Chapter 5

Conclusions and Future Works

5.1 Conclusions

We apply cellular neural networks for seismic horizon linking and pattern recognition. For horizon linking, we establish the energy function by setting four different constraints of peak distribution. Then we compare this energy function and the standard energy function of a cellular neural network, and then we can get the weightings A(i, j; k, l). Then we use this network which is trained to deal with seismic horizon linking. Experimental results show that the extracted horizons are the same with the results of our visual observation.

For pattern recognition, we design discrete-time cellular neural networks to behave as associative memories first. Discrete-time cellular neural networks work as associative memories when the memory patterns correspond to asymptotically stable equilibrium points of the network dynamics. And then we use the associative memories to recognize patterns. Seismic pattern recognition will help the analysis and interpretation of seismic data.

5.2 Future Works

For seismic horizon linking, we can link the broken horizons which have the short broken parts now. The broken part with length 2 can be linked by our algorithm now. In the future work, we can add some other constraints to link the broken horizons which have the longer broken parts.

In the future work, we may use the high-order CNN to design associative memories with better performance. High order neural networks for associative memories with a greater storage capability with respect to the first-order ones have been introduced [1]-[6]. M. Brucoli, L. Carnimeo and G. Grassi proposed a design method for associative memories using discrete-time high-order neural networks which includes local interconnections among neurons [7]. Simulation results show that the performance of [7] is quite better than the first-order cellular neural associative memories [8]. We perhaps can apply the high-order structure to the synthesis procedure of cellular neural networks for associative memories. A discrete-time second-order CNN is considered.

$$x_i(h+1) = \sum_{j=1}^{n} a_{ij} y_j(h) + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} g_{ijk} y_j(h) y_k(h) + \sum_{j=1}^{n} b_{ij} u_j + I_i \qquad \text{(5-1a)}$$

$$y_i(h) = f(x_i(h)) = \frac{1}{2}\left(\left|x_i(h)+1\right| - \left|x_i(h)-1\right|\right) \quad i = 1, \cdots, n \qquad \text{(5-1b)}$$

Equation (5-1) can be rewritten in vector form as

$$\mathbf{x}(h+1) = \mathbf{A}\,\mathbf{y}(h) + \mathbf{G}\,\mathbf{z}(h) + \mathbf{B}\,\mathbf{u} + \mathbf{e} \qquad \text{(5-2a)}$$

$$\mathbf{y}(h) = \mathbf{f}(\mathbf{x}(h)) \qquad \text{(5-2b)}$$

where $\mathbf{x} = [x_1\ x_2 \cdots x_n]^{\mathrm{T}} \in \Re^{n\times 1}$; $\mathbf{y} = [y_1\ y_2 \cdots y_n]^{\mathrm{T}} \in \Re^{n\times 1}$; $\mathbf{u} = [u_1\ u_2 \cdots u_n]^{\mathrm{T}} \in \Re^{n\times 1}$; $\mathbf{e} = [I_1\ I_2 \cdots I_n]^{\mathrm{T}} \in \Re^{n\times 1}$; $\mathbf{f}(\mathbf{x}) = [f(x_1)\ f(x_2) \cdots f(x_n)]^{\mathrm{T}} \in \Re^{n\times 1}$ and $\mathbf{z} = [y_1 y_2\ y_1 y_3 \cdots y_1 y_n\ y_2 y_3 \cdots y_2 y_n \cdots y_{n\text{-}1} y_n]^{\mathrm{T}} \in \Re^{p}$.

In order to design **G** as a circulant matrix, we must rearrange the elements of **z**. According to linear neighboring (1-D template), Equation (5-1a) can be rewritten as

$$x_i(h+1) = \sum_{j=1}^{n} a_{ij} y_j(h) + \sum_{(j,k)\in NS_i} g_{ijk} y_j(h) y_k(h) + \sum_{j=1}^{n} b_{ij} u_j + I_i$$

where $NS_i = \left\{(j,k) \middle| C_j \in N_r(i) \text{ and } C_k \in N_r(i)\right\}$, C_j is the jth cell, $N_r(i)$ is the r-neighborhood of the cell C_i.

G can be decomposed into 2r circulant matrices. For r = 1, **G** can be decomposed into 2 circulant matrices. Suppose that these two circulant matrices are $\mathbf{G}_1$ and $\mathbf{G}_2$. We can design $\mathbf{G}_1$ and $\mathbf{G}_2$ as the following.

