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The randomized vacation policy for a batch arrival queue

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ABSTRACT

This paper examines an $M^{[X]}/G/1$ queueing system with a randomized vacation policy and at most J vacations. Whenever the system is empty, the server immediately takes a vacation. If there is at least one customer found waiting in the queue upon returning from a vacation, the server will be immediately activated for service. Otherwise, if no customers are waiting for service at the end of a vacation, the server either remains idle with probability p or leaves for another vacation with probability $1-p$. This pattern continues until the number of vacations taken reaches J. If the system is empty by the end of the Jth vacation, the server is dormant idly in the system. If there is one or more customers arrive at server idle state, the server immediately starts his services for the arrivals. For such a system, we derive the distributions of important characteristics, such as system size distribution at a random epoch and at a departure epoch, system size distribution at busy period initiation epoch, idle period and busy period, etc. Finally, a cost model is developed to determine the joint suitable parameters (p^*, J^*) at a minimum cost, and some numerical examples are presented for illustrative purpose.

1. Introduction

We consider an $M^{[X]}/G/1$ queueing system in which the server operates a randomized vacation policy with at most J vacations. The server leaves for a vacation when the system becomes empty. When the server returns from the vacation, he applies a randomized vacation policy and decides to take another vacation, to remain dormant in the system or to provide service for the waiting customers. The randomized vacation policy presented in this paper is described as follows: when the system is empty, the server immediately takes for a vacation. If there is at least one customer found waiting in the queue upon returning from a vacation, the server will be immediately activated for service. Otherwise, if no customers are waiting for service at the end of a vacation, the server remains idle in the system with probability p and leaves for another vacation with probability $1-p$. This pattern continues until the number of vacations taken reaches J. If the system is empty by the end of the Jth vacation, the server becomes idle in the system. If there is one or more customers arrive at server idle state, the server immediately starts his services.

The modeling analysis for the vacation queueing models has been done by a considerable amount of work in the past and successfully used in various applied problems such as production/inventory systems, communication systems, computer networks and etc. (see survey paper by Doshi [1]). A comprehensive and excellent study on the vacation models can be found in Levy and Yechiali [2] and Takagi [3]. Baba [4] studied the $M^{[X]}/G/1$ queueing model with multiple vacations. The first study of vacation models with control policy was done by Kella [5]. The variations and extensions of these vacation models with control policy can be referred to Lee et al. [6,7], Choudhury and Madan [8], Ke [9], Choudhury and Paul [10], Yang et al. [11],

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Ke [12] and others. Recently, Yang et al. [11] analyzed the optimal randomized control policy of an unreliable server M/G/1 system with second optional service and startup. Ke [12] examined the two thresholds of a batch arrival $M^{[X]}/G/1$ queueing system under modified T vacation policy with startup and closedown. The developments and applications on the optimal control of queueing systems are rich and varied (see Tadj and Choudhury [13]). Moreover, Takagi [3] first proposed the concept of a variant vacation (a generalization of the multiple and single vacation) for the single arrival M/G/1 regular system. Zhang and Tian [14] treated the discrete time Geo/G/1 system with variant vacations, where the server will take a random maximum number of vacations after serving all customers in the system. Ke and Chu [15] examined the variant policy for an $M^{[X]}/G/1$ queueing system by stochastic decomposition property. Recently, Ke [16] used supplementary variable technique to study an $M^{[X]}/G/1$ queueing system with balking under a variant vacation. Some important distributions of system performance measures were also derived in [16].

Existing literature focused on vacation policy depending on queue size or timer, so far very few authors have studied the comparable work on the vacation queueing problems with randomized control policy, in which the server may take a sequence of finite vacations in the idle time and apply a randomized vacation policy. This motivates us to develop the variant vacation policy for an $M^{[X]}/G/1$ queueing system, where the server operates a randomized vacation policy and takes at most J vacations when the system is empty. Conveniently, we represent this variant vacation system as $M^{[X]}/G/1/VAC(J)$ queueing system.

The objectives of this paper are as follows: Firstly, we develop differential equations governing the variant vacation system and derive the probability generating functions of system size in various server states. Secondly, we also derive other system characteristics such as the system size distribution at busy period initiation epoch, the busy and idle period distributions, etc. Thirdly, a long-run expected cost function per unit time is constructed to determine the optimal control policy. Fourthly, we provide a decision criterion to find the joint suitable value of (p, J), and some numerical examples are presented for illustrative purpose. Finally, some conclusions are drawn.

2. The system

We consider an $M^{[X]}/G/1$ system in which the server operates a randomized vacation policy and takes at most J vacations when he serves all customers exhaustively. The detailed description of the model is given as follows:

Customers arrive in batches occurring according to a compound Poisson process with mean arrival rate λ. Let X_k denote the number of customers belonging to the kth arrival batch, where X_k, $k = 1, 2, 3, \ldots$, are with a common distribution

$$Pr(X_k = n) = \chi_n, \quad n = 1, 2, 3, \ldots$$

The service time provided by a single server is an independent and identically distributed random variable S with distribution function $S(x)$ and Laplace-Stieltjes transform (LST) $S^*(\theta)$. Arriving customers who join the system form a single waiting line based on the order of their arrivals; that is, they are queued according to the first-come, first-served (FCFS) discipline. The server can serve only one customer at a time, and that the service is independent of the arrival of the customers. If the server is busy or on vacation, arrivals in the queue must wait until the server is available. When the system becomes empty, the server leaves for a vacation with random length V having distribution function $V(x)$ and LST $V^*(\theta)$. If at least one customer is found waiting in the queue upon returning from the vacation, the server is immediately activated for service. Alternatively, if no customers are found in the queue at the end of a vacation, the server remains idle in the system with probability $p(0 \leqslant p \leqslant 1)$ and leaves for another vacation with probability $\bar{p}\ (= 1 - p)$. This pattern continues until the number of vacations taken reaches J. If the system is still empty by the end of the Jth vacation, the server remains idle in the system. If there is at least one customer arrives at server idle state, the server immediately starts providing his services for the arrivals. It is assumed that various stochastic processes involved in the system are independent of each other.

Our model can be applied to model many real world systems. In particular, some stochastic production and inventory control systems with a multi-purpose production facility can be effectively studied using such a vacation queueing model. In such systems, the demand for the product is random and can be modeled as a compound Poisson process. The production time of each unit of the product is a random variable with general distribution. An application example is considered for illustrative purpose: *Production-to-Order* is a production policy in production planning and management. It is assumed that customer orders for this product arrive according to a compound Poisson process. Whenever all orders are completed and no new orders arrive, the production will be stopped and the facility is performed a closedown task before closing (can be referred to an essential vacation). After the production facility is completely closed, it may be available to perform some optional jobs. The optional jobs can be referred to other second tasks or a sequence of finite maintenances. Upon completion of each optional job, the manager check the orders and decides whether or not to resume the major production. If at this moment the major orders are empty, a decision may be made for taking other optional jobs to be performed. The $M^{[X]}/G/1/VAC(J)$ queueing system presented in this paper is a good approximation of such a production system.

3. The analysis

We first develop the steady-state differential-difference equations for the $M^{[X]}/G/1/VAC(J)$ queueing system by treating the elapsed service time and the elapsed vacation time as supplementary variables. Then we solve these system equations and derive the probability generating functions of various server states at a random epoch.

