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Shilnikov's Cross-map Method and Hyperbolic Dynamics of Three-dimensional Hénon-like Maps

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Abstract—We study the hyperbolic dynamics of three-dimensional quadratic maps with constant Jacobian the inverse of which are again quadratic maps (the so-called 3D Hénon maps). We consider two classes of such maps having applications to the nonlinear dynamics and find certain sufficient conditions under which the maps possess hyperbolic nonwandering sets topologically conjugating to the Smale horseshoe. We apply the so-called Shilnikov's cross-map for proving the existence of the horseshoes and show the existence of horseshoes of various types: (2,1)- and (1,2)-horseshoes (where the first (second) index denotes the dimension of stable (unstable) manifolds of horseshoe orbits) as well as horseshoes of saddle and saddle-focus types.

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1. INTRODUCTION

The Theory of Dynamical Systems has been formed as a separate branch of science in the 1960s–1970s. Fundamentally, this event happened as the result of the so-called "hyperbolic revolution", i.e. very interesting and quite short period (of the 60-s) when numerous problems of natural science got a solid mathematical base. In turn, this base – the Hyperbolic Theory – was born so rapidly due to long standing scrupulous work of such distinguish scientists as H. Poincaré, J. Hadamard, J. Birkhoff, A.A. Andronov etc. Due to this activity, there were collected a lot of scientific facts and important problems which have required an explanation by the mathematical language. One can point out three such famous problems which have led tightly to the creation of Hyperbolic Theory. The first one is the *Poincaré–Birkhoff problem* on the structure of orbits entirely lying near a transverse Poincaré homoclinic orbit. The second problem (one of the most famous problems in differential equations) was connected with the study of *geodesic flows on the manifolds of constant negative curvature*. Finally, the third problem selected by us is the *Andronov hypothesis* on the existence of structurally stable systems with infinitely many periodic orbits.

The first two problems had a quite long-time history, many important partial results were obtained by H. Poincaré, J. Hadamard, J. Birkhoff, M. Morse, G. Hedlund et al. However, the complete solution of these problems was found only in the 1960s: by L.Shilnikov who solved the Poincaré–Birkhoff problem, [1], and by D.Anosov who solved the problem on geodesic flows, [2]. As is well known, the positive answer on the Andronov hypothesis was obtained, in fact, by S. Smale who introduced the famous "horseshoe C^∞-homeomorphism" on S^2, [3]. Indeed, in this horseshoe map, the "evident" structural stability was combined with the presence of infinitely many (saddle) periodic orbits. However, a justification of the pointed out "evidence" required a

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certain mathematical formalization including, in particular, the introduction, by D. Anosov [4], of such basic notation as the *hyperbolic set*. On the basis of this, many mathematical methods for establishing hyperbolicity were created, e.g. the cone theory, the shadow lemma (also referred to as the (Anosov) lemma on ε-trajectories) etc.

In this series of methods, a special value have the so-called *Shilnikov's cross-map method*. Namely, this method was used by L. Shilnikov for the complete proof of the mentioned Poincaré–Birkhoff problem. The essence of the method can be illustrated by the simplest example of a map having a fixed point. Let this C^1-map have a form $\bar{x} = f(x,y)$, $\bar{y} = g(x,y)$ and let $O(0,0)$ be its the fixed point. Suppose that the second equation can be resolved with respect to the coordinate y (near O), i.e. $y = \tilde{g}(x,\bar{y})$. Then, the map can be rewritten in the cross form as $\bar{x} = \tilde{f}(x,\bar{y})$, $y = \tilde{g}(x,\bar{y})$, where $\tilde{f}(x,\bar{y}) \equiv f(x,\tilde{g}(x,\bar{y}))$. Then, it is evident that

- *if the cross map $(x,\bar{y}) \mapsto (\bar{x},y)$ is contracting (near $(0,0)$), then the fixed point O is a hyperbolic saddle.*

However, the given example corresponds only to one of the simplest and not interesting applications of the method. There are more important applications including non-uniformly hyperbolic situations (see, e.g. [5]). For more information on the Shilnikov's method see Section 3.2 (see also [6]).

In this paper, we will apply the Shilnikov's cross-map method for finding some sufficient conditions for hyperbolicity of certain three-dimensional quadratic maps.

We will say that a map (diffeomorphism, in fact) defined on R^3 is a *three-dimensional Hénon map* (or a *3D Hénon map*) if the following conditions hold: (i) the map has the constant Jacobian; (ii) the map and its inverse map are both quadratic; (iii) the coordinates are not decoupled by the action of the map.[1)]

The class of 3D Hénon maps is sufficiently vast: any three-dimensional Hénon map can be brought (by affine transformations) to the following standard form, [7, 8],

$$\begin{aligned} &\bar{x} = y, \ \ \bar{y} = z, \\ &\bar{z} = M_1 + Bx + M_2^{(1)} y + M_2^{(2)} z + \hat{a} y^2 + \hat{b} yz + \hat{c} z^2, \end{aligned} \tag{1.1}$$

where B is the Jacobian. Formally speaking, map (1.1) depends on 7 parameters. However, in any case, only 5 of them can be considered as independent. Indeed, by an appropriate rescaling $(x,y,z) \mapsto \alpha(x,y,z)$, we can make equal to ± 1 one of the coefficients $\hat{a}, \hat{b}$ or $\hat{c}$ of the quadratic form; if $\hat{a}$ or $\hat{c}$ is non-zero, we can vanish identically (by a coordinate shift) the coefficient $M_2^{(1)}$ or $M_2^{(2)}$, respectively.

In the paper we will consider two classes of 3D Hénon maps. The first class corresponds to the case where $\hat{a} = \hat{b} = 0, \hat{c} \neq 0$. Thus, in this case, equations (1) define the following three parameter family

$$\bar{x} = y, \ \ \bar{y} = z, \ \ \bar{z} = M + Bx - by - z^2. \tag{1.2}$$

The second class corresponds to the case where $\hat{b} = \hat{c} = 0, \hat{a} \neq 0$ and, thus, equations (1) define here the following three parameter family[2)]

$$\bar{x} = y, \ \ \bar{y} = z, \ \ \bar{z} = M + Bx + Cz - y^2. \tag{1.3}$$

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1) Note that not all quadratic maps with constant Jacobian are of Hénon type. So, for example, the map $(\bar{x},\bar{y},\bar{z}) = (y, x+y^2, z+x^2)$ has an inverse map which is not quadratic; the maps $(\bar{x},\bar{y},\bar{z}) = (x, -z, y+z+x^2)$ and $(\bar{x},\bar{y},\bar{z}) = (y+x, -x, z+y^2)$ decouple the coordinates: the image $\bar{x}$ depends only on x for the first map and the image $(\bar{x},\bar{y})$ depends only on (x,y) for the second map.

2) In the case of equation (1.2), we take $\hat{c} \equiv -1$, $\hat{M}_2^{(2)} \equiv 0$, $M = M_1$ and $\hat{M}_2^{(1)} = -b$ in (1.1); in the case of equation (1.3), we take $M = M_1, \hat{a} \equiv -1, \hat{M}_2^{(1)} \equiv 0$ and $C = \hat{M}_2^{(2)}$, respectively. Thus, if $B = 0$ the variable x decouples and map (1.2) degenerates, in fact, into the standard Hénon map $\bar{y} = z$, $\bar{z} = M - by - z^2$, whereas, map (1.3) degenerates into the Mirá map $\bar{y} = z$, $\bar{z} = M + Cz - y^2$.

Formally speaking, maps (1.2) and (1.3) are quite specific cases (codimension 2 subfamilies) of the general family (1.1) of 3D Hénon maps. However, they both have an exceptional value for the nonlinear dynamics.

First, these maps play an important role in the bifurcation theory of multidimensional systems with homoclinic orbits. It was shown in [8–11] that maps (1.2) and (1.3) are normal forms of first return maps near certain quadratic homoclinic tangencies. It means that both the maps can be considered as *universal quadratic maps* like the well-known maps: the parabola map $\bar{x} = M - x^2$, the standard Hénon map $\bar{x} = y,\ \bar{y} = M - bx - y^2$ and the Mirá map $\bar{x} = y,\ \bar{y} = M + Cy - x^2$; as well as the generalized Hénon maps (GHM) of the form

$$\bar{x} = y, \quad \bar{y} = M - bx - y^2 + \epsilon_1 xy + \epsilon_2 y^3. \tag{1.4}$$

The latter map was introduced in [12, 13] as the rescaled "homoclinic map" (see also [10]). Note that, unlike the standard Hénon map, it demonstrates nondegenerate Andronov-Hopf bifurcations occurring near $b = 1$; see [13, 14].