Index of cell	Indexes of neighboring cells	Pairs of indexes of neighboring cells
1	1, 2, *n*	(1, 2), (1, *n*), (2, *n*)
2	1, 2, 3	(1, 2), (1, 3), (2, 3)
3	2, 3, 4	(2, 3), (2, 4), (3, 4)
n	1, *n*-1, *n*	(1, *n*-1), (1, *n*), (*n*-1, *n*)

We can design $\mathbf{G}_1$ consist of the coefficients corresponding to the following pairs:
(1, 2)→(2, 3)→(3, 4)→⋯→(*n*-1, *n*)→(1, *n*)→(1, 2)
$\mathbf{G}_2$ consists of the coefficients corresponding to the following pairs:
(1, 3)→(2, 4)→(3, 5)→⋯→(*n*-2, *n*)→(1, *n*-1)→(2, *n*)→(1, 3)
$\mathbf{z}_1$ multiples $\mathbf{G}_1$, so $\mathbf{z}_1 = [y_1y_2\ y_2y_3\ y_3y_4 \cdots y_{n-1}y_n\ y_1y_n]$
$\mathbf{z}_2$ multiples $\mathbf{G}_2$, so $\mathbf{z}_2 = [y_1y_3\ y_2y_4\ y_3y_5 \cdots y_{n-2}y_n\ y_1y_{n-1}\ y_2y_n]$

$\mathbf{G}_1 \cdot \mathbf{z}_1 =$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} (1,2) & (2,3) & (3,4) & \cdots & (n-1,n) & (1,n) \end{array} \\ \begin{bmatrix} g_1(1) & 0 & 0 & \cdots & 0 & g_1(2) \\ g_1(2) & g_1(1) & 0 & \cdots & 0 & 0 \\ \vdots & & \cdots & & & \vdots \\ 0 & 0 & 0 & \cdots & g_1(2) & g_1(1) \end{bmatrix} \end{array} \cdot \begin{bmatrix} y_1y_2 \\ y_2y_3 \\ \vdots \\ y_1y_n \end{bmatrix}$$

$\mathbf{G}_1 \in \Re^{n \times n}$

$\mathbf{G}_2 \cdot \mathbf{z}_2 =$

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccccc} (1,3) & (2,4) & (3,5) & \cdots & (n-2,n) & (1,n-1) & (2,n) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & g_2(1) \\ g_2(1) & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & \cdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & g_2(1) & 0 \end{bmatrix} \end{array} \cdot \begin{bmatrix} y_1y_3 \\ y_2y_4 \\ \vdots \\ y_2y_n \end{bmatrix}$$

$\mathbf{G}_2 \in \Re^{n \times n}$

So Equation (5-2a) can be rewritten as

$$\mathbf{x}(h+1) = \mathbf{A}\mathbf{y}(h) + \mathbf{G}_1\mathbf{z}_1(h) + \mathbf{G}_2\mathbf{z}_2(h) + \mathbf{B}\mathbf{u} + \mathbf{e}$$

For $r = 2$, we design $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{G}_3$ and $\mathbf{G}_4$ as circulant matrices, then

Index of cell	Indexes of neighboring cells	Pairs of indexes of neighboring cells
1	1, 2, 3, *n*-1, *n*	(1, 2), (1, 3), (1, *n*-1), (1, *n*), (2, 3), (2, *n*-1), (2, *n*), (3, *n*-1), (3, *n*), (*n*-1, *n*)
2	1, 2, 3, 4, *n*	(1, 2), (1, 3), (1, 4), (1, *n*), (2, 3), (2, 4), (2, *n*), (3, 4), (3, *n*), (4, *n*)
3	1, 2, 3, 4, 5	(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)
n	1, 2, *n*-2, *n*-1, *n*	(1, 2), (1, *n*-2), (1, *n*-1), (1, *n*), (2, *n*-2), (2, *n*-1), (2, *n*), (*n*-2, *n*-1), (*n*-2, *n*), (*n*-1, *n*)

We can design $\mathbf{G}_1$ consist of the coefficients corresponding to the following pairs:
(1, 2)→(2, 3)→(3, 4)→⋯→(*n*-1, *n*)→(1, *n*)→(1, 2)

$\mathbf{G}_2$ consists of the coefficients corresponding to the following pairs:
(1, 3)→(2, 4)→(3, 5)→⋯→(*n*-2, *n*)→(1, *n*-1)→(2, *n*)→(1, 3)

$\mathbf{G}_3$ consists of the coefficients corresponding to the following pairs:
(1, 4)→(2, 5)→(3, 6)→⋯→(*n*-3, *n*)→(1, *n*-2)→(2, *n*-1)→(3, *n*)→(1, 4)

$\mathbf{G}_4$ consists of the coefficients corresponding to the following pairs:
(1, 5)→(2, 6)→(3, 7)→⋯→(*n*-4, *n*)→(1, *n*-3)→(2, *n*-2)→(3, *n*-1)→(4, *n*)→(1, 5)