3.1. System size distribution at a random epoch

In steady-state, let us assume that $S(x) = 0$, for $x \leqslant 0$, $S(\infty) = 1$, $V(x) = 0$, for $x \leqslant 0$, $V(\infty) = 1$ and these two distribution functions are continuous at $x = 0$, so that $\mu(x)\,dx = \frac{dS(x)}{1-S(x)}$ and $\omega(x)\,dx = \frac{dV(x)}{1-V(x)}$, where $\mu(x)\,dx$ and $\omega(x)\,dx$ are the first order differential (hazard rate) functions of S and V, respectively.

We define the state of the system at time t as follows:

$Q(t) \equiv$ number of customers in the system,
$S^{-}(t) \equiv$ the elapsed service time, and
$V_j^{-}(t) \equiv$ the elapsed time of the jth vacation.

The following random variables are used for the development of $M^{[x]}$/G/1/VAC(J) queueing system:

$$\Delta(t) = \begin{cases} 0, & \text{if the server is idle in the system at time } t, \\ 1, & \text{if the server is busy at time } t, \\ 2, & \text{if the server is on the 1th vacation at time } t, \\ & \vdots \\ j+1, & \text{if the server is on the } j\text{th vacation at time } t, \\ & \vdots \\ J+1, & \text{if the server is on the } J\text{th vacation at time } t, \end{cases}$$

Thus the supplementary variables $S^{-}(t)$ and $V_j^{-}(t)$ are introduced in order to obtain a tri-variate Markov process $\{Q(t), \Delta(t), \delta(t)\}$, where $\delta(t) = 0$ if $\Delta(t) = 0, \delta(t) = S^{-}(t) if \Delta(t) = 1$, and $\delta(t) = V_j^{-}(t)$ if $\Delta(t) = j+1 (j = 1, 2, \ldots, J)$.

Furthermore, let us define the following probabilities:

$$R_0(t) = P_r\{Q(t) = 0, \delta(t) = 0\},$$
$$P_n(x,t)dx = P_r\{Q(t) = n, \delta(t) = S^{-}(t);\; x < S^{-}(t) \leqslant x + dx\}, \quad x > 0, \quad n \geqslant 1,$$
$$\Omega_{j,n}(x,t)dx = P_r\{Q(t) = n, \delta(t) = V^{-}(t);\; x < V_j^{-}(t) \leqslant x + dx\}, \quad x > 0, \quad n \geqslant 0, \quad 1 \leqslant j \leqslant J.$$

In steady-state, we can set $R_0 = lim_{t\to\infty} R_0(t)$ and limiting densities $P_n(x) = lim_{t\to\infty} P_n(x,t)$ and $\Omega_{j,n}(x) = lim_{t\to\infty} \Omega_{j,n}(x,t)$. According to Cox [17], the steady-state Kolmogorov forward equations that govern the system can be written as follows:

$$\lambda R_0 = \int_0^{\infty} \Omega_{J,0}(x)\omega(x)dx + p\sum_{j=1}^{J-1} \int_0^{\infty} \Omega_{j,0}(x)\omega(x)\,dx, \tag{1}$$

$$\frac{d}{dx}P_n(x) + [\lambda + \mu(x)]P_n(x) = \lambda \sum_{k=1}^{n-1} \chi_k P_{n-k}(x), \quad x > 0, \quad n \geqslant 1, \tag{2}$$

$$\frac{d}{dx}\Omega_{j,0}(x) + [\lambda + \omega(x)]\Omega_{j,0}(x) = 0, \quad x > 0, \quad 1 \leqslant j \leqslant J, \tag{3}$$

$$\frac{d}{dx}\Omega_{j,n}(x) + [\lambda + \omega(x)]\Omega_{j,n}(x) = \lambda \sum_{k=1}^{n} \chi_k \Omega_{j,n-k}(x), \quad x > 0, \quad n \geqslant 1, \quad 1 \leqslant j \leqslant J. \tag{4}$$

We solve the above equations by means of the following boundary conditions at $x = 0$

$$P_n(0) = \sum_{j=1}^{J} \int_0^{\infty} \Omega_{j,n}(x)\omega(x)\,dx + \int_0^{\infty} P_{n+1}(x)\mu(x)dx + \lambda\chi_n R_0, \quad n \geqslant 1, \tag{5}$$

$$\Omega_{1,n}(0) = \begin{cases} \int_0^{\infty} P_1(x)\mu(x)\,dx, & n = 0, \\ 0, & n \geqslant 1. \end{cases} \tag{6}$$

$$\Omega_{j,n}(0) = \begin{cases} \bar{p}\int_0^{\infty} \Omega_{j-1,n}(x)\omega(x)\,dx, & n = 0, \quad j = 2,3,\ldots,J \\ 0, & n \geqslant 1, \quad j = 2,3,\ldots,J \end{cases} \tag{7}$$

and the normalization condition

$$R_0 + \sum_{n=1}^{\infty} \int_0^{\infty} P_n(x)\,dx + \sum_{j=1}^{J} \left[\sum_{n=0}^{\infty} \int_0^{\infty} \Omega_{j,n}(x)\,dx \right] = 1. \tag{8}$$

Let us define the probability generating functions for $\{\chi_n\}\{P_n(\cdot)\}$ and $\{\Omega_{j,n}(\cdot)\}$ as follows:

$$X(z) = \sum_{n=1}^{\infty} z^n \chi_n, \quad |z| \leqslant 1,$$
$$P(x;z) = \sum_{n=1}^{\infty} z^n P_n(x), \quad |z| \leqslant 1,$$
$$\Omega_j(x;z) = \sum_{n=0}^{\infty} z^n \Omega_{j,n}(x), \quad |z| \leqslant 1, \quad 1 \leqslant j \leqslant J.$$

Now multiplying (2) by z^n $(n = 1, 2, 3, \ldots)$ and then adding the equations up term by term, it gives

$$\frac{\partial P(x;z)}{\partial x} + [a(z) + \mu(x)]P(x;z) = 0, \tag{9}$$

where $a(z) = \lambda(1 - X(z))$.

Similar proceeding in the usual manner with (3)–(5), we have

$$\frac{\partial \Omega_j(x;z)}{\partial x} + [a(z) + \omega(x)]\Omega_j(x;z) = 0 \tag{10}$$

and

$$P(0;z) = \sum_{j=1}^{J} \int_0^{\infty} \Omega_j(x;z)\omega(x)\,dx + \frac{1}{z}\int_0^{\infty} P(x;z)\mu(x)\,dx + \lambda X(z)R_0 - \sum_{j=1}^{J} \Omega_j(0;z) - \lambda R_0, \tag{11}$$

where $x > 0$.