Second, the dynamics of maps (1.2) and (1.3) is quite interesting by itself, since, for example, both these maps possess strange attractors. Notice that map (1.3) was introduced in [9] and also this map written in a specific form is known as the ACT-map (Arneodo–Coullet–Tresser map), whose chaotic dynamics was studied in [15]. We note especially that map (1.2) demonstrates very interesting type of chaotic dynamics. Namely, it was shown in [8, 16] that map (1.2) has the so-called *wild Lorenz-like attractor* for values of the parameters from some open region. The question of the existence of such attractors in map (1.3) is still open (note that this map possesses a wild Lorenz-like repeller, since map (1.3) is the inverse of (1.2)). The notation of wild hyperbolic attractor was introduced in [17]: briefly, it is a strange attractor which allows homoclinic tangencies but does not contain stable periodic orbits, and this property holds for all close systems.[3)] The mentioned wild Lorenz-like attractors compose very important class of wild hyperbolic attractors: in the sense that they can be considered as periodically perturbed classical Lorenz attractors of flows; see [16, 19] for more detail.

And finally, maps (1.2) and (1.3) can be regarded as the simplest in form three-dimensional nonlinear maps. Therefore, one can apply various standard qualitative, analytical and numerical methods when studying their dynamics. Our previous expertise shows that one can obtain rather interesting results in this way; see e.g. [8, 15, 16].

Note that the present paper was essentially stimulated by our recent results, [20], obtained when studying hyperbolic dynamics of GHM (1.4) at small ϵ_1, ϵ_2 and b. It was shown in [20] that horseshoes of new types, the so-called half-orientable horseshoes, appear here, see also [21, 22]. In contrast to the standard Smale horseshoes, the half-orientable ones are divided into infinitely many equivalence classes with respect to such an equivalence relation as the local topological conjugacy.

These results cause natural questions about types of multidimensional horseshoes. In this paper we study these questions for three-dimensional Hénon-like maps. We show, in particular, that 3D Hénon maps (1.2) and (1.3) can possess horseshoes of different types. Map (1.2) has (2,1)-horseshoes, whereas, map (1.3) has (1,2)-horseshoes, where the first (second) index denotes the dimension of the stable (unstable) manifolds of points of horseshoes.

2. STATEMENT OF THE PROBLEM AND MAIN RESULTS

In the present paper, we study hyperbolic dynamics of 3D Hénon maps (1.2) and (1.3). We find certain regions of hyperbolicity (in the parameter space) and show that, in these regions, the nonwandering sets, namely Λ_1 and Λ_2, respectively, for maps (1.2) and (1.3) are uniformly hyperbolic sets. Both Λ_1 and Λ_2 are Smale horseshoes in the sense that the restriction of the

[3)] It is not the case for the well-known Hénon-like strange attractors that are born under homoclinic bifurcations of two-dimensional diffeomorphisms [18], as they do not seem to be robust, i.e. an arbitrarily small perturbation may lead to the birth of stable periodic orbits. The same holds true for many "physical" attractors observed in numerical experiments, where a seeming chaotic behavior can easily correspond to some periodic orbit with a very large period (plus inevitable noise).

corresponding map onto Λ_i is topologically conjugate to the topological Bernoulli scheme (shift) with two symbols.

However, geometrically, the horseshoes Λ_1 and Λ_2 are quite different; see Fig. 1. Both of them can be obtained by means of the standard procedure of the Smale horseshoe creation: we start from some initial cube Q; the image of Q under the map has a form of three-dimensional horseshoe; this horseshoe intersects Q into two connected and disjoint components etc. However, map (1.2) contracts Q along two directions and expands it along one direction. Therefore, we say that the horseshoe Λ_1 has type (2,1): it contains hyperbolic orbits having two-dimensional stable and one-dimensional unstable invariant manifolds. On the contrary, map (1.3) contracts Q along only one direction and expands it along two directions. Therefore, the horseshoe Λ_2 has type (1,2): it contains hyperbolic orbits having one-dimensional stable and two-dimensional unstable invariant manifolds.

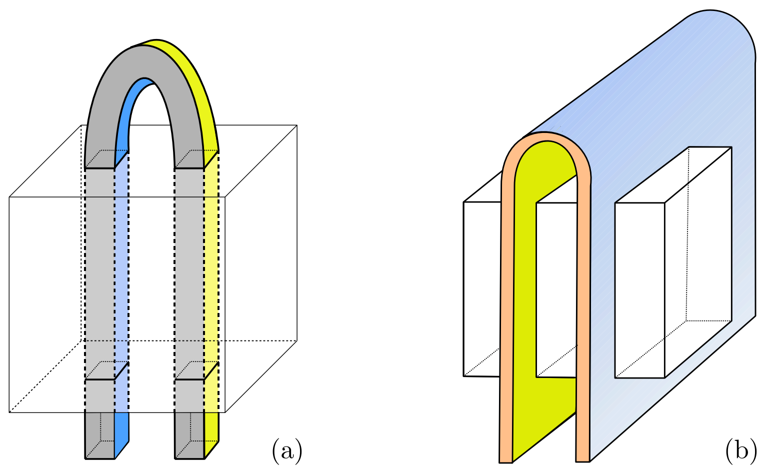

Fig. 1. A geometry of three-dimensional horseshoes of (a) type (2,1) and (b) type (1,2).

The following two theorems describe the mentioned hyperbolic regions for maps (1.2) and (1.3), respectively.

Theorem 1. *Consider map* (1.2). *Let M satisfy the inequality*

$$M > \left(\rho + \sqrt{\rho^2 + \frac{1}{4}}\right)^2 - (1 + b - B)\left(\rho + \sqrt{\rho^2 + \frac{1}{4}}\right), \tag{2.1}$$

where

$$\rho = \frac{3 + 5(|B| + |b|)}{3 + 4(|B| + |b|)}(1 + |B| + |b|).$$

Then the nonwandering set of map (1.2) *is the hyperbolic* $(2,1)$*-horseshoe* Λ_1.

Theorem 2. *Consider map* (1.3). *Let M satisfy the inequality*

$$M > \left(\tilde\rho + \sqrt{\tilde\rho^2 + \frac{1}{4}}\right)^2 - (1 - C - B)\left(\tilde\rho + \sqrt{\tilde\rho^2 + \frac{1}{4}}\right), \tag{2.2}$$

where

$$\tilde\rho = \frac{3 + 5(|B| + |C|)}{3 + 4(|B| + |C|)}(1 + |B| + |C|).$$

Then the nonwandering set of map (1.3) *is the hyperbolic* $(1,2)$*-horseshoe* Λ_2.

Although our goal is to study hyperbolic property of maps (1.2) and (1.3) directly, the most of our proofs will be related to these maps written in the so-called parabola-like form. Namely, we can represent map (1.2) in the following form (for those values of the parameters M, B and b at which map (1.2) has fixed points):

$$\bar{x} = y, \;\; \bar{y} = z, \;\; \bar{z} = \gamma z(1-z) + Bx - by. \tag{2.3}$$

Analogously, the parabola-like form of map (1.3) is as follows

$$\bar{x} = y, \;\; \bar{y} = z, \;\; \bar{z} = \gamma y(1-y) + Bx + Cz. \tag{2.4}$$

Here, $\gamma = (1+b-B) \pm \sqrt{4M + (1+b-B)^2}$ in the case of map (2.3) and $\gamma = (1-C-B) \pm \sqrt{4M + (1-C-B)^2}$ in the case of map (2.4).

In Section 3, we will find relations between the parameters γ, B, b and γ, B, C of maps (2.3) and (2.4), respectively, which guarantee the existence of horseshoes Λ_1 for (2.3) and Λ_2 for (2.4) as the unique nonwandering sets. Then, as applications, in Section 2.6, we prove Theorems 1 and 2.

Note that the parabola-like form of the maps is quite suitable for detecting nonwandering sets, since they always occupy, for all γ, a rather restricted domain of the phase space. So, the initial box Q, for the maps written in the parabola-like form, can be always chosen as the cube with sizes of length $1+\beta$ centered at the point $(1/2, 1/2, 1/2)$, where β is positive and not very large ($\beta \to +0$ as $|B| + |b| \to 0$ or $|B| + |C| \to 0$ and always $\beta < 1/4$). However, when the maps are written in the standard form we must consider some "flatting" cube $Q(\gamma, \beta)$ of sizes $\gamma(1+\beta)$ centered at the point (γ, γ, γ), where $\gamma \to +\infty$ as $M_1 \to +\infty$.

