that, for an $n$-dimensional contraction $A$, $k_A$ is either infinity or less than or equal to $n - 1$.

**Theorem 3.1.** *If $A$ is a c.n.u. contraction, then $k(A) \leqslant \lceil (k_A + 1)/2 \rceil$. If, in addition, $A$ is compact, then the equality $k(A) = \lceil (k_A + 1)/2 \rceil$ holds.*

The proof depends on the following lemma relating $H_j(A) \cap H_k(A^*)$ and $H_{j+k}(A)$.

**Lemma 3.2.** *If $A$ is a contraction, then $A^{k*}$ maps $H_j(A) \cap H_k(A^*)$ onto $H_{j+k}(A)$ isometrically for any $j$, $0 \leqslant j \leqslant \infty$, and $k$, $0 \leqslant k < \infty$, and, in particular, $j(A) \leqslant 2k(A)$.*

**Proof.** We first check that $A^{k*}$ maps $H_j(A) \cap H_k(A^*)$ to $H_{j+k}(A)$ for $0 \leqslant j, k < \infty$. Indeed, if $x$ is in $H_j(A) \cap H_k(A^*)$, then

$$\|x\| \geqslant \|A^{k*}x\| \geqslant \|A^{j+k}A^{k*}x\| = \|A^j(A^k A^{k*}x)\| = \|A^j x\| = \|x\|, \tag{b}$$

where the last two equalities follow from the assumptions that $x \in H_k(A^*)$ and $x \in H_j(A)$, respectively. It follows that $\|A^{j+k}(A^{k*}x)\| = \|A^{k*}x\|$ and thus $A^{k*}x$ is in $H_{j+k}(A)$. (b) above also implies that $\|A^{k*}x\| = \|x\|$ and thus $A^{k*}: H_j(A) \cap H_k(A^*) \to H_{j+k}(A)$ is isometric. To show the surjectivity of the asserted map, let $y$ be in $H_{j+k}(A)$. Then

$$\|y\| \geqslant \|A^k y\| \geqslant \|A^j(A^k y)\| = \|y\|,$$

showing that $A^k y$ is in $H_j(A)$. We now prove that $x \equiv A^k y$ is also in $H_k(A^*)$. Indeed, this is so because

$$A^k A^{k*}x = A^k(A^{k*}A^k y) = A^k y = x,$$

where the second equality follows from $y \in H_{j+k}(A) \subseteq H_k(A)$. Finally, the equalities

$$A^{k*}x = A^{k*}A^k y = y$$

show that $A^{k*}$ maps $x$ in $H_j(A) \cap H_k(A^*)$ to $y$. This proves our assertion for finite $j$ and $k$. The case for $j = \infty$ can be easily shown to be true from above.

To prove $j(A) \leqslant 2k(A)$, we decompose $A$ as $U \oplus A'$, where $U$ is unitary and $A'$ is c.n.u. Since $j(A) = j(A')$ and $k(A) = k(A')$ by Proposition 1.3(8) and (5), we need only check $j(A') \leqslant 2k(A')$. Assuming $k = k(A') < \infty$, we have $H_k(A') \cap H_k(A'^*) = \{0\}$. Hence $H_{2k}(A') = \{0\}$ from above. Thus $j(A') \leqslant 2k = 2k(A')$ as desired. $\quad\square$

**Proof of Theorem 3.1.** Since $A$ is c.n.u., we have $H_\infty(A) \cap H_\infty(A^*) = \{0\}$. If $k(A) = \infty$, then $H_k(A) \cap H_k(A^*) \neq \{0\}$ for any $k$, $0 \leqslant k < \infty$. Thus $H_k(A) \neq \{0\}$ or $\|A^k\| = 1$ for all $k \geqslant 0$. It follows that $k(A) = \infty = \lceil (k_A + 1)/2 \rceil$.

Now assume that $k \equiv k(A)$ is finite (resp., $k \equiv k(A)$ is finite and $A$ is compact). We have $H_{k-1}(A) \cap H_{k-1}(A^*) \neq \{0\}$ and $H_k(A) \cap H_k(A^*) = \{0\}$. Apply Lemma 3.2 to infer that $H_{2k-2}(A) \neq \{0\}$ and $H_{2k}(A) = \{0\}$. Hence $\|A^{2k-2}\| = 1$ (resp., $\|A^{2k-2}\| = 1$ and $\|A^{2k}\| < 1$ because the compact $A^{2k}$ attains its norm). It follows that $k_A \geqslant 2k - 2$ (resp., $k_A = 2k - 1$ or $2k - 2$). Therefore, $k(A) \leqslant (k_A + 2)/2$ (resp., $k(A) = (k_A + 1)/2$ or $(k_A + 2)/2$). We conclude that $k(A) \leqslant \lceil (k_A + 1)/2 \rceil$ (resp., $k(A) = \lceil (k_A + 1)/2 \rceil$). $\quad\square$

Note that, in this theorem, the equality is in general false. One example is the c.n.u. normal contraction $A = \text{diag}(1/2, 2/3, \ldots, (n-1)/n, \ldots)$ on $l^2$. In this case, $k(A) = 1$ and $k_A = \infty$.