Solving the partial differential Eqs. (9) and (10), we obtain

$$P(x;z) = P(0;z)[1 - S(x)]e^{-a(z)x}, \tag{12}$$

and

$$\Omega_j(x;z) = \Omega_j(0;z)[1 - V(x)]e^{-a(z)x}, \quad j = 1, 2, \ldots, J. \tag{13}$$

Solving the differential Eq. (3) yields

$$\Omega_{j,0}(x) = \Omega_{j,0}(0)(1 - V(x))e^{-\lambda x}, \quad j = 1, 2, \ldots, J. \tag{14}$$

Now Eq. (14) is multiplied by $\omega(x)$ on both sides for and integrating with x from 0 to ∞, we then have

$$\int_0^{\infty} \Omega_{j,0}(x)\omega(x)\,dx = \Omega_{j,0}(0)\alpha_0, \tag{15}$$

where $\alpha_0 = V^*(\lambda)$. Inserting (15) in (7), we can recursively obtain

$$\Omega_{j,0}(0) = \frac{\Omega_{J,0}(0)}{(\bar{p}\alpha_0)^{J-j}}, \quad j = 1, 2, \ldots, J-1. \tag{16}$$

Substituting (15) and (16) into (1) and after some algebraic manipulation, we have

$$\Omega_{J,0}(0) = \frac{\lambda R_0}{\alpha_0\left[1 + \frac{p(1-(\bar{p}\alpha_0)^{J-1})}{(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)}\right]}. \tag{17}$$

From (16) and (17) we finally obtain

$$\Omega_j(0;z) = \Omega_{j,0}(0) = \frac{\lambda R_0}{(\bar{p}\alpha_0)^{J-j}\alpha_0\left[1 + \frac{p(1-(\bar{p}\alpha_0)^{J-1})}{(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0))}\right]}, \quad j = 1, 2, \ldots, J. \tag{18}$$

Integrating (14) with respect to x from 0 to ∞ we have

$$\Omega_{j,0} = \Omega_{j,0}(0)\int_0^{\infty} [1 - V(x)]e^{-\lambda x}\,dx = \frac{1}{\lambda}\Omega_{j,0}(0)(1 - \alpha_0). \tag{19}$$

From (18) and (19), it finally yields

$$\Omega_{j,0} = \frac{R_0(1 - \alpha_0)}{(\bar{p}\alpha_0)^{J-j}\alpha_0\left[1 + \frac{p(1-(\bar{p}\alpha_0)^{J-1})}{(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)}\right]}, \quad j = 1, 2, \ldots, J. \tag{20}$$

Noting that $\Omega_{j,0}$ represents the steady-state probability that there are no customers in the system when the server is on the j^{th} vacation. Let us define Ω_0 the probability that no customers appear in the system when the server is on vacation. Then we have

$$\Omega_0 = \sum_{j=1}^{J} \Omega_{j,0} = \frac{R_0(1-\alpha_0)}{\alpha_0\left[1+\frac{p\left(1-(\bar{p}\alpha_0)^{J-1}\right)}{(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)}\right]} \times \frac{1-(\bar{p}\alpha_0)^J}{(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)}. \tag{21}$$

Inserting (12), (13) and (18) into (11) we get on simplification

$$P(0;z) = \frac{\lambda R_0\left(1-(\bar{p}\alpha_0)^J\right)V^*(a(z))}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p\left(1-(\bar{p}\alpha_0)^{J-1}\right)\right]} + \frac{P(0;z)S^*(a(z))}{z} + \lambda X(z)R_0 - \sum_{j=1}^{J}\Omega_{j,0}(0) - \lambda R_0. \tag{22}$$

Solving $P(0;z)$ from (22) and using (18) yields

$$P(0;z) = \frac{\lambda R_0 z\left(\frac{\left(1-(\bar{p}\alpha_0)^J\right)(V^*(a(z))-1)}{(\alpha_0)\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]} - 1 + X(z)\right)}{z - S^*(a(z))}. \tag{23}$$

It follows from (12) and (23) that

$$P(x;z) = \frac{\lambda R_0 z\left(\frac{\left(1-(\bar{p}\alpha_0)^J\right)(V^*(a(z))-1)}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]} - 1 + X(z)\right)}{z - S^*(\lambda(1-X(z)))} \times [1-S(x)]e^{-a(z)x}, \tag{24}$$

which leads to

$$P(z) = \int_0^\infty P(x;z)\,dx = \frac{R_0 z\left(\frac{\left(1-(\bar{p}\alpha_0)^J\right)(V^*(a(z))-1)}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]} - 1 + X(z)\right)}{z - S^*(a(z))} \times \frac{S^*(a(z))-1}{X(z)-1}. \tag{25}$$

Using (13) and (18) and the well-known result of renewal theory

$$\int_0^\infty e^{-\lambda x}(1-V(x))\,dx = \frac{1-\alpha_0}{\lambda},$$

we have

$$\Omega_j(z) = \frac{R_0(V^*(a(z))-1)}{(\bar{p}\alpha_0)^{J-j}\alpha_0[X(z)-1]\left[1+\frac{p(1-(\bar{p}\alpha_0)^{J-1})}{(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)}\right]}, \quad j = 1,2,3,\ldots,J. \tag{26}$$

The unknown constant R_0 can be determined by using the normalization condition (8), which is equivalent to $R_0 + P(1) + \sum_{j=1}^{J}\Omega_j(1) = 1$. Thus we obtain

$$R_0 = \frac{1-\rho}{1+\frac{\lambda E[V]\left(1-(\bar{p}\alpha_0)^J\right)}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p\left(1-(\bar{p}\alpha_0)^{J-1}\right)\right]}}, \tag{27}$$

where $\rho = \lambda E[X]E[S]$.

Note that Eq. (27) represents the steady state probability that the server is idle but available in the system. Also from Eq. (27), we have $\rho < 1$ which is the necessary and sufficient condition under which steady state solution exists.

Let $\Phi(z) = R_0 + P(z) + \sum_{j=1}^{J}\Omega_j(z)$ be the probability generating function of the system size distribution at stationary point of time, we then have

$$\Phi(z) = \frac{(1-\rho)S^*(a(z))(z-1)}{z-S^*(a(z))} \times \frac{(1-(\bar{p}\alpha_0)^J)(V^*(a(z))-1) + \alpha_0[X(z)-1]\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]}{\left(\lambda E[V](1-(\bar{p}\alpha_0)^J) + \alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]\right)[X(z)-1]}. \tag{28}$$

Remark 1. Special one of our model is the ordinary $M^{[X]}/G/1$ queueing system with at most J vacations. That is, if we let $p = 0$ in our model, then R_0 can be reduced to

$$\frac{1-\rho}{\frac{\lambda E[V]}{\alpha_0^J}\cdot\frac{1-\alpha_0^J}{1-\alpha_0}+1},$$

which agrees with Ke and Chu [15].

Remark 2. Letting $p = 0$ and $J = 1$, our model can be simplified to the ordinary $M/G/1$ queueing system with single vacation. $\Phi(z)$ can be rewritten as

$$\left(\frac{(1-\rho)(1-z)S^*(a(z))}{S^*(a(z))-z}\right)\left(\frac{1-V^*(a(z))+\alpha_0(1-X(z))}{[1-X(z)][\lambda E[V]+\alpha_0]}\right),$$

which confirms the result in Section 6 of Choudhury's system [18]. It should be noted that if we let $p=1$, our model can be also reduced to the ordinary $M/G/1$ queueing system with single vacation. That is, when $p=1$, the results are in accordance with those of letting $p=0$ and $J=1$.

Remark 3. Letting $p=0$ and $J=\infty$, our model becomes the ordinary $M/G/1$ queueing system with miltiple vacations. $\Phi(z)$ can be rewritten as

$$\left(\frac{(1-\rho)(1-z)S^*(a(z))}{S^*(a(z))-z}\right)\left(\frac{1-V^*(a(z))}{\lambda E[V][1-X(z)]}\right)$$

and the result in in accordance with Takagi [3].