In the last part of the paper, Section 4, we will prove that the 3D Hénon maps under consideration can possess horseshoes of two different smooth types: saddle horseshoes and saddle-focus horseshoes. We will say that a horseshoe is of *saddle* type, if its both fixed points are saddles, i.e. all their three multipliers are real; and a horseshoe is of *saddle-focus* type, if at least one of its fixed points is a saddle-focus, i.e. it has a pair of complex conjugate multipliers. See also definition 2 relating to a more general situation.

Denote as H_1 and H_2 the domains of hyperbolicity (2.1) and (2.2) for maps (1.2) and (1.3), respectively. Geometrically, H_1 and H_2 look, in the space of the corresponding parameters, like similar infinite curvilinear cones. However, divisions of these regions into domains corresponding to the existence of saddle and saddle-focus horseshoes are quite different, which the following result shows (see the proof in Section 4).

Theorem 3. *The regions H_1 and H_2 are divided into 6 open subregions $S_1^+, S_1^-, S_2^+, S_2^-$ and SF^+, SF^- according to Fig. 2 such that the horseshoe (Λ_1 for H_1 or Λ_2 for H_2) is of saddle type for $S_i^+ \cup S_i^-, i = 1, 2$, and of saddle-focus type for $SF^+ \cup SF^-$.*

Note that map (1.2) for $(b, B, M) \in H_1$ or map (1.3) for $(C, B, M) \in H_2$ has two saddle hyperbolic fixed points O_1 and O_2, except for the plane $B = 0$ where the points have a zero multiplier. For map (1.2), both points are of type (2,1), i.e. they have one unstable multiplier (greater than 1 in the absolute value) and two stable multipliers (resp., less than 1). We define concretely the point O_1 (resp., O_2) as that fixed point which has the positive (resp., negative) unstable multiplier. For map (1.3), both the points are of type (1,2), i.e. they have two unstable multipliers and a stable one. We show (see Lemma 10) that one of the fixed points has always (in H_2) real multipliers and, moreover, one of its unstable multipliers is positive and the other is negative. We label this point as O_1 and, accordingly, the other point is labelled as O_2.

For the region H_1, the point O_1, resp. O_2, is a saddle-focus (2,1) in SF^+, resp. in SF^-. Note that, in this case, the regions SF^+ and SF^- touch one another along the axis ($B = 0, b = 0$) (when $B = 0$ and $b = 0$ map (1.3) becomes one-dimensional and, thus, fixed points have two zero multipliers). Regions S_1 and S_2 do not intersect but adjoin "by angle" the line ($B = 0, b = 0$). For the region H_2, the point O_2 is a saddle-focus (1,2) in both SF^+ and SF^-. In this case, the regions SF^+ and SF^- adjoin one another along some area element of the plane $B = 0$ (on which the corresponding two-dimensional map $\bar{y} = z, \bar{z} = M + Cz - y^2$ has a fixed point, which is an unstable focus). Regions S_1 and S_2 are disjoint in H_2.

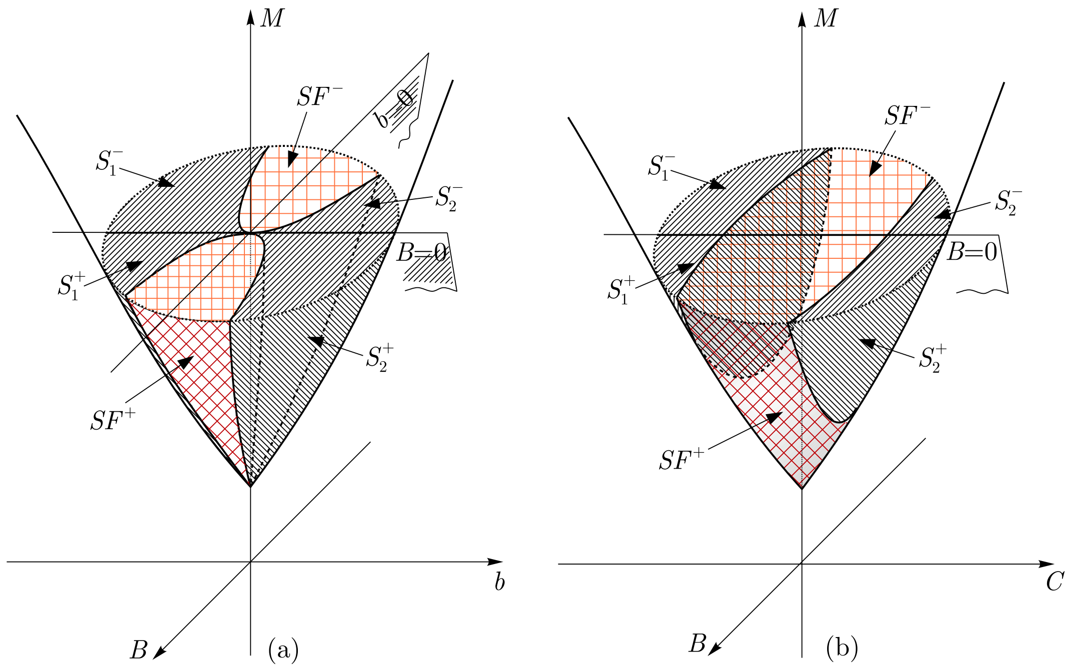

Fig. 2. The partition of the hyperbolicity regions into sub-regions corresponding the existence of horseshoes of saddle type and saddle-focus type in the cases of (a) map (1.2); (b) map (1.3).

3. ON THE EXISTENCE OF HORSESHOES IN 3D HÉNON MAPS

In this section we prove Theorems 1 and 2. However, all our calculations, except for Section 3.5, will be related to 3D Hénon maps (2.3) and (2.4) written in the parabola-like form. In what follows, we will call map (2.3) *3HM1-map* and denote it as $\tilde{T}$; analogously, we use notations *3HM2-map* and $\hat{T}$ for map (2.4).

3.1. Geometrical Horseshoes in 3HM1-map

It is well known that in the case of the classical Hénon map $\bar{y} = z, \bar{z} = 1 - by + az^2$, some sufficient condition for hyperbolicity is given by the inequality $a > \frac{1}{4}(5 + 2\sqrt{5})(1 + |b|)^2$; refer to [23, 24]. For the Hénon map written in the parabola-like form, $\bar{y} = z, \bar{z} = \gamma z(1 - z) - by$, it gives the condition $\gamma > (2 + \sqrt{5})(1 + |b|)$. We need results of such a type for the case of 3HM1-map. However, at first, we will find some conditions for the existence of a geometrical horseshoe.

Let β be a positive constant. Consider a box $Q_\beta = I_x \times I_y \times I_z$ where $I_x = [-(1 + \beta), (1 + \beta)], I_y = I_z = [-\beta, 1 + \beta]$ centered at the point $(0, 1/2, 1/2)$. Denote the faces $z = -\beta$ and $z = 1 + \beta$ of Q_β as q_0 and q_1, respectively.

Lemma 1. *Let β and γ be such that*

$$\gamma\beta^2 + (\gamma - 1 - |B| - |b|)\beta - (|B| + |b|) > 0 \tag{3.1}$$

and

$$\gamma > 4(1 + \beta)(1 + |B| + |b|) \tag{3.2}$$

for any given B and b. Then map (2.3) has a geometric horseshoe in Q_β, that is,

1. *the set $\tilde{T}(Q_\beta) \cap Q_\beta$ consists of two connected components;*
2. *$\tilde{T}(q_i) \cap Q_\beta = \emptyset$, $i = 0, 1$;*
3. *there is a box $D_\beta \subset Q_\beta$ containing the section $z = 1/2$ such that $\tilde{T}(D_\beta) \cap Q_\beta = \emptyset$.*

Proof. It follows from (2.3) that if $(x, y, z) \in Q_\beta$, then $\bar{x} \in I_x$ and $\bar{y} \in I_y$. However, the coordinate $\bar{z}$ can take values outside I_z. If $(x, y, z) \in q_0$, i.e. $z = -\beta, x \in I_x, y \in I_y$, then $\bar{z} = -\gamma\beta(1+\beta) + Bx - by \leqslant -\gamma\beta(1+\beta) + (|B| + |b|)(1+\beta)$. Thus, the inequality

$$-\gamma\beta(1+\beta) + (|B| + |b|)(1+\beta) < -\beta,$$

implies that $\tilde{T}(q_0) \cap Q_\beta = \emptyset$. This inequality can be recast as (3.1).

The same relations are obtained for $(x, y, z) \in q_1$, i.e. for $z = 1+\beta$, i.e. inequality (3.1) also implies that $\tilde{T}(q_1) \cap Q_\beta = \emptyset$.

Let now $z = 1/2$ (it is the maximum point for function $z(1-z)$). Then

$$\bar{z} = \frac{1}{4}\gamma + Bx - by \geqslant \frac{1}{4}\gamma - (|B| + |b|)(1+\beta).$$

The condition $\bar{z} > 1 + \beta$, i.e. $\bar{z} \notin I_z$ for all $(x, y) \in I_x \times I_y$, implies inequality (3.2).