**Corollary 3.3.** *Let $A$ be a contraction on a finite-dimensional space. Then $k(A) = 1$ if and only if $A = U \oplus A'$, where $U$ is unitary, $A'$ is not missing, and $\|A'\| < 1$ or $\|A'\| = 1$ and $\|A'^2\| < 1$.*

This is an easy consequence of Proposition 1.3 and Theorem 3.1. It is comparable to the corresponding condition for $j(A) = 1$, namely, $A = U \oplus A'$, where $U$ is unitary, $A'$ is not missing and $\|A'\| < 1$ (cf. Proposition 1.3(6)).

The next theorem is the main result of this section.

**Theorem 3.4.** *If $A$ is a contraction on an $n$-dimensional space, then $k(A) \leqslant \lceil n/2 \rceil$. For an operator $A$ of class $\mathcal{S}_n$, we have the equality $k(A) = \lceil n/2 \rceil$.*

**Proof.** Let $A = U \oplus A'$, where $U$ is unitary and $A'$ is c.n.u. Then $k(A) = \max\{k(U), k(A')\} = k(A')$ by Proposition 1.3. Since $k(A') = \lceil (k_{A'} + 1)/2 \rceil$ by Theorem 3.1 and $k_{A'} \leqslant n - 1$ by [2, Proposition 2.1(c) or Theorem 2.2], we infer that $k(A) \leqslant \lceil ((n-1)+1)/2 \rceil = \lceil n/2 \rceil$ as asserted.

If $A$ is of class $\mathcal{S}_n$, then $k_A = n - 1$ by [2, Theorem 3.1] and hence $k(A) = \lceil ((n-1)+1)/2 \rceil = \lceil n/2 \rceil$ from above.  $\square$

We now give another proof for the first assertion in Theorem 3.4 which is more in line of the arguments in Section 2 and does not involve the norm-one index.

For the $n$-dimensional contraction $A$, let $k = \lceil n/2 \rceil$. We show that $A^{k*}$ maps $H_k(A) \cap H_k(A^*)$ onto itself isometrically. Indeed, if $x$ is any vector in $H_k(A) \cap H_k(A^*)$, then

$$\|x\| \geqslant \|A^{k*}x\| \geqslant \|A^n(A^{k*}x)\| = \|A^{n-k}(A^k A^{k*}x)\| = \|A^{n-k}x\| = \|x\|,$$

where the last two equalities follow from the assumptions that $x \in H_k(A^*)$ and $x \in H_k(A) \subseteq H_{n-k}(A)$, respectively. It follows that $\|A^n(A^{k*}x)\| = \|A^{k*}x\| = \|x\|$ and thus, in particular, $A^{k*}x$ is in $H_n(A)$. Hence

$$A^{k*}(H_k(A) \cap H_k(A^*)) \subseteq H_n(A) = H_\infty(A) \cap H_\infty(A^*) \subseteq H_k(A) \cap H_k(A^*), \tag{c}$$

where the equality in the middle is by Theorem 2.1. Therefore, $A^{k*}$ maps $H_k(A) \cap H_k(A^*)$ onto itself isometrically and thus we have equalities throughout (c). This yields $H_k(A) \cap H_k(A^*) = H_\infty(A) \cap H_\infty(A^*)$ and thus $k(A) \leqslant k = \lceil n/2 \rceil$.

The final result of this section is a characterization of $n$-dimensional contractions $A$ with $k(A) = \lceil n/2 \rceil$.

**Theorem 3.5.** *Let $A$ be a contraction on an $n$-dimensional space. Then $k(A) = \lceil n/2 \rceil$ if and only if one of the following holds*:

(1) *$A$ is of class $\mathcal{S}_n$,*
(2) *$n$ is even and $A$ is c.n.u. with $\|A^{n-2}\| = 1$ and $\|A^{n-1}\| < 1$, and*
(3) *$n$ is even and $A = U \oplus A'$, where $U$ is unitary on a one-dimensional space and $A'$ is of class $\mathcal{S}_{n-1}$.*

**Proof.** Assume that $k(A) = \lceil n/2 \rceil$. Let $A = U \oplus A'$, where $U$ is unitary on a space of dimension $n - m$ and $A'$ is c.n.u. on a space of dimension $m$, $0 \leqslant m \leqslant n$. Since

$$\lceil n/2 \rceil = k(A) = \max\{k(U), k(A')\} = k(A')$$
$$= \lceil (k_{A'} + 1)/2 \rceil \leqslant \lceil m/2 \rceil \leqslant \lceil n/2 \rceil,$$

where the first inequality is by [2, Proposition 2.1(c) or Theorem 2.2], we have equalities throughout. In particular, $\lceil (k_{A'} + 1)/2 \rceil = \lceil m/2 \rceil$ and $\lceil m/2 \rceil = \lceil n/2 \rceil$ hold. The latter implies that either $m = n$ or $n$ is even and $m = n - 1$:

  (i) If $n = m = 2k + 1$ is odd, then $\lceil (k_{A'} + 1)/2 \rceil = k + 1$, which implies that $k_{A'} = 2k$ or $2k + 1$. In the former case, we have $k_{A'} = n - 1$ and hence $A = A'$ is of class $\mathcal{S}_n$ by [2, Theorem 3.1]. For the latter, we have $k_{A'} = n$, which is impossible by [2, Proposition 2.1(c) or Theorem 2.2]. Thus, in this case, (1) holds.
 (ii) If $n = m = 2k$ is even, then $\lceil (k_{A'} + 1)/2 \rceil = k$, which implies that $k_{A'} = 2k - 1$ or $2k - 2$. For the former, we have $k_{A'} = n - 1$ and thus $A = A'$ is of class $\mathcal{S}_n$, that is, (1) holds. For the latter, $k_{A'} = n - 2$, which yields (2).
(iii) If $n = 2k$ is even and $m = n - 1$, then $\lceil (k_{A'} + 1)/2 \rceil = k$, and hence $k_{A'} = 2k - 1$ or $2k - 2$. The former says that $k_{A'} = m$, which is impossible by [2, Proposition 2.1(c) or Theorem 2.2]. The latter implies that $k_{A'} = m - 1$ and hence $A'$ is of class $\mathcal{S}_{n-1}$ by [2, Theorem 3.1]. Thus, in this case, (3) holds.

That (1) and (3) give $k(A) = \lceil n/2 \rceil$ is easily seen from Theorem 3.4 while (2) yields this equality is by Theorem 3.1.  $\square$

We conclude this paper by remarking that, analogous to the situation for $j(A)$ in Section 2, we can (1) slightly improve $k(A) \leqslant \lceil n/2 \rceil$ in Theorem 3.4 by proving that, *for a finite-dimensional contraction $A$, $k(A) \leqslant \lceil m/2 \rceil$, where $m$ is the degree of the minimal polynomial of $A$*, and (2) relate the dimensions of $H_{k+1}(A) \cap H_{k+1}(A^*)$ and $H_k(A) \cap H_k(A^*)$ as

$$\dim(H_{k+1}(A) \cap H_{k+1}(A^*)) \leqslant \max\{\dim(H_k(A) \cap H_k(A^*)) - 2, \dim(H_\infty(A) \cap H_\infty(A^*))\}$$

*for a finite-dimensional contraction $A$ and $k \geqslant 0$.* To prove (1), we let $l = \lceil m/2 \rceil$. Then $2l \geqslant m$ and hence $H_{2l}(A) \subseteq H_m(A)$. Since $H_m(A) = H_\infty(A) \cap H_\infty(A^*)$ as was proved in Section 2, we have $H_{2l}(A) = H_\infty(A) \cap H_\infty(A^*)$. Thus $H_l(A) \cap H_l(A^*) = H_\infty(A) \cap H_\infty(A^*)$ by Lemma 3.2. Therefore, $k(A) \leqslant l = \lceil m/2 \rceil$. For the proof of (2), note that $\dim(H_k(A) \cap H_k(A^*)) = \dim H_{2k}(A)$ for

all $k \geqslant 0$ by Lemma 3.2. Hence

$$
\begin{aligned}
\dim\big(H_{k+1}(A) \cap H_{k+1}\big(A^*\big)\big) \\
&= \dim H_{2k+2}(A) \\
&\leqslant \max\big\{\dim H_{2k+1}(A) - 1, \dim\big(H_\infty(A) \cap H_\infty\big(A^*\big)\big)\big\} \quad \text{(by (a))} \\
&\leqslant \max\big\{\max\big\{\dim H_{2k}(A) - 1, \dim\big(H_\infty(A) \cap H_\infty\big(A^*\big)\big)\big\} - 1, \dim\big(H_\infty(A) \cap H_\infty\big(A^*\big)\big)\big\} \quad \text{(by (a))} \\
&= \max\big\{\dim H_{2k}(A) - 2, \dim\big(H_\infty(A) \cap H_\infty\big(A^*\big)\big)\big\} \\
&= \max\big\{\dim\big(H_k(A) \cap H_k\big(A^*\big)\big) - 2, \dim\big(H_\infty(A) \cap H_\infty\big(A^*\big)\big)\big\}.
\end{aligned}
$$

As before, this can be used to give an alternative proof of $k(A) \leqslant \lceil n/2 \rceil$ in Theorem 3.4.

## Acknowledgments

## References

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