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ORIGINAL PAPER

Complete identification of chaos of nonlinear nonholonomic systems

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Abstract The chaos of nonholonomic systems with two external nonlinear nonholonomic constraints where the magnitude of velocity is a constant and the magnitude of the velocity is a constant with a periodic disturbance, respectively, is completely identified for the first time. The scope of the chaos study is extended to nonlinear nonholonomic systems. By applying the nonlinear nonholonomic form of Lagrange's equations, the dynamic equation is expressed. The existence of chaos in these two nonlinear nonholonomic systems is first wholly proved by all numerical criteria of chaos, i.e., the most reliable Lyapunov exponents, phase portraits, Poincaré maps, and bifurcation diagrams. Furthermore, it is found that the Feigenbaum number still holds for nonlinear nonholonomic systems.

Keywords Chaos · Nonlinear nonholonomic system

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1 Introduction

The study of the nonholonomic system [1, 2] has been developed over one hundred years since Hertz [3] distinguished the nonholonomic system from the holonomic system in 1894. There are a great number of studies in this field connected with the extension of the developed analytical methods for the holonomic system and for the systems with nonholonomic constraints. Many applications of the dynamics of the nonholonomic system can be found in the problems of modern technology, such as the pursuit problems, the motion of automobiles, landing devices of airplanes, railway wheels, etc. However, it is still deficient for the complete study of chaos in nonholonomic systems. As far as we know, the only paper which studies the chaos of the nonholonomic system with an external linear holonomic constraint is [4]; the chaotic phenomena of rattleback dynamics. But only Poincaré maps are given in this paper. It is well known that only the Poincaré map cannot identify the existence of chaos completely.

By applying the nonlinear nonholonomic form of Lagrange's equations, chaos of nonholonomic systems with external nonlinear nonholonomic constraint for two types of problems, the magnitude of velocity is a constant, and the magnitude of velocity is a constant with a periodic disturbance, is studied in this paper. The existence of chaos is first completely identified by all numerical criteria of chaos, i.e., the most reliable Lyapunov exponents [5], phase portraits, Poincaré maps, and bifurcation diagrams. Furthermore, it is found that the Feigenbaum number still holds for nonlinear nonholonomic systems. The structure of this paper is as follows. The chaos of the nonlinear nonholonomic problem where the magnitude of velocity is a constant is studied in Sect. 2. In Sect. 3, the chaos of the nonlinear nonholonomic problem where the magnitude of velocity is a constant with a periodic disturbance is studied. Finally, conclusions are drawn in Sect. 4.

2 The magnitude of velocity is a constant

From Newton's law, the dynamic equations of a free particle with unit mass moving in a horizontal smooth (x_1, y_1) plane are

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1, \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1,
\end{aligned} \tag{1}$$

where $-a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1$ and $-(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1$ are x_1 and y_1 components of the forces applied on the particle, respectively, and a, b, ω are constants. Equation (1) consists of two linearly coupled nonlinear Mathieu systems.

Now an external nonlinear nonholonomic constraint

$$\dot{x}_1^2 + \dot{y}_1^2 = c \tag{2}$$

is added; the particle is not free anymore but a constrained particle. c is a constant, i.e., the constraint makes the magnitude of the velocity constant. Since the constraint is nonlinear nonholonomic, the particle becomes a nonlinear nonholonomic system, of which the dynamic equations can be obtained as follows.

From the nonholonomic constraint equations,

$$f_j(q_i, \dot{q}_i, t) = 0 \quad (j = 1, \ldots, m; i = 1, \ldots, n), \tag{3}$$

the constraint conditions of the virtual velocities can be derived:

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial \dot{q}_i} \delta \dot{q}_i = 0 \quad (j = 1, \ldots, m). \tag{4}$$

According to Jourdain's principle [6]

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i \right] \delta \dot{q}_i = 0 \tag{5}$$

and the constraint conditions of the virtual velocities, Lagrange multiplier method is used to obtain the nonlinear nonholonomic form of Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial \dot{q}_i} \quad (i = 1, \ldots, n), \tag{6}$$

where q_i is the generalized coordinate, T is the kinetic energy of the system, Q_i is the generalized applied force, λ_j is the Lagrange multiplier, and f_j is the nonholonomic constraint function in (3). Together with m nonholonomic constraint equations, there are $(m+n)$ equations solving for n generalized coordinates and m Lagrange multipliers.