3.2. Saddle Maps, Cross-Maps, Infinite Sequences of Saddle Maps

We will find sufficient hyperbolicity conditions for map (2.3) in Section 3.3. Our consideration follows the Shilnikov method [1] when, instead of directly checking hyperbolicity conditions for map(s) (like the well-known "cone conditions"), we consider the so-called "cross-form" of the map and find its contraction conditions. In the section below, we give some necessary facts. We use the following definition as in [1, 6].

Definition 1. *A map $(x_1, y_1) \mapsto (x_2, y_2)$ (where $(x_1, y_1) \in X_1 \times Y_1$, $(x_2, y_2) \in X_2 \times Y_2$ and X_1, X_2, Y_1, Y_2 are some closed subsets of complete metric spaces) is* a saddle map*, if the coordinates x_2 and y_1 are uniquely defined by any $x_1 \in X_1$, $y_2 \in Y_2$ and the correspondence map $(x_1, y_2) \mapsto (x_2, y_1)$ (the so-called* cross-map*) is a contraction map in the metric* $\max\{\|x\|, \|y\|\}$.

Recall that if we have a map $\bar{\xi} = f(\xi, \eta)$, $\bar{\eta} = g(\xi, \eta)$, then its *cross-form* is given by the map $\bar{\xi} = \tilde{f}(\xi, \bar{\eta}), \eta = \tilde{g}(\xi, \bar{\eta})$, where $\bar{\eta} \equiv g(\xi, \tilde{g}(\xi, \bar{\eta})), \tilde{f}(\xi, \bar{\eta}) \equiv f(\xi, \tilde{g}(\xi, \bar{\eta}))$.[4)]

This notation is effectively used in various problems related to the description of nontrivial hyperbolic subsets of dynamical systems with homoclinic or heteroclinic orbits. The basis of the corresponding theory consists in construction of criteria for the existence and uniqueness of both periodic and non-periodic (nonwandering) orbits in terms of theorems on saddle and stable fixed points of operators acting in a countable product of metric spaces. The corresponding exact statements were proved by Shilnikov in [1]. Recall some principal moments.

Let us have infinitely many complete metric spaces $X_i, i = 0, \pm 1, \ldots,$ with metric $\rho_i(x_i', x_i'')$ and operators A_i such that $A_i X_i \subset X_{i+1}$ and $\rho_{i+1}(A_i x_i', A_i x_i'') < q\rho_i(x_i', x_i'')$. Assume also that $\sup\limits_{x_i', x_i'' \in X_i} \rho_i(x_i', x_i'') < d < \infty$.

Consider a direct product $X = \prod\limits_{i=-\infty}^{+\infty} X_i$ of the spaces and introduce the metric in X as follows $\rho_i(x', x'') = \sup\limits_{-\infty < i < +\infty} \rho_i(x_i', x_i'')$, where $x = (\ldots, x_{-i}, \ldots, x_{-1}, x_0, x_1, \ldots, x_i, \ldots)$. Consider the operator A acting in X, $A : x \mapsto \bar{x}$, by the rule

$$\bar{x} = (\ldots, \bar{x}_{-i}, \ldots, \bar{x}_{-1}, \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_i, \ldots), \text{ where } \bar{x}_i = A_{i-1} x_{i-1}.$$

Lemma 2. [1] *If $q < 1$, the operator A is contracting.*

4) Usually, the cross-form is constructed in such a way: we find the relation $\eta = \tilde{g}(\xi, \bar{\eta})$ between coordinates η and $\bar{\eta}$ from the equation $\bar{\eta} = g(\xi, \eta)$ (using, for example, that $\|\partial g/\partial \eta\| \neq 0$), then this relation is also substituted into the equation $\bar{\xi} = f(\xi, \eta)$).

Let $x^* = (\dots, x_i^*, \dots)$ be a fixed point of A, i.e. $x^* = Ax^*$. Writing the latter equality by components, we obtain that there is only one sequence $\{x_i^*\}_{i=-\infty}^{+\infty}$ satisfying conditions $x_{i+1}^* = A_i x_i^*$.

Note that since return maps (near homoclinic orbits, for example) are saddle maps, this lemma is used here in the spirit of ideas of Hadamard. That is, first, there are constructed stable leaves $\{W_i^s : i = 0, \pm 1, \dots, \}$. In this case, we consider as X_i suitable sets of Lipschitz functions with C^0-metric (or C^r-functions with C^r-metric, as in [6]). Next, in an analogous way, a set $\{W_i^u; i = 0, \pm 1, \dots, \}$ of unstable leaves is constructed. The sought saddle orbit is $\{W_i^s\} \cap \{W_i^u\}$. Usually, such an approach is used when we need to know not only the existence (and uniqueness) of the orbit but also the behavior of its stable and unstable manifold. If the latter is not required, then one can use the following result (see Lemma 3 below).

Let us consider two sequences of complete metric spaces X_i and Y_i $(i = 0, \pm 1, \dots)$ and assume that

$$\sup_{x_i', x_i'' \in X_i} \rho_{X_i}(x_i', x_i'') < d, \quad \sup_{y_i', y_i'' \in Y_i} \rho_{Y_i}(y_i', y_i'') < d.$$

Let us define on $X_i \times Y_{i+1}$ the operators A_i and B_{i+1} such that

$$A_i(X_i \times Y_{i+1}) \subset X_{i+1}, \quad B_{i+1}(X_i \times Y_{i+1}) \subset Y_i.$$

Assume that

$$\rho_{X_{i+1}}(\bar{x}_{i+1}', \bar{x}_{i+1}'') < \frac{q}{2}\left(\rho_{X_i}(x_i', x_i'') + \rho_{Y_{i+1}}(y_{i+1}', y_{i+1}'')\right),$$
$$\rho_{Y_i}(\bar{y}_i', \bar{y}_i'') < \frac{q}{2}\left(\rho_{X_i}(x_i', x_i'') + \rho_{Y_{i+1}}(y_{i+1}', y_{i+1}'')\right),$$

where $\bar{x}_{i+1} = A_i(x_i, y_{i+1}),\ \bar{y}_i = B_{i+1}(x_i, y_{i+1})$.

Consider the space $Z = \prod_{i=-\infty}^{+\infty} (X_i \times Y_i)$. The metric in Z is defined as follows

$$\rho_Z\left((x', y'), (x'', y'')\right) = \sup_{-\infty < i < +\infty} \left(\rho_{X_i}(x_i', x_i'') + \rho_{Y_i}(y_i', y_i'')\right),$$

where $z = (x, y) = (\dots, (x_i, y_i), \dots)$. Introduce the operator[5)] $C^{1,-1}$

$$C^{1,-1} : Z = (x, y) \mapsto \bar{Z} = (\bar{x}, \bar{y})$$

(by the rule: $(\bar{x}, \bar{y}) = (\dots, (\bar{x}_i, \bar{y}_i), \dots)$ where $\bar{x}_{i+1} = A_i(x_i, y_{i+1}),\ \bar{y}_i = B_{i+1}(x_i, y_{i+1})$).

Lemma 3. [1] *If $q < 1$, the operator $C^{1,-1}$ is contracting.*

Let $z^* = (x^*, y^*)$ be a fixed point of $C^{1,-1}$. Writing this condition by components, we obtain that there exists only one sequence $\left(\dots, (x_{-1}^*, y_{-1}^*), (x_0^*, y_0^*)(x_1^*, y_1^*) \dots\right)$ satisfying conditions $x_{i+1}^* = A_i(x_i^*, y_{i+1}^*),\ \ y_i^* = B_{i+1}(x_i^*, y_{i+1}^*)$.