$\mathbf{z}_1$ multiples $\mathbf{G}_1$, so $\mathbf{z}_1 = [y_1y_2 \ y_2y_3 \ y_3y_4 \cdots y_{n-1}y_n \ y_1y_n]$
$\mathbf{z}_2$ multiples $\mathbf{G}_2$, so $\mathbf{z}_2 = [y_1y_3 \ y_2y_4 \ y_3y_5 \cdots y_{n-2}y_n \ y_1y_{n-1} \ y_2y_n]$
$\mathbf{z}_3$ multiples $\mathbf{G}_3$, so $\mathbf{z}_3 = [y_1y_4 \ y_2y_5 \ y_3y_6 \cdots y_{n-3}y_n \ y_1y_{n-2} \ y_2y_{n-1} \ y_3y_n]$
$\mathbf{z}_4$ multiples $\mathbf{G}_4$, so $\mathbf{z}_4 = [y_1y_5 \ y_2y_6 \ y_3y_7 \cdots y_{n-4}y_n \ y_1y_{n-3} \ y_2y_{n-2} \ y_3y_{n-1} \ y_4y_n]$

$$\mathbf{G}_1 \cdot \mathbf{z}_1 =$$

$$\begin{array}{cc}
 & \begin{array}{ccccccc} (1,2) & (2,3) & (3,4) & \cdots & (n-2,n-1) & (n-1,n) & (1,n) \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{bmatrix} g_1(1) & g_1(3) & 0 & \cdots & 0 & g_1(4) & g_1(2) \\ g_1(2) & g_1(1) & g_1(3) & \cdots & 0 & 0 & g_1(4) \\ \vdots & & \cdots & & & & \vdots \\ g_1(3) & 0 & 0 & \cdots & g_1(4) & g_1(2) & g_1(1) \end{bmatrix}
\end{array} \cdot \begin{bmatrix} y_1 y_2 \\ y_2 y_3 \\ \vdots \\ y_1 y_n \end{bmatrix}$$

$\mathbf{G}_1 \in \Re^{n \times n}$

$$\mathbf{G}_2 \cdot \mathbf{z}_2 =$$

$$\begin{array}{cc}
 & \begin{array}{ccccccc} (1,3) & (2,4) & (3,5) & \cdots & (n-2,n) & (1,n-1) & (2,n) \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{bmatrix} g_2(1) & 0 & 0 & \cdots & 0 & g_2(2) & g_2(3) \\ g_2(3) & g_2(1) & 0 & \cdots & 0 & 0 & g_2(2) \\ \vdots & & \cdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & g_2(2) & g_2(3) & g_2(1) \end{bmatrix}
\end{array} \cdot \begin{bmatrix} y_1 y_3 \\ y_2 y_4 \\ \vdots \\ y_2 y_n \end{bmatrix}$$

$\mathbf{G}_2 \in \Re^{n \times n}$

$$\mathbf{G}_3 \cdot \mathbf{z}_3 =$$

$$\begin{array}{cc}
 & \begin{array}{ccccccc} (1,4) & (2,5) & (3,6) & \cdots & (1,n-2) & (2,n-1) & (3,n) \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & g_3(1) & g_3(2) \\ g_3(2) & 0 & 0 & \cdots & 0 & 0 & g_3(1) \\ \vdots & & \cdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & g_3(1) & g_3(2) & 0 \end{bmatrix}
\end{array} \cdot \begin{bmatrix} y_1 y_4 \\ y_2 y_5 \\ \vdots \\ y_3 y_n \end{bmatrix}$$

$\mathbf{G}_3 \in \Re^{n \times n}$

$$\mathbf{G}_4 \cdot \mathbf{z}_4 =$$

$$\begin{array}{cc}
 & \begin{array}{ccccccc} (1,5) & (2,6) & (3,7) & \cdots & (2,n-2) & (3,n-1) & (4,n) \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & g_4(1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & g_4(1) \\ \vdots & & \cdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & g_4(1) & 0 & 0 \end{bmatrix}
\end{array} \cdot \begin{bmatrix} y_1 y_5 \\ y_2 y_6 \\ \vdots \\ y_4 y_n \end{bmatrix}$$

$\mathbf{G}_4 \in \Re^{n \times n}$

So Equation (5-2a) can be rewritten as

$$\mathbf{x}(h+1) = \mathbf{A}\mathbf{y}(h) + \mathbf{G}_1\mathbf{z}_1(h) + \mathbf{G}_2\mathbf{z}_2(h) + \mathbf{G}_3\mathbf{z}_3(h) + \mathbf{G}_4\mathbf{z}_4(h) + \mathbf{B}\mathbf{u} + \mathbf{e}$$

Equation (5-2a) can be rewritten as the general form:

$$\mathbf{x}(h+1) = \mathbf{A}\mathbf{y}(h) + \mathbf{G}_1\mathbf{z}_1(h) + \mathbf{G}_2\mathbf{z}_2(h) + \cdots + \mathbf{G}_{2r}\mathbf{z}_{2r}(h) + \mathbf{B}\mathbf{u} + \mathbf{e} \qquad (5\text{-}3)$$

where

$$\mathbf{G}_k = \begin{bmatrix} g_{k0} & g_{k1} & g_{k2} & & \cdots & g_{k(n-1)} \\ g_{k(n-1)} & g_{k0} & g_{k1} & g_{k2} & & \vdots \\ & g_{k(n-1)} & g_{k0} & g_{k1} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & g_{k2} \\ & & & & & g_{k1} \\ g_{k1} & \cdots & & g_{k(n-1)} & & g_{k0} \end{bmatrix}, \ k = 1, 2, \ldots, 2r$$