In our case, choose $(q_1, q_2) = (x_1, y_1)$. The kinetic energy is $T = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2)$, the generalized applied forces are $Q_1 = -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - a\dot{x}_1 + by_1$, $Q_2 = -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - a\dot{y}_1 + bx_1$, and the nonholonomic constraint equation is $f = \dot{x}_1^2 + \dot{y}_1^2 - c$. Then the nonlinear nonholonomic form of Lagrange's equations is obtained:

$$\begin{aligned}
\ddot{x}_1 = &-a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 \\
&- a\dot{x}_1 + by_1 + 2\lambda\dot{x}_1,
\end{aligned} \tag{7}$$

$$\begin{aligned}
\ddot{y}_1 = &-(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 \\
&- a\dot{y}_1 + bx_1 + 2\lambda\dot{y}_1.
\end{aligned} \tag{8}$$

Together with nonholonomic constraint equation

$$\dot{x}_1^2 + \dot{y}_1^2 = c, \tag{9}$$

there are three equations solving two generalized coordinates and one Lagrange multiplier. In order to solve λ, differentiate the nonholonomic constraint equation (9) with respect to time and get

$$\dot{x}_1\ddot{x}_1 + \dot{y}_1\ddot{y}_1 = 0. \tag{10}$$

Substitute (7) and (8) into (10), and λ can be solved:

$$\begin{aligned}
\lambda = \frac{a}{2} + \big\{ &\big[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1\big]\dot{x}_1 \\
&+ \big[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 \\
&- bx_1\big]\dot{y}_1 \big\}/2c.
\end{aligned} \tag{11}$$

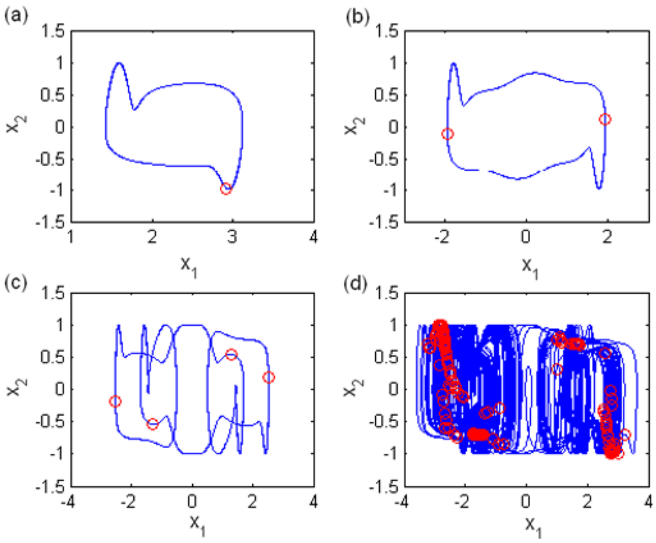

Fig. 1 Phase portraits and Poincaré maps for nonlinear nonholonomic system where the magnitude of velocity is a constant: (**a**) period 1 for $b = 5.8$, (**b**) period 2 for $b = 2.5$, (**c**) period 4 for $b = 4.1$, (**d**) chaotic for $b = 5.3$