3.2. The expected number of customers in the system and the expected waiting time

In (28), we evaluate $\frac{d}{dz}\Phi(z)|_{z=1}$ by using L'hopital rule which leads to the expected number of customers, L_s, in the system given by

$$L_s=\rho+\frac{\lambda E[X(X-1)]E[S]+(\lambda E[X])^2E[S^2]}{2(1-\rho)}+\frac{\lambda^2E[X](1-(\bar{p}\alpha_0)^J)E[V^2]}{2\left(\lambda E[V](1-(\bar{p}\alpha_0)^J)+\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]\right)}. \tag{29}$$

Noting that, the first and second terms in (29) represent the expected number of customers in the system for the ordinary $M^{[X]}/G/1$ queueing system.

By using Little's formula, we obtain the expected waiting time in the queue, W_q, given by

$$W_q=\frac{E[X(X-1)]E[S]+\lambda(E[X])^2E[S^2]}{2E[X](1-\rho)}+\frac{\lambda(1-(\bar{p}\alpha_0)^J)E[V^2]}{2\left(\lambda E[V](1-(\bar{p}\alpha_0)^J)+\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]\right)}. \tag{30}$$

The expected waiting time in the system W_s can be obtained by $W_s=W_q+E[S]$.

Remark 4. Suppose that we have $p=0$ and $J=1$, then if we put $Pr[X=1]=1$, our model can be reduced to the ordinary M/G/1 queueing system with single vacation. It follows from (30) that the expected waiting time in the system is given by

$$W_s=\frac{\lambda E[V^2]}{2(\lambda E[V]+\alpha_0)}+\frac{\lambda E[S^2]}{2(1-\rho)},$$

which is in accordance with Takagi's system [3, Section 2.2, p.126].

Remark 5. Letting $p=0$, our model can be recovered to the $M^{[X]}/G/1$ queueing system with at most J vacations. Using (30) we have the expected waiting time in the system as:

$$W_s=\frac{(1-\alpha_0^J)\lambda E[V^2]}{2((1-\alpha_0^J)\lambda E[V]+(1-\alpha_0)\alpha_0^J)}+\frac{\lambda E[X]E[S^2]}{2(1-\rho)}+\frac{E[X(X-1)]E[S]}{2E[X](1-\rho)},$$

which is in accordance with Ke and Chu [15].

3.3. Queue size distribution at a departure epoch

In this section, we derived the probability generating function of queue size distribution for the $M^{[X]}/G/1/VAC(J)$ queueing system. Following the arguments by Wolff [19], we state that a departing customer will see l customers in the queue just after a departure if and only if there are $(l+1)$ customers in the queue just before the departure. Thus we can write the following:

$$\Phi_l^+=C_0\int_0^\infty\mu(x)P_{l+1}(x)\,dx,\quad l=0,1,\ldots \tag{31}$$

where $\Phi_l^+=\Pr\{$A departing customer will see$l$ customers in the system$\}$, and C_0 is the normalizing constant.

Let $\Phi^+(z)$ be the probability generating function of $\{\Phi_l^+, l=0,1,2,\ldots\}$. Using (12) yields

$$\Phi^+(z)=C_0\times\frac{\lambda R_0\left(\frac{(1-(\bar{p}\alpha_0)^J)(V^*(a(z))-1)}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]}-1+X(z)\right)S^*(a(z))}{z-S^*(a(z))}. \tag{32}$$

Using the normalization condition $\Phi^+(1) = 1$, it gives

$$C_0 = \frac{1-\rho}{\lambda R_0 E[X]\left(1 + \frac{\lambda E[V](1-(\bar{p}\alpha_0)^J)}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]}\right)}, \tag{33}$$

which leads to the probability generating function of the departure point queue size distribution as

$$\Phi^+(z) = \frac{(1-\rho)\left(\frac{(1-(\bar{p}\alpha_0)^J)(V^*(a(z))-1)}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]} - 1 + X(z)\right)S^*(a(z))}{E[X]\left(1 + \frac{\lambda E[V](1-(\bar{p}\alpha_0)^J)}{\alpha_0\left[(\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)+p(1-(\bar{p}\alpha_0)^{J-1})\right]}\right)(z - S^*(a(z)))}. \tag{34}$$

From (34), one see that $\Phi^+(z)$ can be decomposed into two independent terms:

$$\Phi^+(z) = \frac{1-X(z)}{E[X](1-z)} \times \Phi(z). \tag{35}$$

It should be noted that the departure point queue size distribution given by Eq. (35) can be decomposed into two independent random variables: one (the first term) is the number of customers placed before a tagged customer in a batch in which the tagged customer arrives and the other (the second term) is the stationary system size of the $M^{[X]}$/G/1/VAC(J) queueing system.

Remark 6. Substituting $p = 0$ and $J = 1$ into (35), our system can be reduced to the ordinary $M^{[X]}$/G/1 single vacation policy queue. $\Phi^+(z)$ can be rewritten as

$$\left(\frac{1 - V^*(a(z)) + \alpha_0(1-X(z))}{E[X](\lambda E[V]+\alpha_0)(1-z)}\right)\left(\frac{(1-\rho)(1-z)S^*(a(z))}{S^*(a(z))-z}\right) = \beta(z)\Phi^+(z; M^{[X]}/G/1),$$

where $\beta(z) = \frac{1-V^*(a(z))+\alpha_0(1-X(z))}{E[X](\lambda E[V]+\alpha_0)(1-z)}$, and $\Phi^+(z; M^{[X]}/G/1) = \frac{(1-\rho)(1-z)S^*(a(z))}{S^*(a(z))-z}$ is the p.g.f. of the stationary queue size distribution of an ordinary $M^{[X]}$/G/1 queue, which is accordance with the stochastic decomposition property demonstrated in Choudhury's system [18].

3.4. System size distribution at busy period initiation epoch

First, we define ϕ_n $(n = 1, 2, \ldots)$ as the steady-state probability that an arbitrary (tagged) customer finds n customers in the system at the busy initiation epoch (or completion epoch of the idle period). This implies that t_l $(l = 0, 1, 2, \ldots)$ are the initiation epochs of the busy period and $Q(t_l)$ is the number of customers in the system at the time instant t_l, then we have

$$\phi_n = lim_{l\to\infty}\Pr(N(t_l) = n), \quad n = 1, 2, \ldots.$$

Conditioning on the number of customers which arrive during the first vacation, from the concept of *Poisson Arrivals See Time Average* (*PASTA*) [19], we have the following steady-state equation

$$\begin{aligned}\phi_n &= \left(1 + \bar{p}\alpha_0 + \bar{p}^2\alpha_0^2 + \cdots + \bar{p}^{J-1}\alpha_0^{J-1}\right)\sum_{k=1}^{n}\alpha_k\chi_n^{(k)*} + \left(p\left(\alpha_0 + \bar{p}\alpha_0^2 + \ldots + \bar{p}^{J-2}\alpha_0^{J-1}\right) + \bar{p}^{J-1}\alpha_0^J\right)\chi_n \\ &= \sum\nolimits_{m=0}^{J-1}(\bar{p}\alpha_0)^m\sum_{k=1}^{n}\alpha_k\chi_n^{(k)*} + \left(p\sum_{m=0}^{J-2}\bar{p}^m\alpha_0^{m+1} + \bar{p}^{J-1}\alpha_0^J\right)\chi_n,\end{aligned} \tag{36}$$

where $\chi_n^{(k)*} = \Pr(X_1 + X_2 + \cdots + X_k = n)$ is the k-fold convolution of χ_n, and $\chi_n^{(0)}$ is defined to be 1, and $\alpha_k = \Pr$ (k batches arrive during a vacation time).