We can deduce from Lemma 3, for example, the following:

Corollary 1. *If the conditions of Lemma 3 hold, then the set $\{W_i^s\}$ consists of leaves of form $x_i = h_i(y_i)$, where h_i are Lipschitz functions with the Lipschitz constant $L_i < L < 1$.*

Proof. Consider the spaces $\mathcal{X}_i$ of curves of form $x_i = h_i(y_i)$ given on $X_i \times Y_i$. Assume that $|h_i(y_i') - h_i(y_i'')| < L_0 |y_i' - y_i''|$ for all i, where $L_0 < 1$. Consider the operator $\mathcal{A}$ which acts on $\mathcal{X} = \prod_{i=-\infty}^{+\infty} \mathcal{X}_i$

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5)The notation $C^{1,-1}$ belongs to L.Shilnikov and it has a sense: the operator is given in such a form which is "standard" with respect to one group coordinates and "cross" with respect to other coordinates. Note that when there are no "cross"-coordinates the operator is normal; when there are no "standard"-coordinates the operator can be considered as an "inverse limit".

and transforms curves $x_i = h_i(y_i)$ into $x_{i+1} = \bar{h}_{i+1}(y_{i+1})$ by the rule $x_{i+1} = A_i(h_i(y_i), y_{i+1}),\ y_i = B_{i+1}(h_i(y_i), y_{i+1})$. Then we have

$$|y_i' - y_i''| \leqslant \frac{q}{2}\left(L_0|y_i' - y_i''| + |y_{i+1}' - y_{i+1}''|\right) \Rightarrow |y_i' - y_i''| < |y_{i+1}' - y_{i+1}''|,$$

since $0 < q, L_0 < 1$. Thus,

$$|x_{i+1}' - x_{i+1}''| \leqslant \frac{q}{2}\left(L_0|y_i' - y_i''| + |y_{i+1}' - y_{i+1}''|\right) < L_0|y_{i+1}' - y_{i+1}''|.$$

It means that $\mathcal{A}(\mathcal{X}) \subset \mathcal{X}$. Let us show that the operator $\mathcal{A}$ is contracting. We write the following equalities

$$h_{i+1}'(y_{i+1}) = A_i(h_i(y_i'), y_{i+1}),\ y_i' = B_{i+1}(h_i'(y_i'), y_{i+1}),$$
$$h_{i+1}''(y_{i+1}) = A_i(h_i(y_i''), y_{i+1}),\ y_i'' = B_{i+1}(h_i''(y_i''), y_{i+1}).$$

Then we have $|y_i' - y_i''| \leqslant \frac{q}{2}\,|h_i'(y_i') - h_i''(y_i'')| \leqslant \frac{q}{2}\,(|h_i' - h_i''| + L_0|y_i' - y_i''|)$. Since $q < 1, L_0 < 1$, it implies that $|y_i' - y_i''| < |h_i' - h_i''|$ and, hence,

$$|h_{i+1}' - h_{i+1}''| \leqslant \frac{q}{2}\left|h_i'(y_i') - h_i''(y_i'')\right| \leqslant \frac{q}{2}\left(|h_i' - h_i''| + L_0|y_i' - y_i''|\right)$$
$$< \frac{q}{2}(1 + L_0)|h_i' - h_i''| < q\,|h_i' - h_i''|.$$

Analogously, we can prove that the set $\{W_i^u\}$ consists of leaves of form $y_i = h_i(x_i)$, where h_i are Lipschitz functions with the Lipschitz constant $L_i < L < 1$. For this goal, we consider the operator $C^{-1,1}$. Moreover, applying methods of [6], for example, we can prove that $\{W_i^s\}$ and $\{W_i^u\}$ consist of the corresponding C^r-leaves.

Thus, Lemma 3 allows to be deduced all necessary facts for applying the hyperbolic theory. Note also that we can use various contraction conditions in particular calculations (taking suitable metrics, for example). In particular, we will use the following reformulation of Lemma 3; see [6].

Lemma 4. *Let complete metric spaces X_i and Y_i and operators A_i and $B_i, i = 0, \pm 1, \ldots$, satisfy the following conditions:*

1. *diam* $X_i < d$, *diam* $Y_i < d$;
2. $A_i(X_i \times Y_{i+1}) \subset X_{i+1},\ \ B_{i+1}(X_i \times Y_{i+1}) \subset Y_i$;
3. $\rho_{X_{i+1}}(\bar{x}_{i+1}^1, \bar{x}_{i+1}^2) < q_1\ \rho_{X_i}(x_i^1, x_i^2) + L_1 q_1\ \ \rho_{Y_{i+1}}(y_{i+1}^1, y_{i+1}^2)$,
   $\rho_{Y_i}(\bar{y}_i^1, \bar{y}_i^2) < q_2\ \rho_{X_i}(x_i^1, x_i^2) + L_2 q_2\ \ \rho_{Y_{i+1}}(y_{i+1}^1, y_{i+1}^2)$,

where d and $q_l, L_l, l = 1, 2$, are constants with $d > 0, q_l > 0, L_l > 0$ and $0 < q_l(1 + L_l) < 1$, and for all i, $x_i \in X_i$, $y_i \in Y_i$, and $\bar{x}_{i+1} = A_i(x_i, y_{i+1})$, $\bar{y}_i = B_{i+1}(x_i, y_{i+1})$. Then there exists the unique sequence of form $(\ldots, (x_{i-1}^*, y_{i-1}^*), (x_i^*, y_i^*), \ldots)$, satisfying the conditions $x_{i+1}^* = A_i(x_i^*, y_{i+1}^*)$, $y_i^* = B_{i+1}(x_i^*, y_{i+1}^*)$.

3.2.1. Abstraction.

Note that the bases of the theory of cross-maps were constructed by L.Shilnikov as far back as 1960s. However, this theory seems to be not very popular. The reason may be that some basic constructions (like Lemmas 2 and 3) look quite abstract and, as a result, have no "transparent" geometrical sense. However, this theory must be considered as an alternative and complementary one to such "standard" branches of the hyperbolic theory as "cone technique", "shadow lemma" etc. Moreover, there is a series of classical results which have been proved by means of the Shilnikov method (and it is not clear how these results could be obtained by other methods). For example, the Poincaré–Birkhoff problem on a structure of a neighborhood of transverse Poincaré homoclinic orbit was solved, [1]; analogous results were obtained for the cases of transverse heteroclinic cycles, [1], and a homoclinic tube of an invariant torus, [26]; in [5, 27, 28] a description was given of nontrivial hyperbolic subsets in the case of systems close to systems having homoclinic tangencies.[6)] Therefore, in this paper, we use the Shilnikov method, since it seems to be rather convenient for our goals.

3.3. Hyperbolic Horseshoes for 3HM1-map

Lemma 5. *Let the conditions of Lemma* 1 *be fulfilled. If*

$$\gamma > 2(1+\beta)(1+|b|+|B|) + \sqrt{4(1+\beta)^2(1+|b|+|B|)^2+1}, \tag{3.3}$$

then the 3HM1-map (2.3) *has in* Q_β *a hyperbolic* $(2,1)$*-horseshoe* Λ_1.

Proof. Introduce a new coordinate x by the formula $x_{new} = sx$ with $s > 1$. Then map (2.3) takes the following form

$$\bar{x} = s^{-1}y, \quad \bar{y} = z, \quad \bar{z} = \gamma z(1-z) + sBx - by. \tag{3.4}$$

Now we rewrite map (3.4) in the so-called *cross-form* [1, 6] with respect to the "unstable" coordinate z. For this goal, we express the coordinate z in the third equation of (3.4) in terms of all other coordinates: x, y and $\bar{z}$. We obtain

$$z = \frac{1}{2} + (-1)^{i+1}\frac{1}{2}\sqrt{1 - \frac{4}{\gamma}(\bar{z} + by - sBx)} \equiv P_i(x,y,\bar{z}), \quad i = 0,1. \tag{3.5}$$

Then the action of map (3.4) on Q_β is equivalent to the actions of two cross-maps

$$\begin{aligned} \tilde{T}_0: \quad & \bar{x} = s^{-1}y, \quad \bar{y} = P_0(x,y,\bar{z}), \quad z = P_0(x,y,\bar{z}), \quad \text{and} \\ \tilde{T}_1: \quad & \bar{x} = s^{-1}y, \quad \bar{y} = P_1(x,y,\bar{z}), \quad z = P_1(x,y,\bar{z}). \end{aligned} \tag{3.6}$$

Both these maps are defined in $J_\beta = I_x \times I_y \times I_{\bar{z}}$ where I_x and I_y are defined as before and $I_{\bar{z}} = [-\beta, 1+\beta]$. However, unlike $\tilde{T}$, the image of any point of J_β under $\tilde{T}_i, i = 0,1$, is, by Lemma 1, a point of J_β. Moreover, there are no points of J_β which are mapped onto D_β under $\tilde{T}_i, i = 0,1$. Indeed, by Lemma 1, all such points do not belong to J_β, since the image of D_β under $\tilde{T}$ is positioned outside Q_β (i.e. all points from $\tilde{T}(D_\beta)$ have the $\bar{z}$-coordinate outside $I_{\bar{z}}$). Note that Q_β is divided into three connected and disjoint components: Q_0, Q_1 and D_β which are, respectively,low, upper and middle parts of Q_β and, moreover, $\tilde{T}_0(Q_\beta) \subset Q_0$ and $\tilde{T}_1(Q_\beta) \subset Q_1$; see Fig. 3. We need now to find conditions under which both maps $\tilde{T}_0$ and $\tilde{T}_1$ are contracting. Then we can conclude that the initial map $\tilde{T}$ is a saddle map on Q_β.