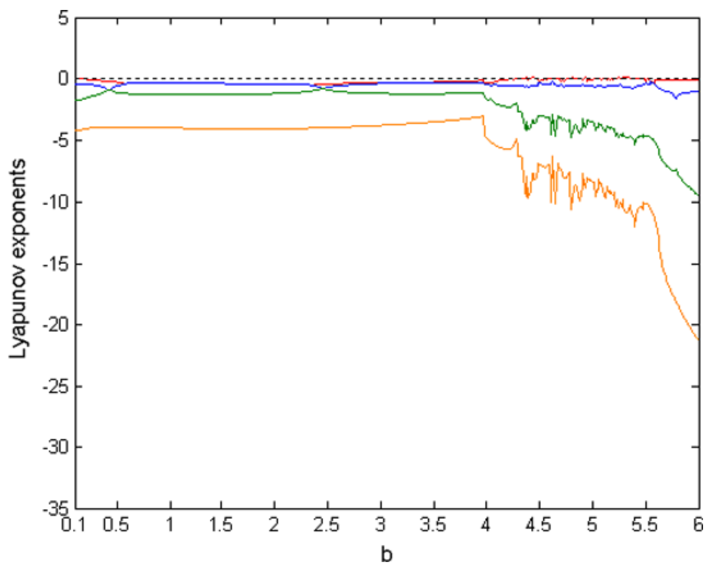

Fig. 2 Lyapunov exponents for nonlinear nonholonomic system where the magnitude of velocity is a constant

Finally, the differential equations of nonholonomic system can be expressed as

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3\\
&\quad + by_1 + \frac{h_1(x_1,x_2,y_1,y_2,t)}{c},\\
\dot{y}_1 &= y_2,\\
\dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3\\
&\quad + bx_1 + \frac{h_2(x_1,x_2,y_1,y_2,t)}{c},
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
h_1 &= \big[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1\big]x_2^2\\
&\quad + \big[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3\\
&\quad - bx_1\big]x_2y_2,\\
h_2 &= \big[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1\big]x_2y_2\\
&\quad + \big[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1\big]y_2^2.
\end{aligned} \tag{13}$$

The parameters in simulation are $a = 0.5$, $b =$ 0.1–6, $c = 1$, $\omega = 1$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $y_1(0) = 0.1$, $y_2(0) = \sqrt{0.99}$.

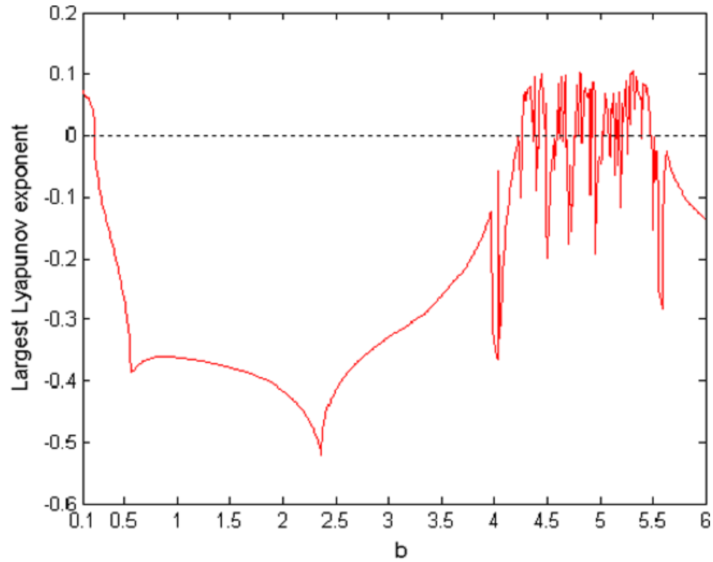

Fig. 3 Largest Lyapunov exponent for nonlinear nonholonomic system where the magnitude of velocity is a constant