Analysis of $\mathbf{G}_1 \sim \mathbf{G}_{2r}$

If neighborhood radius is r, then each cell has $2r + 1$ neighboring cells, and $\binom{2r+1}{2}$ pairs. We distribute one pair to $\mathbf{G}_{2r}$, two pairs to $\mathbf{G}_{2r-1}$,…, $2r$ pairs to $\mathbf{G}_1$. The total number of all distributed pairs is $1 + 2 + \cdots + 2r = \binom{2r+1}{2}$. We can design $\mathbf{G}_1$ consist of the coefficients corresponding to the following pairs:

(1, 2)→(2, 3)→(3, 4)→⋯→(n-1, n)→(1, n)→(1, 2)

According to r, we right cyclic shift the list of these pairs r times, so the list of these pairs becomes:

If $r = 1$:

(1, n)→(1, 2)→(2, 3)→(3, 4)→⋯→(n-1, n)

If $r > 1$:

(n-r+1, ξ(n-r+2))→⋯→(n-r, n-r+1)

where $\xi(\mathrm{a}) = \begin{cases} \mathrm{a}, \text{if } \mathrm{a} \le n \\ \mathrm{a} - n, \text{if } \mathrm{a} > n \end{cases}$.

So if $r = 1$, $\mathbf{G}_1$'s element index is as the following:

$$\mathbf{G}_1 =$$
$$\begin{array}{cc} & \begin{array}{ccccc} (1,n) & (1,2) & (2,3) & \cdots & (n-1,n) \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \left[\begin{array}{ccccc} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \end{array} \right] \end{array}$$

Other matrices $\mathbf{G}_k$, k = 2,…, $2r$ are the same, according to r, right cyclic shift its own pairs r times.

So in general:

$$\mathbf{G}_1 =$$
$$\begin{array}{cc} & \begin{array}{ccccc} (n-r+1,\xi(n-r+2)) & \cdots & \cdots & \cdots & (n-r,n-r+1) \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \left[\begin{array}{cccccc} g_1(1) & \cdots & g_1(2r) & 0 & \cdots & 0 \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \right] \end{array}$$

$$\mathbf{G}_2 =$$
$$\begin{array}{cc} & \begin{array}{ccccc} (n-r+1,\xi(n-r+3)) & \cdots & \cdots & \cdots & (n-r,\xi(n-r+2)) \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \left[\begin{array}{cccccc} g_2(1) & \cdots & g_2(2r-1) & 0 & \cdots & 0 \\ & & & & & \\ & & & & & \\ & & & & & \end{array} \right] \end{array}$$

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$$\mathbf{G}_{2r} =$$
$$\begin{array}{cc} & \begin{array}{ccccc} (n-r+1,\xi(n+r+1)) & \cdots & \cdots & \cdots & (n-r,\xi(n+r)) \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \end{array} & \left[\begin{array}{ccccccc} g_{2r}(1) & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{array} \right] \end{array}$$

Globally Asymptotically Stable Condition

The discrete time cellular neural network with matrices **A**, $\mathbf{G}_1 \sim \mathbf{G}_{2r}$ given in the above is globally asymptotically stable, if and only if

$$|S(2\pi q/n)| + |P_1(2\pi q/n)| + \cdots + |P_{2r}(2\pi q/n)| < 1,\ q = 0, 1, 2, \ldots, n\text{-}1 \qquad (5\text{-}4)$$

where

$$S(2\pi q/n) = \sum_{h=-r}^{r} a(h) e^{-j2\pi hq/n},\quad P_k(2\pi q/n) = \sum_{h=0}^{n-1} g_{kh} e^{-j2\pi hq/n}, \qquad (5\text{-}5)$$

$k = 1, 2, \ldots, 2r$

Proof:

The eigenvalues of matrix **A** can be derived as follows [9]:

$$\lambda(\mathbf{A}) = \sum_{h=-r}^{r} a(h) e^{-j2\pi hq/n},\quad q = 0, 1, 2, \ldots, n\text{-}1$$

The eigenvalues of $\mathbf{G}_k$ are

$$\lambda(\mathbf{G}_k) = \sum_{h=0}^{n-1} g_{kh} e^{-j2\pi hq/n},\quad q = 0, 1, 2, \ldots, n\text{-}1$$