It follows from (3.6) that $|\bar{x}_1 - \bar{x}_2| < s^{-1}|y_1 - y_2|$, and since $0 < s^{-1} < 1$, it gives us a contraction with respect to the first coordinate. Evidently, the sought contraction along y and $\bar{z}$ will be guaranteed if the following estimates hold

$$R = \left\|\frac{\partial P_i}{\partial x}\right\| + \left\|\frac{\partial P_i}{\partial y}\right\| + \left\|\frac{\partial P_i}{\partial \bar{z}}\right\| < q < 1, \quad i = 0,1, \tag{3.7}$$

where q is some positive constant. We obtain from (3.5) that

$$R = \frac{1+|b|+s|B|}{\gamma}\left|1 - \frac{4}{\gamma}(\bar{z} + by - sBx)\right|^{-1/2}.$$

Since $\bar{z} + by - sBx = \gamma z(1-z)$, we have

$$R = \frac{1+|b|+s|B|}{\gamma}|z - \frac{1}{2}|^{-1}.$$

Thus, the condition $R < q$ can be rewritten as the following inequality

$$|z - \frac{1}{2}| > \frac{1+|b|+s|B|}{2q\gamma}.$$

[6] For example, in [27] (see also [29]), the existence of nontrivial hyperbolic subsets was proved for systems having homoclinic tangencies of arbitrary finite orders and their description was given; these results cover widely the results obtained (by other methods) in [30, 31], for example (note that many proofs in [31] are not correct).

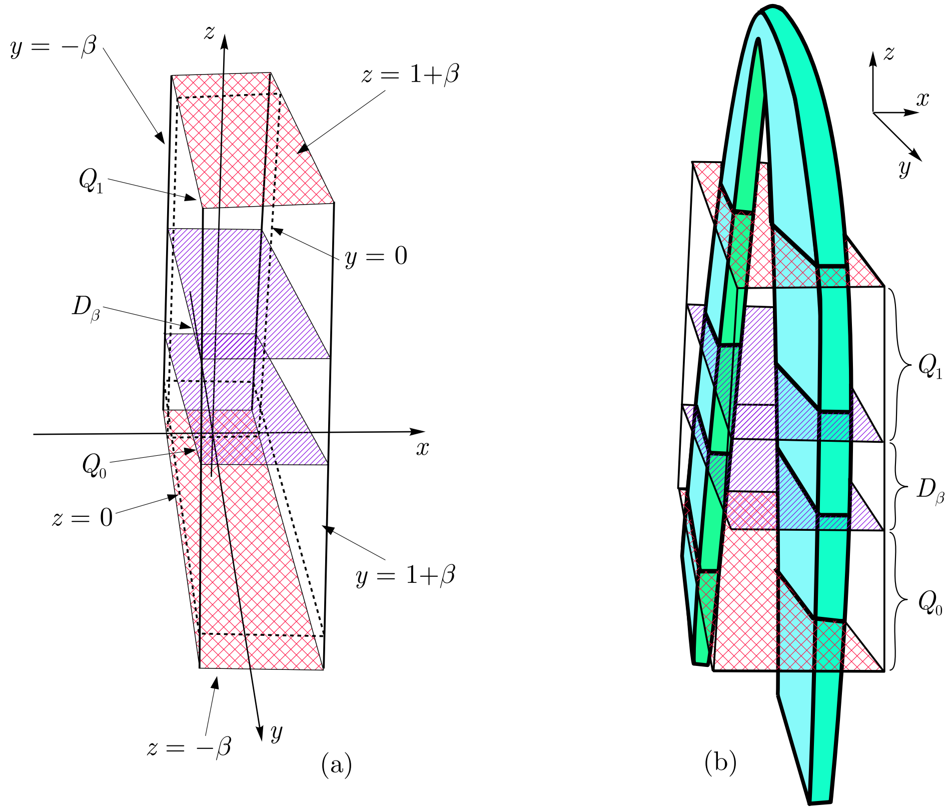

Fig. 3. The initial box Q_β (a) and its image $\tilde{T}(Q_\beta)$ (b) under the 3HM1-map.

Since this inequality holds for any $s > 1, 0 < q < 1$ we can take the limit $s \to 1, q \to 1$. Then we obtain

$$|z - \frac{1}{2}| > \frac{1 + |b| + |B|}{2\gamma}. \tag{3.8}$$

Inequality (3.8) gives, in fact, some estimate for the vertical size of the sub-box D_β. The corresponding $\bar{z}$ for these values of z must be greater than $(1 + \beta)$. Since $\bar{z} = \gamma z(1 - z) + Bx - by$ (we put here $s = 1$) and $z(1 - z) = \frac{1}{4} - (z - \frac{1}{2})^2$, the condition $\bar{z} > 1 + \beta$ is satisfied (when (3.8) holds), if

$$\frac{\gamma}{4} - \frac{1}{4\gamma} - |B|(1 + \beta) - |b|(1 + \beta) > 1 + \beta.$$

This gives the final estimate (3.3).

It remains now to show that Lemma 1 and estimate (3.3) provide sufficient conditions for the existence of the Smale horseshoe Λ_1 on Q_β. Note that the set of orbits which do not leave Q_β under both forward and backward iterations of $\tilde{T}$ is defined as $\Lambda = \bigcap_{n=-\infty}^{+\infty} \tilde{T}^n(Q_\beta)$. We have proved that the set $\tilde{T}(Q_\beta)$ consists of two connected components $\mathcal{D}_0$ and $\mathcal{D}_1$; see Fig. 3. Therefore, for a complete description of Λ_1 we need to use (at least) two symbols in order to point out to which component, $\mathcal{D}_0$ or $\mathcal{D}_1$, belongs a point of an orbit from Λ. Let some orbit of Λ intersect successively components

$$(\ldots, \mathcal{D}_{\alpha_i},\ \mathcal{D}_{\alpha_{i+1}}, \ldots), \tag{3.9}$$

where $i = 0, \pm 1, \ldots,$ and $\alpha_i \in \{0, 1\}$. Let $M_i(x_i, y_i, z_i) \in \mathcal{D}_{\alpha_i}$ be the corresponding sequence of points of this orbit. By our construction, we have such relations between coordinates of the points

M_i and M_{i+1} (see (3.6)): for $i = 0, \pm 1, \dots$,

$$x_{i+1} = s^{-1} y_i, y_{i+1} = P_{\alpha_i}(x_i, y_i, z_{i+1}),\ z_i = P_{\alpha_i}(x_i, y_i, z_{i+1}), \tag{3.10}$$

where the functions P_{α_i} are defined, for all i, on $I_x \times I_y \times I_{\bar{z}}$. Let us prove that the system (3.10) has a unique solution where the conditions of Lemma 5 hold. Consider the following sequence of operators and spaces: for $i = 0, \pm 1, \dots$,

$$\begin{aligned} &A_i : I_x \times I_y \times I_{\bar{z}} \to I_x \times I_y \ \text{ given by } \ \bar{x}_{i+1} = s^{-1} y_i,\ \bar{y}_{i+1} = P_{\alpha_i}(x_i, y_i, z_{i+1}), \\ &B_{i+1} : I_x \times I_y \times I_{\bar{z}} \to I_{\bar{z}} \ \text{ given by } \ \bar{z}_i = P_{\alpha_i}(x_i, y_i, z_{i+1}). \end{aligned} \tag{3.11}$$

However, we have proved that this sequence satisfies the conditions of Lemma 4. Thus, it has a unique fixed point $M^* = (\dots, (x_i^*, y_i^*, z_i^*), \dots)$ and this point corresponds to a saddle orbit from Λ: its coordinates satisfy relations (3.10) and the orbit intersect successively components $\mathcal{D}_\prime$ and $\mathcal{D}_\infty$ of Q_β in accordance with rule (3.9). Besides, we have proved that when the conditions of Lemma 5 hold the 3HM1-map is hyperbolic on Q_β, more exactly, it is (exponentially) contracting along y-direction and expanding along z-direction. It gives a way to construct stable and unstable leaves (see Corollary 1). These leaves compose two invariant families of stable and unstable invariant manifolds of Λ (having on Q_β the structure of direct products of Cantor sets on interval). It completes the proof.