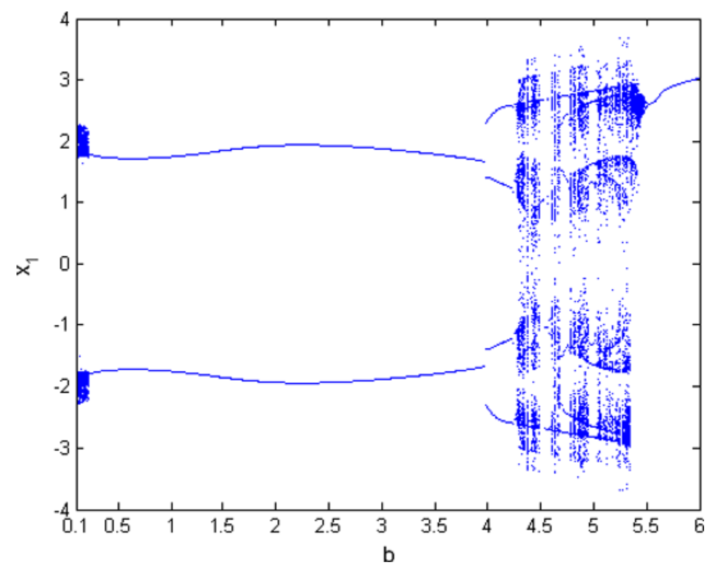

Fig. 4 Bifurcation diagram for nonlinear nonholonomic system where the magnitude of velocity is a constant

The phase portraits, Poincaré maps, Lyapunov exponents, and bifurcation diagram for nonholonomic system are shown in Figs. 1, 2, 3, 4. It can be observed that the motion is period 1 for $b = 5.8$, period 2 for $b = 2.5$, and period 4 for $b = 4.1$. For $b = 5.3$, the motion is chaotic. All numerical criteria of chaos prove that the chaotic phenomena exist in nonlinear nonholonomic system where the magnitude of velocity of the particle is a constant.

From bifurcation diagram, Fig. 4, it shows that the period-doubling phenomenon occurs from $b = 6$ to $b = 5$. Take enlargement of Fig. 4; we can clearly observe the period-doubling phenomenon as shown in Fig. 5. Then the Feigenbaum number [7] can be calculated. Feigenbaum number δ is defined as

$$\delta = \lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}, \tag{14}$$

where μ_k is the kth bifurcation point. The results are shown in Table 1; it can be found that the Feigenbaum number approaches the universal number $\delta = 4.6692016091029909\ldots$. This means that the Feigenbaum number still holds for nonlinear nonholonomic system where the magnitude of velocity is a constant.

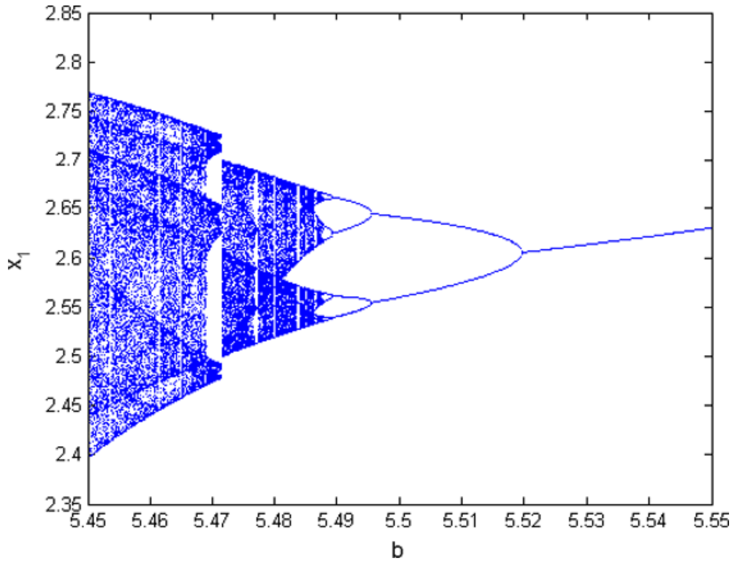

Fig. 5 Period-doubling for nonlinear nonholonomic system where the magnitude of velocity is a constant

Table 1 Calculation of Feigenbaum number for system (12)