So $S(2\pi q/n) = \lambda(\mathbf{A})$, $P_k(2\pi q/n) = \lambda(\mathbf{G}_k)$. We want to prove that if $|\lambda(\mathbf{A})| + |\lambda(\mathbf{G}_1)| + \cdots + |\lambda(\mathbf{G}_{2r})| < 1$, then the cellular neural network is globally asymptotically stable. Consider nonlinear mapping $\mathbf{H}(\mathbf{x}) = \mathbf{A}\mathbf{f}(\mathbf{x}) + \mathbf{G}_1\mathbf{t}_1(\mathbf{x}) + \cdots + \mathbf{G}_{2r}\mathbf{t}_{2r}(\mathbf{x}) + \mathbf{B}\mathbf{u} + \mathbf{e}$. For every $\mathbf{x}^1, \mathbf{x}^2 \in \Re^n$,

$$\begin{aligned}
&|\mathbf{H}(\mathbf{x}^1) - \mathbf{H}(\mathbf{x}^2)| \\
&= |\mathbf{A}\mathbf{f}(\mathbf{x}^1) - \mathbf{A}\mathbf{f}(\mathbf{x}^2) + \mathbf{G}_1\mathbf{t}_1(\mathbf{x}^1) - \mathbf{G}_1\mathbf{t}_1(\mathbf{x}^2) + \cdots + \mathbf{G}_{2r}\mathbf{t}_{2r}(\mathbf{x}^1) - \mathbf{G}_{2r}\mathbf{t}_{2r}(\mathbf{x}^2)| \\
&= |\mathbf{A}[\mathbf{f}(\mathbf{x}^1) - \mathbf{f}(\mathbf{x}^2)] + \mathbf{G}_1[\mathbf{t}_1(\mathbf{x}^1) - \mathbf{t}_1(\mathbf{x}^2)] + \cdots + \mathbf{G}_{2r}[\mathbf{t}_{2r}(\mathbf{x}^1) - \mathbf{t}_{2r}(\mathbf{x}^2)]| \\
&\le \|\mathbf{A}\|\,|\mathbf{f}(\mathbf{x}^1) - \mathbf{f}(\mathbf{x}^2)| + \|\mathbf{G}_1\|\,|\mathbf{t}_1(\mathbf{x}^1) - \mathbf{t}_1(\mathbf{x}^2)| + \cdots + \|\mathbf{G}_{2r}\|\,|\mathbf{t}_{2r}(\mathbf{x}^1) - \mathbf{t}_{2r}(\mathbf{x}^2)|
\end{aligned}$$

According to the mean value theorem, there is a $\mathbf{x}^3$ in ($\mathbf{x}^1$, $\mathbf{x}^2$), it makes $\mathbf{f}(\mathbf{x}^1) - \mathbf{f}(\mathbf{x}^2) = \mathbf{f}'(\mathbf{x}^3)(\mathbf{x}^1 - \mathbf{x}^2)$ and $\mathbf{t}_k(\mathbf{x}^1) - \mathbf{t}_k(\mathbf{x}^2) = \mathbf{t}'_k(\mathbf{x}^3)(\mathbf{x}^1 - \mathbf{x}^2)$. And because the slopes of $\mathbf{f}(\cdot)$ and $\mathbf{t}_k(\cdot)$ are less than or equal to 1, namely $\left|\mathbf{f}'(\mathbf{x})\right| \le 1$ and $\left|\mathbf{t}'_k(\mathbf{x})\right| \le 1$, so

$$\begin{aligned}
\left|\mathbf{H}(\mathbf{x}^1) - \mathbf{H}(\mathbf{x}^2)\right| &\le \|\mathbf{A}\|\left|\mathbf{f}'(\mathbf{x}^3)(\mathbf{x}^1 - \mathbf{x}^2)\right| + \|\mathbf{G}_1\|\left|\mathbf{t}'_1(\mathbf{x}^3)(\mathbf{x}^1 - \mathbf{x}^2)\right| + \cdots + \|\mathbf{G}_{2r}\|\left|\mathbf{t}'_{2r}(\mathbf{x}^3)(\mathbf{x}^1 - \mathbf{x}^2)\right| \\
&\le \|\mathbf{A}\|\left|\mathbf{f}'(\mathbf{x}^3)\right|\left|\mathbf{x}^1 - \mathbf{x}^2\right| + \|\mathbf{G}_1\|\left|\mathbf{t}'_1(\mathbf{x}^3)\right|\left|\mathbf{x}^1 - \mathbf{x}^2\right| + \cdots + \|\mathbf{G}_{2r}\|\left|\mathbf{t}'_{2r}(\mathbf{x}^3)\right|\left|\mathbf{x}^1 - \mathbf{x}^2\right| \\
&\le \|\mathbf{A}\|\left|\mathbf{x}^1 - \mathbf{x}^2\right| + \|\mathbf{G}_1\|\left|\mathbf{x}^1 - \mathbf{x}^2\right| + \cdots + \|\mathbf{G}_{2r}\|\left|\mathbf{x}^1 - \mathbf{x}^2\right| \\
&= \left(\|\mathbf{A}\| + \|\mathbf{G}_1\| + \cdots + \|\mathbf{G}_{2r}\|\right)\left|\mathbf{x}^1 - \mathbf{x}^2\right|
\end{aligned}$$

where | • | represents a arbitrary vector norm, ||**A**|| represents spectral norm of **A**.