Remark 1. Inequality (3.1) has a particular solution of form[7)]

$$\beta = \frac{|B| + |b|}{\gamma - 1}. \tag{3.12}$$

This solution can be regarded as "good"(it gives a quite small value of β: note that $\beta = (|B| + |b|)/\gamma$ is not a solution of (3.1)). Thus, we can always take (3.12) as a very suitable value of β. Let β be given by (3.12).[8)]

Remark 2. Since $\gamma > 4$, we can use the estimate $\beta < \frac{1}{3}(|B| + |b|)$ which is quite suitable for the case of small $|B| + |b|$. If $|B| + |b|$ are not small, we can estimate β as $\beta < \frac{1}{4}$, since one can obtain from (3.3) that $\gamma > 4(|B| + |b| + 1)$ in the domain of hyperbolicity. It also allows us to take the following value of β:

$$\beta = \frac{|B| + |b|}{3 + 4(|B| + |b|)} \tag{3.13}$$

which is quite suitable. Thus, in any case, the horseshoes of the 3HM1-map are positioned not far from the unit box $Q_0 = [0, 1] \times [0, 1] \times [0, 1]$.

Lemma 6. *Let inequality* (3.3) *be satisfied with* β *given by* (3.12) *and* $|B| + |b| \neq 0$. *Then the hyperbolic set* Λ_1 *from Lemma* 5 *is the unique nonwandering set of the 3HM1-map.*

Proof. Take β satisfying (3.12). Consider the image of the face $z = -\beta$ (or, which is the same, $z = 1 + \beta$) of Q_β under map (2.3). We immediately obtain

$$\bar{z} \leqslant -\gamma\beta(1 + \beta) + (|B| + |b|)(1 + \beta) = (1 + \beta)[-\gamma\beta + |B| + |b|].$$

Since, by (3.12), $|B| + |b| = (\gamma - 1)\beta$, i.e. $[-\gamma\beta + |B| + |b|] = -\beta$, we obtain $\bar{z} \leqslant -\beta(1 + \beta) < -\beta$. For the new boundary $z = -\beta(1 + \beta)$, we obtain

$$\begin{aligned} \bar{z} &\leqslant -\gamma\beta(1 + \beta)\,(1 + \beta(1 + \beta)) + (|B| + |b|)(1 + \beta) \\ &< -\gamma\beta(1 + \beta)^2 + (|B| + |b|)(1 + \beta)^2 = (1 + \beta)^2[-\gamma\beta + |B| + |b|] = -\beta(1 + \beta)^2. \end{aligned}$$

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7) Since we assume that $B \neq 0$, the constant β is always positive. In the specific one-dimensional case $B = b = 0$ we can take, formally, any sufficiently small positive number as β. However, the hyperbolicity here will be "degenerate", as well as for $B = 0$, from the point of view of the three-dimensional setting. Although, our hyperbolic estimates "work" for these cases also.

8) If $|B| + |b| \to 0$, then $\beta \to 0$ and the inequalities (3.2) and (3.3) give, respectively, solutions $\gamma > 4$ and $\gamma > 2 + \sqrt{5}$ that are well-known for the parabola map.

We repeat this procedure in the same way. Let us obtain the boundary $\bar{z}^{(n)} = -\beta(1+\beta)^n$ on the nth step. Then

$$\begin{aligned}\bar{z}^{(n+1)} &\leqslant -\gamma\beta(1+\beta)^n\left(1+\beta(1+\beta)^n\right) + (|B|+|b|)(1+\beta)\\ &\leqslant -\gamma\beta(1+\beta)^{n+1} + (|B|+|b|)(1+\beta)^{n+1}\\ &= (1+\beta)^{n+1}[-\gamma\beta+|B|+|b|] = -\beta(1+\beta)^{n+1}.\end{aligned}$$

Of course, this estimate will be true as long as $|x| < 1+\beta, |y| < 1+\beta$. But the form of map (2.3) is such that these relations are fulfilled0 as long as $|z| < 1+\beta$. However, for some n we can obtain that $\bar{z} < -1-\beta$ (but $|z| < 1+\beta$). Then, taking $z = -1-\beta$, we obtain that the corresponding value of $\bar{z}$ satisfies the estimate $\bar{z} \leqslant (-2\gamma + |B| + |b|)(1+\beta)$. Since, by (3.3), $\gamma > 4(1+|b|+|B|)$, one has that

$$\bar{z} < -8(1+|b|+|B|) + |B| + |b| < -7(1+|b|+|B|).$$

It follows from [25] that in the case of map (2.3) any point lying outside the box $\|(x,y,z)\| \leqslant (1+|b|+|B|)$ is wandering (tends to ∞).

The cases of the faces $x = -\beta, x = 1+\beta$ and $y = -\beta, y = 1+\beta$ are considered in the same way, since $\bar{y} = z$ and $\bar{\bar{x}} = z$.

3.4. Horseshoes for 3HM2-map

Now we consider map (2.4) (called 3HM2-map). It easily to see that maps 3HM1 and 3HM2 are inverse each other. Thus, we can conclude that the 3HM2-map can also have hyperbolic dynamics related to horseshoes, only these horseshoes will be of type (1,2). Naturally, we could transfer the results obtained for the 3HM1-map to 3HM2-map. But the 3HM2-map is quite interesting by itself and, besides, we demonstrate here another application of the cross-map technique. Therefore, we consider 3HM2-map separately.

Take again the box $Q_\beta = I_x \times I_y \times I_z$ and denote the squares $y = -\beta$ and $y = 1+\beta$ of Q_β as r_0 and r_1, respectively.

Lemma 7. *Consider the box Q_β with $\beta > 0$ satisfying the inequality*

$$\gamma\beta^2 + (\gamma - 1 - |B| - |C|)\beta - (|B|+|C|) > 0. \tag{3.14}$$

Then if

$$\gamma > 4(1+\beta)(1+|B|+|C|), \tag{3.15}$$

then the 3HM2-map (2.4) *has a geometric horseshoe in Q_β, i.e.*

1. *the set $\hat{T}(Q_\beta) \cap Q_\beta$ consists of two connected components;*
2. *$\hat{T}(r_i) \cap Q_\beta = \emptyset$, $i = 0, 1$;*
3. *there is a box $\hat{D}_\beta \subset Q_\beta$ such that $\hat{T}(\hat{D}_\beta) \cap Q_\beta = \emptyset$ and $\hat{D}_\beta$ contains the section $y = 1/2$ of Q_β (see Fig. 4).*

Proof. It follows from (2.4) that if $(x,y,z) \in Q_\beta$, then $\bar{x} \in I_x$ and $\bar{y} \in I_y$. However, the coordinate $\bar{z}$ can take values outside I_z since the coordinate y can expand. Indeed, if $(x,y,z) \in r_0$, i.e. $y = -\beta, x \in I_x, z \in I_z$, then $\bar{z} = -\gamma\beta(1+\beta) + Bx + Cz \leqslant -\gamma\beta(1+\beta) + (|B|+|C|)(1+\beta)$. It gives us the condition

$$-\gamma\beta(1+\beta) + (|B|+|C|)(1+\beta) < -\beta \tag{3.16}$$

which is recast as inequality (3.14), the latter having solutions of the form

$$\beta > \frac{1}{2\gamma}\left(1+|B|+|C|-\gamma+\sqrt{(1+|B|+|C|-\gamma)^2+4\gamma(|B|+|C|)}\right).$$

The same relations are obtained for $(x,y,z)\in r_1$, i.e. for $y=1+\beta$.

Let now $y=1/2$. Then

$$\bar z=\frac14\gamma+Bx+Cz\geqslant\frac14\gamma-(|B|+|C|)(1+\beta).$$

The condition $\bar z>1+\beta$, i.e. $\bar z\notin I_z$ for all $(x,y)\in I_x\times I_y$, reads as inequality (3.15).

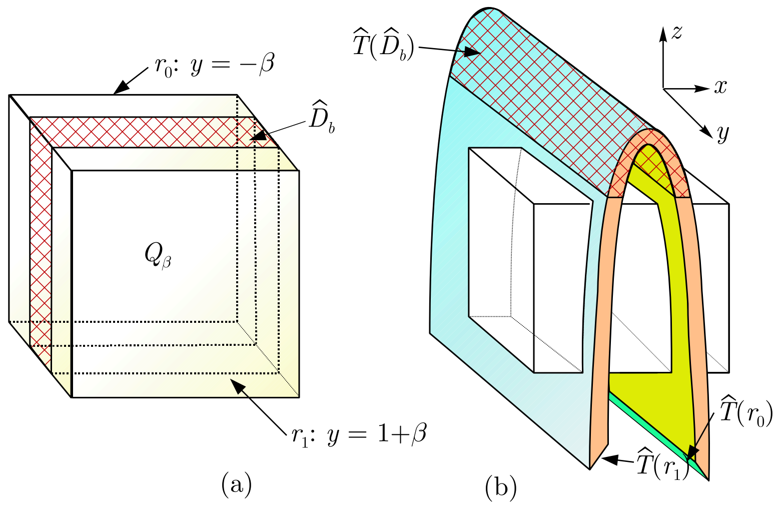

Fig. 4. (a) The initial box $\hat Q_\beta$ and (b) its image $\hat T(\hat Q_\beta)$ under the 3HM2-map.