Now multiplying (36) by appropriate powers of z and then taking summation over all possible values of n, we get the p.g.f. of $[\phi_n]$ given by

$$\phi(z) = \frac{1-(\bar{p}\alpha_0)^J}{1-\bar{p}\alpha_0}(V^*(a(z)) - \alpha_0) + \left(\frac{p\left(\alpha_0 - \bar{p}^{J-1}\alpha_0^J\right)}{1-\bar{p}\alpha_0} + \bar{p}^{J-1}\alpha_0^J\right)X(z), \tag{37}$$

which leads to

$$E[\phi] = \frac{(1-(\bar{p}\alpha_0)^J)\lambda E[X]E[V]}{1-\bar{p}\alpha_0} + \left(\frac{p(\alpha_0 - \bar{p}^{J-1}\alpha_0^J)}{1-\bar{p}\alpha_0} + \bar{p}^{J-1}\alpha_0^J\right)E[X]. \tag{38}$$

Noting that (37) represents the p.g.f. of the number of customers in the system at the completion epoch of the idle period and this is equivalent to the p.g.f. of the system size distribution at busy period initiation epoch.

Remark 7. Substituting $p = 0$ and $J = 1$ into (37), our system can be reduced to the ordinary $M^{[x]}/G/1$ single vacation policy queue and it gives

$$\phi(z) = V^*(a(z)) + \alpha_0(X(z) - 1),$$

which is in accordance with Choudhury's system [18].

Remark 8. As $p = 0$, our system can be simplified to the ordinary $M^{[x]}/G/1$ vacation policy queue and with at most J vacations. Eq. (37) can be rewritten as

$$\phi(z) = \frac{1-\alpha_0^J}{1-\alpha_0}(V^*(a(z)) - \alpha_0) + \alpha_0^J X(z),$$

which is in accordance with Ke and Chu [15].

3.5. System size distribution due to idle period

Let us define ξ_n $(n = 0, 1, 2\ldots)$ as the probability that a batch of n customers arrived before a tagged customer during the forward recurrence time (residual life) of the idle period where the tagged customer arrived. The batch of arriving customers associated with the tagged customer is randomly chosen from the arriving batch that occurs at the completion epoch of the idle period (busy period initiation epoch). Following arguments of Burke [20] and applying renewal theory, we obtain the p.g.f of the number of customers that arrive during the residual life of the idle period given by

$$\xi(z) = \frac{(1-\phi(z))}{(1-z)E[\phi]}. \tag{39}$$

From (37), $\xi(z)$ can be expressed as

$$\xi(z) = \frac{1 - \bar{p}\alpha_0 - (1-(\bar{p}\alpha_0)^J)(V^*(a(z)) - \alpha_0) - (1-\bar{p}\alpha_0)\left[\frac{p(\alpha_0 - \bar{p}^{J-1}\alpha_0^J)}{1-\bar{p}\alpha_0} + \bar{p}^{J-1}\alpha_0^J\right]X(z)}{E[X]\left((1-(\bar{p}\alpha_0)^J)\lambda E[V] + (1-\bar{p}\alpha_0)\left[\frac{p(\alpha_0-\bar{p}^{J-1}\alpha_0^J)}{1-\bar{p}\alpha_0} + \bar{p}^{J-1}\alpha_0^J\right]\right)(1-z)}. \tag{40}$$

Noting that Eq. (40) is the p.g.f. of the number of customers that arrive during a time interval from the beginning of the idle period to a random point in the idle period. We may view it as the system size distribution due to the idle period including vacation times.

3.6. Busy period and idle period distribution

Let $B^*(\theta)$ and $I^*(\theta)$ represent the LST of the busy period and idle period for the $M^{[x]}/G/1/VAC(J)$ queueing system. Utilizing the arguments by Takagi [3, Section 2.2] and system definition, $B^*(\theta)$ and $I^*(\theta)$ can be expressed as

$$B^*(\theta) = \frac{1-(\bar{p}\alpha_0)^J}{1-\bar{p}\alpha_0}(V^*(\lambda(1 - X(B_0^*(\theta)))) - \alpha_0) + \left(\frac{p(\alpha_0 - \bar{p}^{J-1}\alpha_0^J)}{1-\bar{p}\alpha_0} + \bar{p}^{J-1}\alpha_0^J\right)X(B_0^*(\theta)) \tag{41}$$

and

$$I^*(\theta) = \frac{1-(\bar{p}V^*(\theta+\lambda))^J}{1-\bar{p}V^*(\theta+\lambda)}(V^*(\theta) - V^*(\theta+\lambda)) + \left(\frac{p(V^*(\theta+\lambda) - \bar{p}^{J-1}(V^*(\theta+\lambda))^J)}{1-\bar{p}V^*(\theta+\lambda)} + \bar{p}^{J-1}(V^*(\theta+\lambda))^J\right)\left(\frac{\lambda}{\lambda+\theta}\right), \tag{42}$$

where $B_0^*(\theta) = S^*(\theta + \lambda - \lambda X(B_0^*(\theta)))$ is the LST of the busy period initiated by a single customer in the ordinary $M^{[x]}/G/1$ queueing model.

Now, we further define the following:

$E[B] \equiv$ the expected length of busy period,
$E[I] \equiv$ the expected length of idle period,
$E[C] \equiv$ the expected length of busy cycle.

Employing (41) and (42), we obtain

$$E[B] = \left(\frac{(1-(\bar{p}\alpha_0)^J)\lambda E[V]}{1-\bar{p}\alpha_0} + \frac{p(\alpha_0 - \bar{p}^{J-1}\alpha_0^J)}{1-\bar{p}\alpha_0} + \bar{p}^{J-1}\alpha_0^J\right)\left(\frac{E[X]E[S]}{1-\rho}\right), \tag{43}$$

$$E[I] = \frac{1}{(1-\bar{p}\alpha_0)^2}\left\{-J(\bar{p}\alpha_0)^{J-1}x\alpha_0(1-\bar{p}\alpha_0) + [1-(\bar{p}\alpha_0)^J](\bar{p}x\alpha_0)\right\}(1-\alpha_0) + \frac{1-(\bar{p}\alpha_0)^J}{1-(\bar{p}\alpha_0)}(E[V]-x\alpha_0)$$
$$+\frac{1}{(1-\bar{p}\alpha_0)^2}\left(p[x\alpha_0(1-\bar{p}^{J-1}J\alpha_0^{J-1})](1-\bar{p}\alpha_0) + p(\alpha_0-\bar{p}^{J-1}\alpha_0^J)(\bar{p}x\alpha_0)\right) + \frac{1}{\lambda}\bar{p}^{J-1}J\alpha_0^{J-1}x\alpha_0$$
$$+\frac{1}{\lambda}\left\{\frac{p\left(\alpha_0-\bar{p}^{J-1}\alpha_0^J\right)}{1-\bar{p}\alpha_0} + \bar{p}^{J-1}\alpha_0^J\right\}, \tag{44}$$

and

$$E[C] = E[B] + E[I]. \tag{45}$$

Remark 9. In Eq. (42), if we let $p = 0$ and $J \to \infty$, can be reduced to

$$\frac{V^*(\theta) - V^*(\theta+\lambda)}{1 - V^*(\theta+\lambda)},$$

which is in accordance with Takagi [3].

Remark 10. In Eq. (42), if we let $p = 0$ and $J = 1$, can be simplified to

$$V^*(\theta) - \frac{\theta V^*(\theta+\lambda)}{\theta+\lambda},$$

which is in accordance with Takagi [3].