The spectral norm of **A** and the spectral norm of $\mathbf{G}_k$ are as follows [9]:

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^{\mathrm{T}}\mathbf{A})}$$

$$\|\mathbf{G}_k\|_2 = \sqrt{\lambda_{\max}(\mathbf{G}_k{}^{\mathrm{T}}\mathbf{G}_k)},\ k = 1, 2, \ldots, 2r$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue.

From [9], we can know that

$$\lambda_{\max}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \left|\lambda_{\max}(\mathbf{A})\right|^2,\ \lambda_{\max}(\mathbf{G}_k{}^{\mathrm{T}}\mathbf{G}_k) = \left|\lambda_{\max}(\mathbf{G}_k)\right|^2,\ k = 1, 2, \ldots, 2r$$

So the condition (5-4) implies that

$$\begin{aligned}
&\|\mathbf{A}\|_2 + \|\mathbf{G}_1\|_2 + \cdots + \|\mathbf{G}_{2r}\|_2 \\
&= \sqrt{\lambda_{\max}(\mathbf{A}^{\mathrm{T}}\mathbf{A})} + \sqrt{\lambda_{\max}(\mathbf{G}_1{}^{\mathrm{T}}\mathbf{G}_1)} + \cdots + \sqrt{\lambda_{\max}(\mathbf{G}_{2r}{}^{\mathrm{T}}\mathbf{G}_{2r})} \\
&= \sqrt{\left|\lambda_{\max}(\mathbf{A})\right|^2} + \sqrt{\left|\lambda_{\max}(\mathbf{G}_1)\right|^2} + \cdots + \sqrt{\left|\lambda_{\max}(\mathbf{G}_{2r})\right|^2} \\
&= \left|\lambda_{\max}(\mathbf{A})\right| + \left|\lambda_{\max}(\mathbf{G}_1)\right| + \cdots + \left|\lambda_{\max}(\mathbf{G}_{2r})\right| \\
&< 1 \\
&\Rightarrow \quad \left|\mathbf{H}(\mathbf{x}^1) - \mathbf{H}(\mathbf{x}^2)\right| < \left|\mathbf{x}^1 - \mathbf{x}^2\right|
\end{aligned}$$

So **H**(•) is a contraction and there is a unique solution to the nonlinear equation **H**(**x**) **= x**, namely a unique fixed point of system (5-3). It can be shown that for any initial state **x**(0), the system (5-3) converges to this unique fixed point.

Given m bipolar training patterns as input vectors $\mathbf{u}^i$, i = 1, 2, …, m, Equation (5-2a) can be rewritten in compact form as

$$\begin{aligned}\mathbf{X} &= \mathbf{AY} + \mathbf{G}_1\mathbf{Z}_1 + \cdots + \mathbf{G}_{2r}\mathbf{Z}_{2r} + \mathbf{BU} + \mathbf{J} \\ \mathbf{BU} + \mathbf{J} &= \mathbf{X} - \mathbf{AY} - \mathbf{G}_1\mathbf{Z}_1 - \cdots - \mathbf{G}_{2r}\mathbf{Z}_{2r}\end{aligned} \tag{5-6}$$

Define the following matrices:

$\mathbf{R} = [\mathbf{U}^{\mathrm{T}} | \mathbf{h}]$

$\mathbf{h} = [1\ 1\ \cdots\ 1]^{\mathrm{T}}$

$\mathbf{X}_j = [x_j^1\ \ x_j^2\ \cdots\ x_j^m]$

$\mathbf{A}_y = \mathbf{AY} = [\mathbf{Ay}^1\ \ \mathbf{Ay}^2\ \cdots\ \mathbf{Ay}^m] = [\mathbf{d}^1\ \ \mathbf{d}^2\ \cdots\ \mathbf{d}^m] \in \Re^{n\times m}$

$\mathbf{d}^i = [d_1^i\ \ d_2^i\ \cdots\ d_n^i]^{\mathrm{T}} \in \Re^{n\times 1}$, i = 1, …, m

$\mathbf{A}_{y,j} = [d_j^1\ \ d_j^2\ \cdots\ d_j^m]$

$\mathbf{w}_j = [b_{j1}\ \ b_{j2}\ \cdots\ b_{jn}\ \ I_j]$

Equation (5-6) becomes

$$\mathbf{Rw}_j^{\mathrm{T}} = \mathbf{X}_j^{\mathrm{T}} - \mathbf{A}_{y,j}^{\mathrm{T}} - (\mathbf{G}_1\mathbf{Z}_1)_j^{\mathrm{T}} - \cdots - (\mathbf{G}_{2r}\mathbf{Z}_{2r})_j^{\mathrm{T}}, \quad j = 1, 2, \ldots, n \tag{5-7}$$

where $(\mathbf{G}_i\mathbf{Z}_i)_j$ is the jth row of $\mathbf{G}_i\mathbf{Z}_i$.