Period doubling	Bifurcation point	Feigenbaum number
Period 1 to Period 2	5.519779	–
Period 2 to Period 4	5.495639	3.8986
Period 4 to Period 8	5.489447	4.5263
Period 8 to Period 16	5.488079	4.5600
Period 16 to Period 32	5.487779	4.6154
Period 32 to Period 64	5.487714	–

3 The magnitude of velocity is a constant with a periodic disturbance

From Newton's law, the dynamic equations of a free particle with unit mass moving in a horizontal smooth (x_1, y_1) plane are

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1,\\
\dot{y}_1 &= y_2,\\
\dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1,
\end{aligned} \tag{15}$$

where $-a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - ax_2 + by_1$ and $-(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - ay_2 + bx_1$ are x_1 and y_1 components of the forces applied on the particle, respectively, and a, b, ω are constants. Equation (15) consists of two linearly coupled nonlinear Mathieu systems.

Now an external nonlinear nonholonomic constraint

$$\dot{x}_1^2 + \dot{y}_1^2 = c + d\sin t \tag{16}$$

is added, the particle is not free anymore, but a constrained particle. c and d are constants, i.e., the constraint makes the magnitude of the velocity of the particle constant with a periodic function of time. Since the constraint is nonlinear nonholonomic, the particle becomes a nonlinear nonholonomic system, of which the dynamic equations can be obtained as follows.

According to Jourdain's principle [6] and the constraint conditions of the virtual velocities, we can use Lagrange multiplier method to obtain the nonlinear nonholonomic form of Lagrange's equations.

In our case, choose $(q_1, q_2) = (x_1, y_1)$. The kinetic energy is $T = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2)$, the generalized applied forces are $Q_1 = -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - a\dot{x}_1 + by_1$, $Q_2 = -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - a\dot{y}_1 + bx_1$, and the nonholonomic constraint function is $f = \dot{x}_1^2 + \dot{y}_1^2 - c - d\sin t$. Then the nonlinear nonholonomic form of Lagrange's equations is obtained:

$$\begin{aligned}
\ddot{x}_1 = {} & -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 - a\dot{x}_1\\
& + by_1 + 2\lambda\dot{x}_1,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\ddot{y}_1 = {} & -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 - a\dot{y}_1\\
& + bx_1 + 2\lambda\dot{y}_1.
\end{aligned} \tag{18}$$

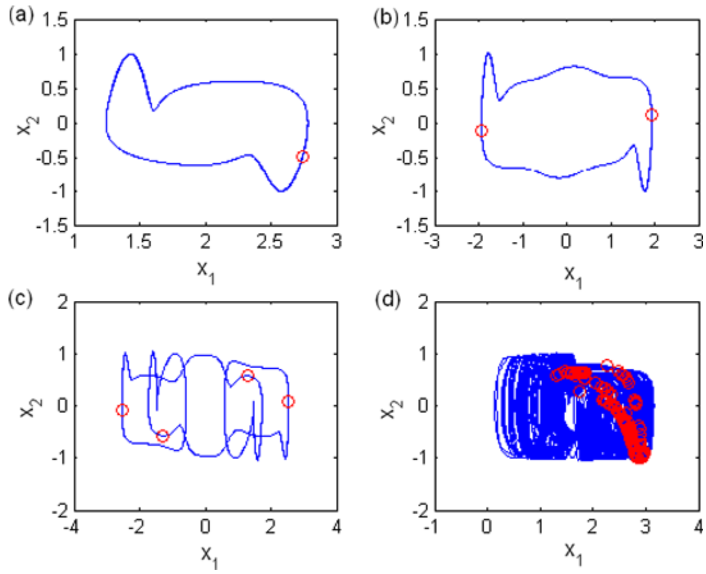

Fig. 6 Phase portraits and Poincaré maps for nonlinear nonholonomic system where the magnitude of velocity is a constant with a periodic disturbance: (**a**) period 1 for $b = 5.9$, (**b**) period 2 for $b = 2.5$, (**c**) period 4 for $b = 4.2$, (**d**) chaotic for $b = 5.6$