4. Optimal randomized control policy

In this section, we develop the long-run expected cost function per unit time for the $M^{[X]}/G/1/VAC(J)$ queueing system, in which p and J are the decision variables. Our objective is to determine the suitable values of the control variables p and J, say p^* and J^*, so as to minimize the cost function. Let us define the following cost elements:

$C_h \equiv$ holding cost per unit time per customer present in the system;
$C_s \equiv$ set-up cost per busy cycle.

By using the renewal reward theory, we know that the long-run expected cost per unit time is given by

$$F(p,J) = C_h L_s + \frac{C_s}{E[C]} = A_1 + \frac{A_2(1-(\bar{p}\alpha_0)^J) + A_3(1-\bar{p}\alpha_0)}{B_1(1-(\bar{p}\alpha_0)^J) + \alpha_0\left(p(1-(\bar{p}\alpha_0)^{J-1}) + (\bar{p}\alpha_0)^{J-1}B_2\right)}, \tag{46}$$

where

$$A_1 = C_h \times L_{M^{[X]}/G/1},$$
$$A_2 = \frac{C_h\lambda^2 E[X]E[V^2]}{2},$$
$$A_3 = C_s\lambda(1-\rho),$$
$$B_1 = \lambda E[V], \quad \text{and}$$
$$B_2 = (1-\bar{p}\alpha_0)$$

with $L_{M^{[X]}/G/1} = \rho + \frac{\lambda E[X(X-1)]E[S] + (\lambda E[X])^2 E[S^2]}{2(1-\rho)}$.

For analysis, J may be treated as a continuous variable greater than zero. Noting that that if J^* is not an integer, the best positive integer value of J is one of the integers surrounding J^*. Differentiating $F(p,J)$ (in Eq. (46)) with respect to p and J, respectively, it gives

$$\frac{\partial F(p,J)}{\partial p} = \frac{D_1}{B_1(1-(\bar{p}\alpha_0)^J) + \alpha_0\left(p(1-(\bar{p}\alpha_0)^{J-1}) + (\bar{p}\alpha_0)^{J-1}B_2\right)} \tag{47}$$

and

$$\frac{\partial F(p,J)}{\partial J} = -(ln(\bar{p}\alpha_0))(\bar{p}\alpha_0)^J \times \frac{D_2}{B_1(1-(\bar{p}\alpha_0)^J) + \alpha_0\left(p(1-(\bar{p}\alpha_0)^{J-1}) + (\bar{p}\alpha_0)^{J-1}B_2\right)}, \tag{48}$$

where

$$D_1 = \left(A_2\alpha_0^J J\bar{p}^{J-1} + A_3\alpha_0\right)\left(B_1(1-(\bar{p}\alpha_0)^J) + \alpha_0\left(p(1-(\bar{p}\alpha_0)^{J-1}) + (\bar{p}\alpha_0)^{J-1}(1-\bar{p}\alpha_0)\right)\right)$$
$$- \left(A_2(1-(\bar{p}\alpha_0)^J) + A_3(1-\bar{p}\alpha_0)\right) \times \left(-B_1\alpha_0^J J\bar{p}^{J-1} + \alpha_0\left((1-(\bar{p}\alpha_0)^{J-1}) + p(J-1)\alpha_0^{J-1}\bar{p}^{J-2}\right)\right.$$
$$\left. +\alpha_0\left(-\alpha_0^{J-1}(J-1)\bar{p}^{J-2}(1-\bar{p}\alpha_0) + (\bar{p}\alpha_0)^{J-1}\alpha_0\right)\right)$$

and

$$D_2 = A_2B_2\alpha_0\left((\bar{p}\alpha_0)^{J-1} + \frac{1-(\bar{p}\alpha_0)^J}{\bar{p}\alpha_0}\right) + A_2\alpha_0 p\left((1-(\bar{p}\alpha_0)^{J-1}) - \frac{1-(\bar{p}\alpha_0)^J}{\bar{p}\alpha_0}\right) - A_3(1-\bar{p}\alpha_0)((B_1-B_2)-(B_1-1)p). \tag{49}$$

For any given J, we know that:

1a. If $D_1 > 0$ in $p \in (0,1)$, by using (47) yields $\frac{\partial F(p,J)}{\partial p} > 0$ which means $F(p,J)$ is an increasing function of p in $p \in (0,1)$,
2a. If $D_1 < 0$ in $p \in (0,1)$, by using (47) yields $\frac{\partial F(p,J)}{\partial p} < 0$ which implies $F(p,J)$ is a decreasing function of p in $p \in (0,1)$.
3a. Noting that $\frac{\partial F(p,J)}{\partial p} = 0$ iff $D_1 = 0$ (in this case, p^* is arbitrary value between 0 and 1). This case is a rare event (see the structure of D_1). That is, the occurrence of D_1 is very small.

For any given p, we also know that:

1b. If $D_2 > 0$, by using (48) yields $\frac{\partial F(p,J)}{\partial J} > 0$ which means $F(p,J)$ is an increasing function of J,
2b. If $D_2 < 0$, by using (48) yields $\frac{\partial F(p,J)}{\partial J} < 0$ which implies $F(p,J)$ is a decreasing function of J.
3b. Noting that $\frac{\partial F(p,J)}{\partial J} = 0$ iff $D_2 = 0$ (in this case, J^* is arbitrary positive integer or $J = \infty$). Noting that the occurrence of the case $D_2 = 0$ is very small.

In order to find the joint optimal values of p and J, say p^* and J^*, we should solve the following equations:

$$\frac{\partial F(p,J)}{\partial p} = 0 \quad \text{and} \quad \frac{\partial F(p,J)}{\partial J} = 0. \tag{50}$$

The solutions $(p,J) = (p^*,J^*)$ attain a local minimum if it satisfies the following:

$$\frac{\partial^2 F(p,J)}{\partial p^2} > 0, \tag{51}$$
$$\frac{\partial^2 F(p,J)}{\partial J^2} > 0 \tag{52}$$

and the determinant of the Hessian matrix is positive definite, that is,

$$det(H) = \frac{\partial^2 F(p,J)}{\partial p^2} \cdot \frac{\partial^2 F(p,J)}{\partial J^2} - \left(\frac{\partial^2 F(p,J)}{\partial p \partial J}\right)^2 > 0. \tag{53}$$

Noting that (50) is just necessary conditions for $F(p,J)$ to attain it's minimum. Although we cannot analytically prove that $F(p,J)$ is a convex function of (p,J) indeed, one heuristic approach is provided to search the joint optimum values of p and J. By the inferences listed above we know that for a given p, the optimal values J^* of J is $J^* = 1$, arbitrary positive integer or $J^* = \infty$ (say M, where M is a sufficiently large number in practice). A heuristic decision is summarized in the following that makes it possible to determine the joint suitable values (p^*,J^*) as follows:

The criterion to search the joint suitable values p^* and J^*:

Case 1 If $D_1 > 0$ and
 i. $D_2 > 0$ then $(p^*,J^*) = (0,1)$
 ii. $D_2 < 0$ then $(p^*,J^*) = (0,\infty)$
 iii. $D_2 = 0$ then $(p^*,J^*) = (0$, any positive integer$)$

Case 2 If $D_1 < 0$ and
 i. $D_2 > 0$ then $(p^*,J^*) = (1,1)$
 ii. $D_2 < 0$ then $(p^*,J^*) = (1,\infty)$
 iii. $D_2 = 0$ then $(p^*,J^*) = (1$, any positive integer$)$

Case 3 If $D_1 = 0$ and

i. $D_2 > 0$ then (p^*, J^*) =(any value between 0 and 1, 1)
ii. $D_2 < 0$ then (p^*, J^*) =(any value between 0 and 1, ∞)
iii. $D_2 = 0$ then (p^*, J^*)=(any value between 0 and 1, any positive integer)

Remark 11. It should be noted that it is a rare event for the case $D_1 = 0$ or $D_2 = 0$.