Define one matrix **S** to represent the locally connected network architecture.
The matrix **S**:

$S_{ij} = 1$, if the jth neuron belongs to the r-neighborhood of the ith neuron;

$S_{ij} = 0$, otherwise (i = 1,…, n; j = 1,…, n).

A new matrix $\mathbf{R}_j$ can be obtained from the matrix **R** by eliminating the following columns: The columns whose indexes correspond to the zero elements in the jth row of $\mathbf{S}$. In the same way, a vector $\tilde{\mathbf{w}}_j$ can be defined as the vector obtained from $\mathbf{w}_j$ by eliminating its zero elements. Thus, Equation (5-7) can be rewritten as

$$\mathbf{R}_j \tilde{\mathbf{w}}_j^{\mathrm{T}} = \mathbf{X}_j^{\mathrm{T}} - \mathbf{A}_{y,j}^{\mathrm{T}} - (\mathbf{G}_1 \mathbf{Z}_1)_j^{\mathrm{T}} - \cdots - (\mathbf{G}_{2r} \mathbf{Z}_{2r})_j^{\mathrm{T}}, \quad j = 1, \ldots, n \qquad (5\text{-}8)$$

From (5-8) it follows:

$$\tilde{\mathbf{w}}_j^{\mathrm{T}} = \mathbf{R}_j^{+} \left[\mathbf{X}_j^{\mathrm{T}} - \mathbf{A}_{y,j}^{\mathrm{T}} - (\mathbf{G}_1 \mathbf{Z}_1)_j^{\mathrm{T}} - \cdots - (\mathbf{G}_{2r} \mathbf{Z}_{2r})_j^{\mathrm{T}} \right], \quad j = 1, \ldots, n \qquad (5\text{-}9)$$

We test the performance of the second-order CNN associative memory with an experiment. The following experiment is using first-order CNN associative memory and second-order CNN associative memory to store five training patterns respectively. And then we compare the performances of these two associative memories.

The information of the experiment:

Pattern size	7×7
The number of training patterns	5
Hamming distance(HD)	3
Neighborhood radius r	1
α	3

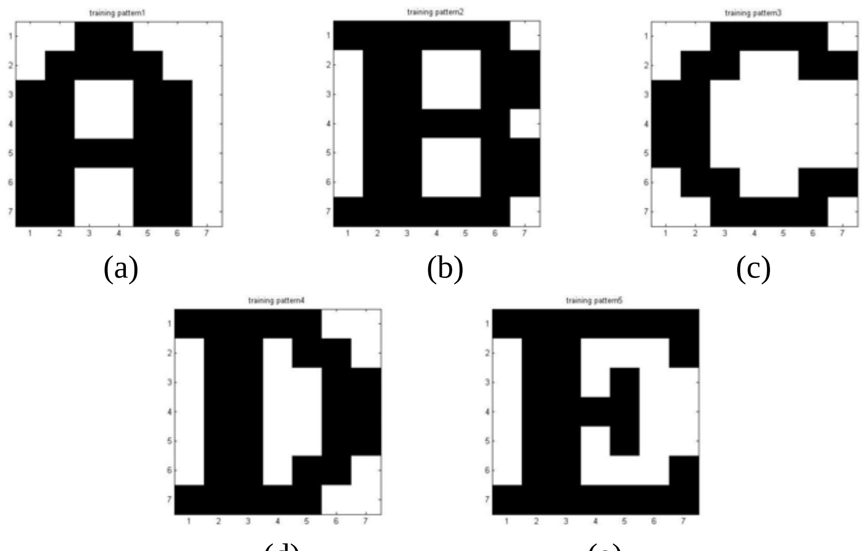


Fig. 5.1 (a) ~ (e) Five training patterns.