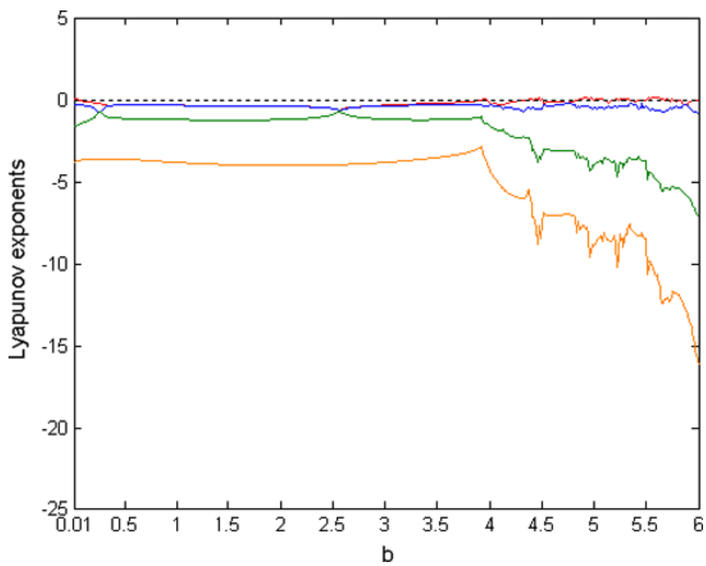

Fig. 7 Lyapunov exponents for nonlinear nonholonomic system where the magnitude of velocity is a constant with a periodic disturbance

Together with nonholonomic constraint equation

$$\dot{x}_1^2 + \dot{y}_1^2 = c + d \sin t, \tag{19}$$

there are three equations solving two generalized coordinates and one Lagrange multiplier.

In order to solve λ, differentiate the nonholonomic constraint equation (19) with respect to time and get

$$\dot{x}_1 \ddot{x}_1 + \dot{y}_1 \ddot{y}_1 = \frac{d \cos t}{2}. \tag{20}$$

Substitute (17) and (18) into (20), and λ can be solved:

$$\begin{aligned} \lambda = \frac{a}{2} + \Big\{ & \big[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 \\ & - b y_1 \big] \dot{x}_1 \\ & + \big[(1+\sin\omega t) y_1 + a(1+\sin\omega t) y_1^3 - b x_1 \big] \dot{y}_1 \\ & - \frac{d \cos t}{2} \Big\} / 2(c + d \sin t). \end{aligned} \tag{21}$$

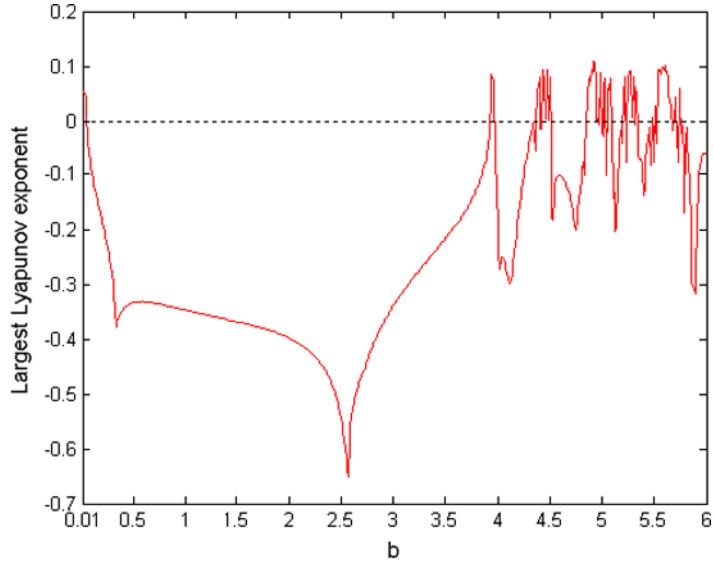

Fig. 8 Largest Lyapunov exponent for nonlinear nonholonomic system where the magnitude of velocity is a constant with a periodic disturbance