5. Numerical Illustration

The first purpose of this section is to study the effects of some parameters on the expected number of customers in the system (L_s) and the expected waiting time of customers in the system (W_s).

For convenience, we first let

1. $p = 0.5$;
2. $\lambda = 0.4$;
3. $X \equiv$ geometric distribution with parameter 0.5 (i.e., $Geo(0.5)$);
4. $S \equiv$ 4-stage Erlang distribution with a mean $E[S] = 0.5$.

Our first set of numerical example performs the above specific parameters by varying J from 1 to 100 and various vacation time distributions with $E[V] = 1$. The vacation times are considered to be exponential (M), 2-stage Erlang (E_2) and hyper-exponential (H_2), respectively. The effects of different values of J and three vacation distributions on L_s are shown in Fig. 1. Fig. 1 reports that L_s first increases as J increases and then becomes stably as J becomes large. One also observes that the three vacation distributions by their relative magnitudes on L_s produce $H_2 > M > E_2$.

A second set of numerical example performs the above specific parameters by varying p from 0.0 to 1.0 and choosing $J = 10$. The setting of vacation parameters are the same as preceding one. From Fig. 2, one sees that L_s decreases as p increases. Also, we observe that the three vacation distributions by their relative magnitudes on L_s produce $H_2 > M > E_2$.

The third set of numerical example is to investigate the cases that the effect of different values of p and different vacation time distributions on L_s and W_s. Three vacation time distributions with $E[V] = 2$ are considered at $J = 10$. Table 1 clearly shows that L_s and W_s decrease as p increases for various vacation time distributions. It also reveals that when p changes from 0.0 to 1.0, the three vacation time distributions by their relative magnitudes on L_s and W_s produce $H_2 > M \geqslant E_2$.

For the fourth set of numerical example, we deal with the cases that the effect of various service time distributions and different service rates on L_s and W_s. The service time distributions are consider to be exponential, 2-stage Erlang and hyper-exponential with vacation rate is 0.5, respectively, at $J = 10$. The effect of the service time distributions and different service rates on L_s and W_s are listed in Table 2. From Table 2, the comparison of L_s for the three service time distributions M, E_2 and H_2, showed the results when service rate changes from 1 to 10 and we observe that the three service distributions by their relative magnitudes on L_s and W_s produce $H_2 > M \geqslant E_2$.

For the last set of numerical example, we study the effect of the vacation time distributions and different vacation rates on L_s and W_s, which are summarized in Table 3. The comparison of L_s and W_s for the three vacation time distributions M, E_2 and

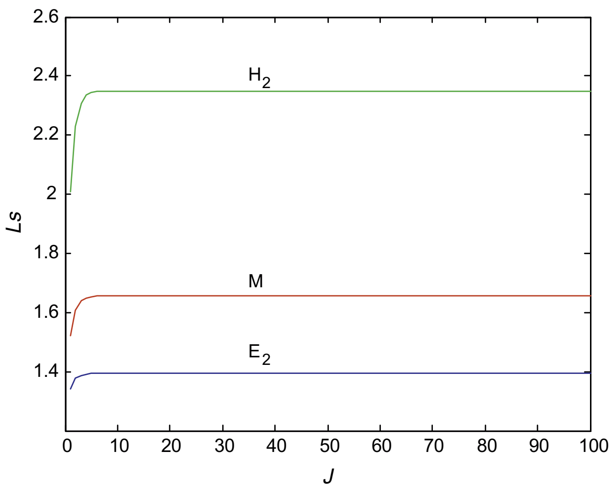

Fig. 1. The expected system sizes L_s for different values of J and three vacation distributions (exponential (M), 2-stage Erlang (E_2) and hyper-exponential (H_2)).

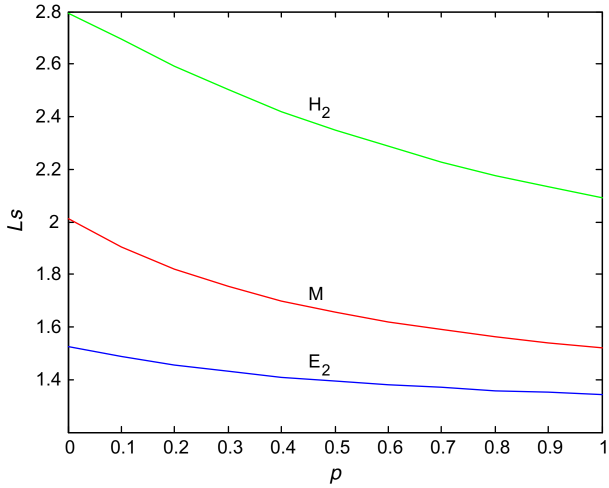

Fig. 2. The expected system sizes Ltextsubscripts for different values of p and three vacation distributions (exponential (M), 2-stage Erlang (E_2) and hyper-exponential (H_2)).

Table 1
The expected system sizes L_s and expected waiting time W_s for different p and different vacation distributions ($\lambda = 0.4$, $X \equiv Geo(0.5)$, $J = 10$, $S \equiv E_4$ with $E[S] = 0.5$).

p		$V \equiv M$	$V \equiv E_2$	$V \equiv H_2$
0.0	L_s	2.83	1.83	4.43
	W_s	3.54	2.29	5.54
0.1	L_s	2.73	1.80	4.33
	W_s	3.41	2.25	5.41
0.2	L_s	2.64	1.77	4.24
	W_s	3.30	2.21	5.30
0.3	L_s	2.56	1.74	4.15
	W_s	3.20	2.18	5.19
0.4	L_s	2.49	1.71	4.07
	W_s	3.11	2.14	5.09
0.5	L_s	2.42	1.69	3.99
	W_s	3.03	2.11	4.99
0.6	L_s	2.36	1.67	3.92
	W_s	2.95	2.09	4.90
0.7	L_s	2.31	1.65	3.85
	W_s	2.89	2.06	4.81
0.8	L_s	2.26	1.63	3.78
	W_s	2.83	2.04	4.73
0.9	L_s	2.22	1.61	3.72
	W_s	2.78	2.01	4.65
1.0	L_s	2.18	1.60	3.66
	W_s	2.73	2.00	4.58

H_2, showed the results when vacation rate changes from 1.0 to 10.0 and we observe that the three service distributions by their relative magnitudes on L_s and W_s produce $H_2 > M \geqslant E_2$.

The above numerical investigations indicate that (i) the vacation times have a significant effect on the expected number of customers (or waiting time of customers) than J or p; and (ii) when all parameters are given, the impacts of the service (or vacation) distributions on system characteristics are not significantly for larger service (vacation) rate.

The second purpose of this section is to perform two extensive examples to illustrate the joint optimum randomized behavior as discussed in Section 4.