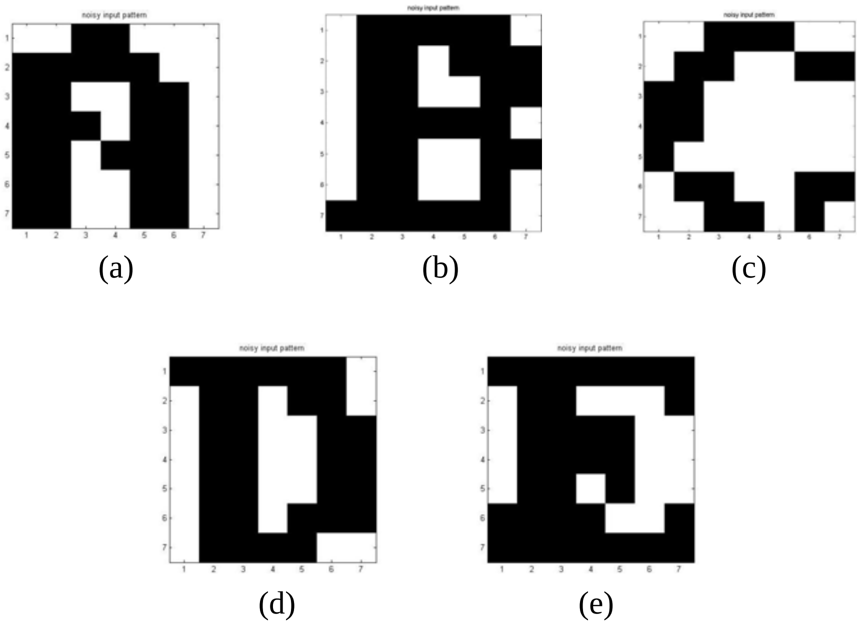


Fig. 5.2 (a) ~ (e) Five noisy input patterns.

Fig. 5.3 is the recognition results of the first-order CNN associative memory:

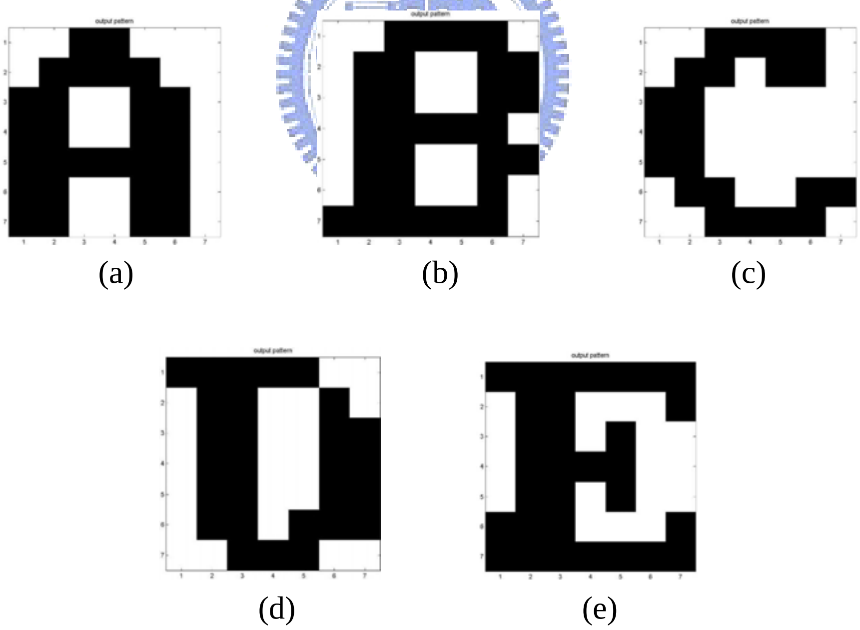
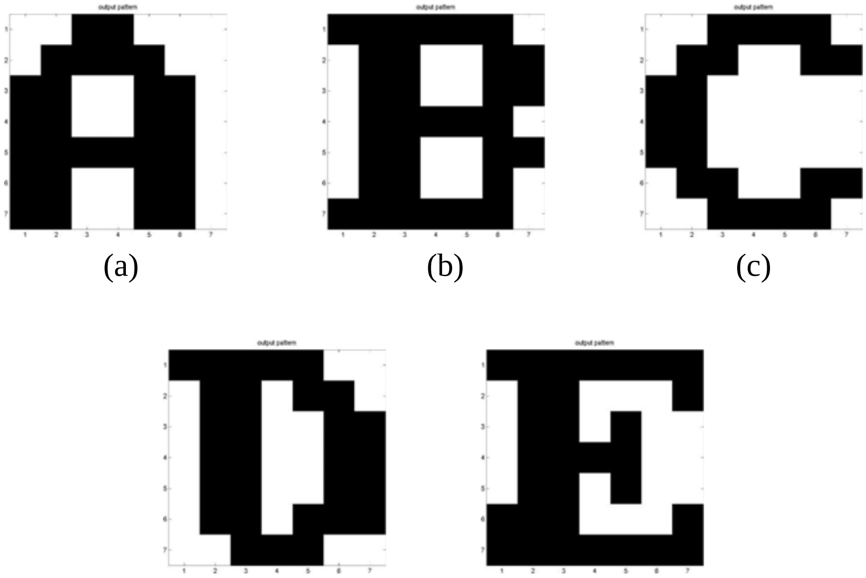


Fig. 5.3 (a) ~ (e) The recognition results of Fig. 5.2(a) ~ (e) outputted by the first-order CNN associative memory.

Fig. 5.4 is the recognition results of the second-order CNN associative memory:



(a) (b) (c)

(e) (e)

Fig. 5.4 (a) ~ (e) The recognition results of Fig. 5.2(a) ~ (e) outputted by the second-order CNN associative memory.

Compare Fig. 5.3 and Fig. 5.4, we can know that the performance of the second-order CNN associative memory is better than that of the first-order CNN associative memory.

In the future work, we may apply the CNN associative memory to the recognition of 3-D seismic patterns.

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