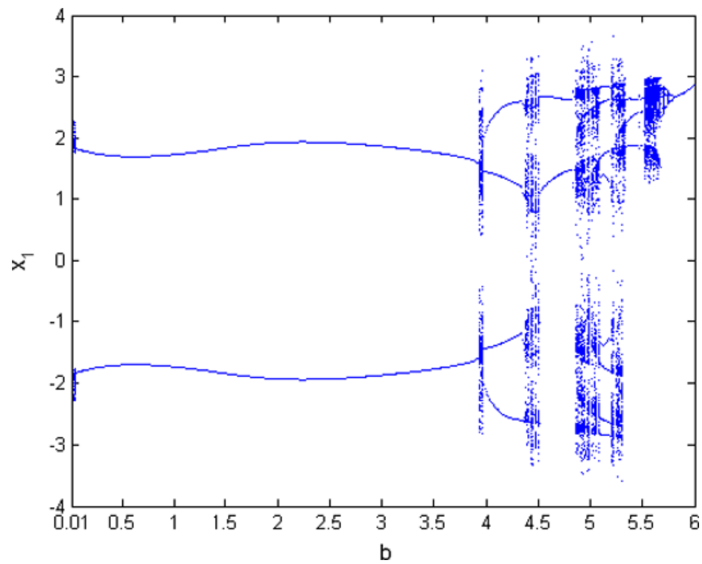

Fig. 9 Bifurcation diagram for nonlinear nonholonomic system where the magnitude of velocity is a constant with a periodic disturbance

Finally, the differential equations of nonholonomic system can be expressed as

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a(1+\sin\omega t)x_1 - (1+\sin\omega t)x_1^3 \\
&\quad + by_1 + \frac{h_1(x_1, x_2, y_1, y_2, t)}{c + d\sin t}, \\
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -(1+\sin\omega t)y_1 - a(1+\sin\omega t)y_1^3 \\
&\quad + bx_1 + \frac{h_2(x_1, x_2, y_1, y_2, t)}{c + d\sin t},
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
h_1 &= \left[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1\right]x_2^2 \\
&\quad + \left[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1\right]x_2 y_2 \\
&\quad + \frac{d\cos t}{2}x_2, \\
h_2 &= \left[a(1+\sin\omega t)x_1 + (1+\sin\omega t)x_1^3 - by_1\right]x_2 y_2 \\
&\quad + \left[(1+\sin\omega t)y_1 + a(1+\sin\omega t)y_1^3 - bx_1\right]y_2^2 \\
&\quad + \frac{d\cos t}{2}y_2.
\end{aligned} \tag{23}$$

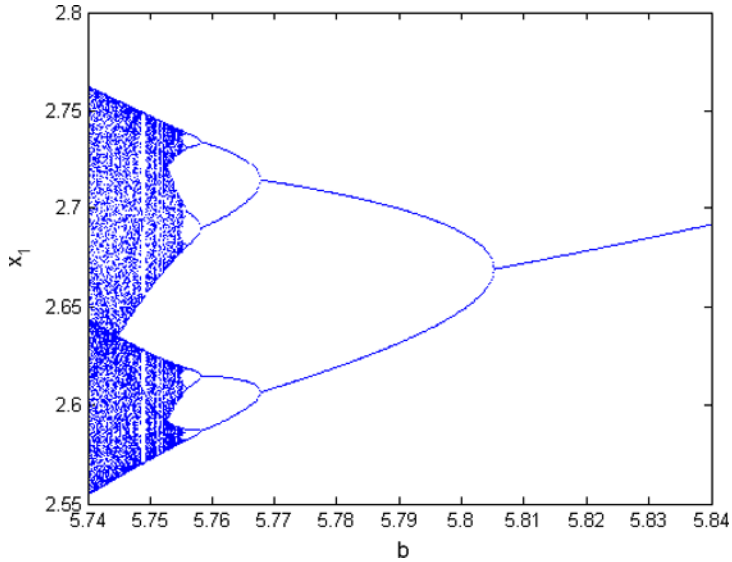

Fig. 10 Period-doubling for nonlinear nonholonomic system where the magnitude of velocity is a constant with a periodic disturbance