We first consider the case of $D_1 > 0$ and $D_2 < 0$ with the setting system's parameters as follows:

1. $p = 0.5$;
2. Batch size distribution of the arrival is geometric with mean $E[X] = 2$;

Table 2
The expected system sizes L_s and expected waiting time W_s for different service distributions ($p = 0.5$, $\lambda = 0.4$, $X \equiv Geo(0.5)$, $J = 10$, $V \equiv M$ with $E[V] = 1$).

$\frac{1}{E[S]}$		$S \equiv M$	$S \equiv E_2$	$S \equiv H_2$
1.0	L_s	8.42	7.62	22.02
	W_s	10.53	9.53	27.53
2.0	L_s	1.76	1.69	3.22
	W_s	2.20	2.11	4.03
3.0	L_s	1.15	1.13	1.80
	W_s	1.44	1.41	2.25
4.0	L_s	0.92	0.91	1.32
	W_s	1.15	1.14	1.65
5.0	L_s	0.80	0.80	1.09
	W_s	1.00	1.00	1.36
6.0	L_s	0.73	0.73	0.95
	W_s	0.91	0.91	1.19
7.0	L_s	0.68	0.68	0.85
	W_s	0.85	0.85	1.06
8.0	L_s	0.64	0.64	0.79
	W_s	0.80	0.80	0.99
9.0	L_s	0.62	0.62	0.74
	W_s	0.78	0.78	0.93
10.0	L_s	0.60	0.59	0.70
	W_s	0.75	0.74	0.88

Table 3
The expected system sizes L_s and expected waiting time W_s for different vacation distributions ($p = 0.5$, $\lambda = 0.4$, $X \equiv Geo(0.5)$, $J = 10$, $S \equiv E_4$ with $E[S] = 0.5$).

$\frac{1}{E[V]}$		$V \equiv M$	$V \equiv E_2$	$V \equiv H_2$
1.0	L_s	1.66	1.39	2.42
	W_s	2.08	1.74	3.03
2.0	L_s	1.36	1.28	1.66
	W_s	1.70	1.60	2.08
3.0	L_s	1.30	1.26	1.45
	W_s	1.63	1.58	1.81
4.0	L_s	1.27	1.25	1.36
	W_s	1.59	1.56	1.70
5.0	L_s	1.26	1.24	1.32
	W_s	1.58	1.55	1.65
6.0	L_s	1.25	1.24	1.30
	W_s	1.56	1.55	1.63
7.0	L_s	1.25	1.24	1.28
	W_s	1.56	1.55	1.60
8.0	L_s	1.24	1.24	1.27
	W_s	1.55	1.55	1.59
9.0	L_s	1.24	1.24	1.26
	W_s	1.55	1.55	1.58
10.0	L_s	1.24	1.24	1.26
	W_s	1.55	1.55	1.58

3. $\lambda = 0.6$;
4. $V \equiv$ exponential distribution with a mean $E[V] = 1$;
5. $S \equiv$ 2-stage Erlang distribution with a mean $E[S] = 0.5$;
6. the holding cost $C_h = 10$;
7. the set-up cost $C_s = 1000$.

The expected cost $F(p,J)$ for this case is shown in Fig. 3. Noting that the minimum cost per unit time of \$188.6047 is achieved at $p^* = 0$ and J^* is 6. The results also make it obvious that (i) the expected cost increases as p increases; and (ii) J decreases as p increases.

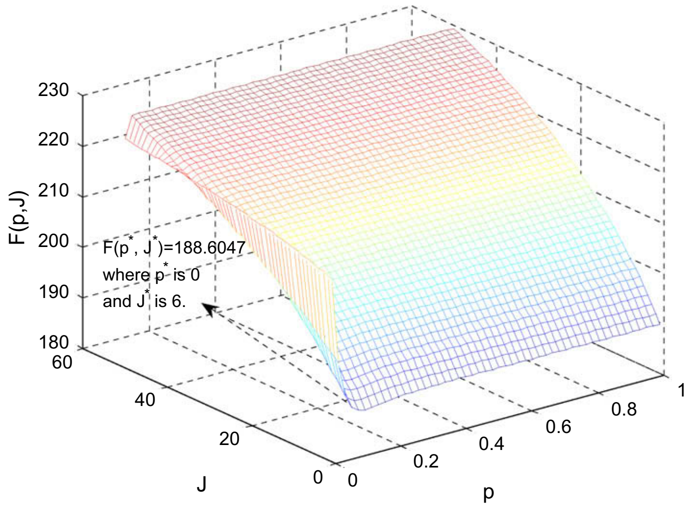

Fig. 3. The expected cost for different values p and J ($D_1 > 0$ and $D_2 < 0$).

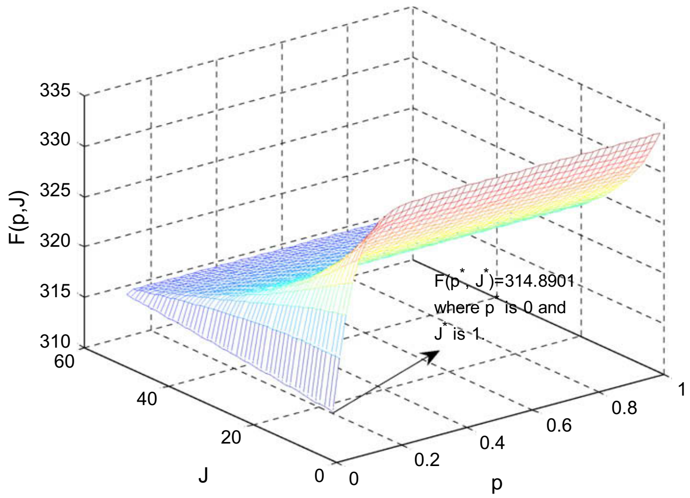

Fig. 4. The expected cost for different values p and J ($D_1 > 0$ and $D_2 > 0$).

The second example is the case of $D_1 > 0$ and $D_2 > 0$ with the following system's parameters:

1. $p = 0.5$;
2. Batch size distribution of the arrival is geometric with mean $E[X] = 3$;
3. $\lambda = 0.6$;
4. $V \equiv$ exponential distribution with a mean $E[V] = 0.5$;
5. $S \equiv$ 2-stage Erlang distribution with a mean $E[S] = 0.5$;
6. the holding cost $C_h = 10$;
7. the set-up cost $C_s = 1000$.

It is seen from Fig. 4, for the $D_1 > 0$ and $D_2 > 0$ case, that a minimum cost value per unit time of $314.8901 is achieved at $p^* = 0$ and $J^* = 1$.

The numerical results agree with the conclusion in preceding section (i.e., the joint optimal values is $(0,1)$-single vacation, $(0,M)$-multiple vacation, $(1,1)$-single vacation, or $(1,M)$-single vacation, where M is a sufficiently large number for practice use). These special policies can be also referred to Remarks 2 and 3 This implies the optimal vacation policy is exactly as single vacation or multiple vacation policy.

6. Conclusions

This paper we address an $M^{[X]}/G/1/VAC(J)$ queueing system, in which the server applies a randomized vacation policy with at most J vacations in his idle period. Some important system characteristics are derived. A cost model is developed to determine the optimum vacation policy. By using the analytic properties of the cost function, we develop an efficient decision criterion for searching the joint suitable value of (p, J). Some numerical examples are performed to investigate the effects of some parameters on the expected number of customers in the system and the expected waiting time of customers in the system. We also perform two extensive numerical examples to illustrate the optimization approach. This research presents an extension of the vacation model theory and the analysis of the model will provide a useful performance evaluation tool for more general situations arising in practical applications.

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