The parameters in simulation are $a = 0.5$, $b = 0.01$–6, $c = 1$, $d = 0.1$, $\omega = 1$, and the initial condition is $x_1(0) = 0.1$, $x_2(0) = 0.1$, $y_1(0) = 0.1$, $y_2(0) = \sqrt{0.99}$. The phase portraits, Poincaré maps, Lyapunov exponents, and bifurcation diagram for nonholonomic system are shown in Figs. 6, 7, 8, 9. It can be observed that the motion is period 1 for $b = 5.9$, period 2 for $b = 2.5$, and period 4 for $b = 4.2$. For $b = 5.6$, the motion is chaotic. All numerical criteria of chaos prove that the chaotic phenomena exist in nonlinear nonholonomic system for which the magnitude of velocity of the particle is a constant with a periodic disturbance.

From bifurcation diagram, Fig. 9, it shows that the period-doubling phenomenon occurs from $b = 6$ to $b = 5.5$. Take enlargement of Fig. 9, we can clearly observe the period-doubling phenomenon as shown in Fig. 10. Then the Feigenbaum number [7] can be calculated. The results are shown in Table 2, it can be found that the Feigenbaum number approaches the universal number $\delta = 4.6692016091029909\ldots$. This means that the Feigenbaum number still holds for nonlinear nonholonomic system where the magnitude of velocity is a constant with a periodic disturbance.

Table 2 Calculation of Feigenbaum number for system (22)

Period doubling	Bifurcation point	Feigenbaum number
Period 1 to Period 2	5.805459	–
Period 2 to Period 4	5.768012	3.7530
Period 4 to Period 8	5.758034	4.4946
Period 8 to Period 16	5.755814	4.6639
Period 16 to Period 32	5.755338	4.6667
Period 32 to Period 64	5.755236	–

4 Conclusions

Chaos of nonlinear nonholonomic systems is first completely identified in this paper. We extend the scope of chaos study to nonlinear nonholonomic systems. In this paper, chaos of two types of nonlinear nonholonomic problems, the magnitude of velocity is a constant, and the magnitude of velocity is a constant with a periodic disturbance, is studied by applying the nonlinear nonholonomic form of Lagrange's equations. Complete identification of chaotic phenomena is obtained in nonlinear nonholonomic systems by Lyapunov exponents, phase portraits, Poincaré maps, and bifurcation diagrams. Furthermore, the Feigenbaum number still holds for two nonlinear nonholonomic systems studied.

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References

1. Mei, F.: Nonholonomic mechanics. Appl. Mech. Rev. **53**, 283–305 (2000)
2. Kolmanovsky, I., McClamroch, N.H.: Developments in nonholonomic control problems. IEEE Control Syst. Mag. **15**, 20–36 (1995)
3. Hertz, H.R.: Die Prinzipien der Mechanik in neuem Zusammenhange dargestellt. Barth, Leipzig (1894)
4. Borisov, A.V., Mamaev, I.S.: Strange attractors in rattleback dynamics. Phys. Usp. **46**, 393–403 (2003)
5. Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos, 2nd edn. Springer, New York (2003)
6. Greenwood, D.T.: Advanced Dynamics, p. 116. Cambridge University Press, Cambridge (2003)
7. Feigenbaum, M.J.: Quantitative universality for a class of nonlinear transformations. J. Stat. Phys. **19**, 25–32 (1978)