Lemma 8. *If*

$$\gamma>2(1+\beta)(1+|b|+|C|)+\sqrt{4(1+\beta)^2(1+|b|+|C|)^2+1},\tag{3.17}$$

then the nonwandering set of map (2.4) *is a hyperbolic* (2,1)*-horseshoe* Λ_2 *which is positioned inside* Q_β.

Proof. Introduce a new coordinate z by the formula $z_{new}=w^{-1}z$ with $w<1$. Then map (2.4) takes the following form

$$\bar x=y,\quad \bar y=w^{-1}z,\quad \bar z=w\gamma y(1-y)+wBx+Cz.\tag{3.18}$$

Now we rewrite map (3.18) in the cross-form with respect to the "unstable" coordinates y and z. For this goal, we express the coordinate y in the third equation of (3.18) in terms of other coordinates: $x, z=\bar y$ and $\bar z$. We obtain

$$y=\frac12+(-1)^{i+1}\frac12\sqrt{1-\frac{4}{w\gamma}(\bar z-Cw\bar y-wBx)}\equiv P_i(x,\bar y,\bar z),\quad i=0,1.\tag{3.19}$$

Then the action of map (3.18) on Q_β is equivalent to the actions of two cross-maps

$$\begin{aligned}\hat{\mathcal T}_0:&\quad \bar x=P_0(x,\bar y,\bar z),\quad z=w\bar y,\quad z=P_0(x,\bar y,\bar z),\quad\text{and}\\ \hat{\mathcal T}_1:&\quad \bar x=P_1(x,\bar y,\bar z),\quad z=w\bar y,\quad z=P_1(x,\bar y,\bar z).\end{aligned}\tag{3.20}$$

Both these maps are defined in $J_\beta=I_x\times I_y\times I_z$. However, unlike $\hat T$, the image of any point of J_β under $\hat{\mathcal T}_i, i=0,1$, is, by Lemma 7, a point of J_β. Moreover, there are no points of J_β which are mapped onto $\hat D_\beta$ under $\hat{\mathcal T}_i, i=0,1$. Indeed, by Lemma 7, all such points do not belong J_β, since image of D_β under $\hat T$ is positioned outside Q_β (i.e. all points from $\hat T(\hat D_\beta)$ have $\bar z$-coordinate not belonging to $I_{\bar{z}}$). Note that the box Q_β is divided onto three connected and disjoined components: $\hat{Q}_0$, $\hat{Q}_1$ and $\hat{D}_\beta$ which are, respectively, left, right and middle parts of Q_β. Then we has that $\hat{\mathcal{T}}_0(Q_\beta) \subset \hat{Q}_0$ and $\hat{\mathcal{T}}_1(Q_\beta) \subset \hat{Q}_1$. We need now to find conditions under which both maps $\hat{\mathcal{T}}_0$ and $\hat{\mathcal{T}}_1$ are contracting.

It follows from (3.20) that $|z_1 - z_2| < w|\bar{y}_1 - \bar{y}_2|$, and since $0 < w < 1$, it gives us a contraction with respect to $\bar{y}$. The sought contraction along $\bar{x}$ and z will be guaranteed if the estimates (3.7) hold with P_i satisfying (3.19). Then we obtain

$$R = \frac{1 + w|C| + w|B|}{w\gamma} \left| 1 - \frac{4}{w\gamma}(\bar{z} - wC\bar{y} - wBx) \right|^{-1/2}.$$

Since $\bar{z} - wC\bar{y} - wBx = w\gamma y(1-y)$, we has

$$R = \frac{1 + w|C| + w|B|}{w\gamma} |y - \frac{1}{2}|^{-1}.$$

Thus, the condition $R < q$ can be rewritten as the following inequality

$$|y - \frac{1}{2}| > \frac{1 + w|C| + w|B|}{2q\gamma}.$$

Taking the limit $w \to 1, q \to 1$ we obtain

$$|y - \frac{1}{2}| > \frac{1 + |C| + |B|}{2\gamma}. \qquad (3.21)$$

Inequality (3.21) gives, in fact, some estimate for y-size of the sub-box $\hat{D}_\beta$. The corresponding $\bar{z}$ for these values of y must be greater than $(1+\beta)$. Since $\bar{z} = \gamma y(1-y) + Bx + Cz$ (we put here $w = 1$), the condition $\bar{z} > 1 + \beta$ is satisfied (when (3.21) holds), if

$$\frac{\gamma}{4} - \frac{1}{4\gamma} - |B|(1+\beta) - |C|(1+\beta) > 1 + \beta.$$

This gives the final estimate (3.17).

The result on the uniqueness of horseshoe is proved quite analogously to Lemma 6. $\square$

Note that both maps (2.3) and (2.4) have the constant Jacobian B. Therefore, the inequality $B \neq 0$ is always considered as a necessary condition for hyperbolicity of the three-dimensional maps. However, in the limit case, $B = 0$, we can also speak of some type of hyperbolic behavior, but only now in the two-dimensional setting. Every map (2.3) and (2.4) has at $B = 0$ "super-attracting" two-dimensional invariant surfaces, S_1 and S_2, respectively. In the case of map (2.3) with $B = 0$, the nonwandering set for $\tilde{T}|_{S_1}$ is (when $b \neq 0$) a two-dimensional Smale horseshoe. In the case of map (2.4) with $B = 0$, the nonwandering set $\hat{T}|_{S_2}$ is a "total snap-back repeller" (i.e., some infinite closed invariant set consisting of completely unstable (node type) orbits and containing a countable subset of periodic orbits having (transverse) homoclinic points).

3.5. The Proofs of Theorems 1 and 2

The proof of Theorem 1. We consider map (2.3) and suppose that Lemma 5 holds. With the following change of coordinates $x_{new} = \gamma(x + \frac{1}{2})$, $y_{new} = \gamma(y + \frac{1}{2})$, $z_{new} = \gamma(z + \frac{1}{2})$ map (2.3) takes the standard form (1.2), where $4M_1 = \gamma^2 - 2\gamma(1 + b - B)$. Since γ is positive (due to Lemma 5), we find also that

$$\gamma = (1 + b - B) + \sqrt{4M_1 + (1 + b - B)^2}. \qquad (3.22)$$

We take as β its value given by formula (3.13). Then Theorem 1 is obtained as the direct reformulation of Lemma 5 in terms of M_1, B and b. $\square$

The proof of Theorem 2. The case of the 3HM2-map in the standard form (1.3) is considered quite analogously and we obtain again Theorem 2 as the direct reformulation of Lemma 8 in terms of M_1, B and C with $\beta = (|B| + |C|)(3 + 4(|B| + |C|))^{-1}$. □

Note that if the initial box Q_β, in the 3D Hénon maps written in the parabola-like form, can be always chosen as the cube with sizes of length $1 + \beta$ centered at the point $(1/2, 1/2, 1/2)$, then in the case of these maps written in the standard form we must consider some "flatting" cube of sizes $\gamma(1 + \beta)$ centered at the point (γ, γ, γ).

4. SADDLE AND SADDLE-FOCUS HORSESHOES IN 3D HÉNON MAPS. THE PROOF OF THEOREM 3.

We show in this section that three-dimensional Hénon maps can possess horseshoes of different smooth types in a sense of the following definitions.

Definition 2. *Let a three-dimensional smooth map g have a nonwandering closed hyperbolic set Λ_g and let the system $g^n|_{\Lambda_g}$ for some positive integer n be conjugate to the topological Bernoulli scheme (shift) with two symbols. We will say that Λ is an n-*horseshoe*; the horseshoe is of* saddle type *or* saddle horseshoe *if its both orbits of period n (fixed points for $n = 1$) are saddles (i.e. all multipliers are real); the horseshoe is of* saddle-focus type *or* saddle-focus horseshoe *if at least one orbit of period n is a saddle-focus (i.e. it has a pair complex conjugate multipliers); the horseshoe is* completely saddle *(resp.,* completely saddle-focus*) if all its periodic orbits are saddles (resp., saddle-foci).*

We show that in 3D Hénon maps, like (2.3) and (2.4) (and, resp., (1.2) and (1.3)) there can exist horseshoes of both saddle and saddle-focus type; they are the horseshoes Λ_1 for (2.3) and Λ_2 for (2.4) existing for distinct values of the parameters from the hyperbolicity regions. The existence of completely saddle horseshoes of type (2,1) is well known. For example, in the case of map (2.3), if the hyperbolicity conditions (see Lemma 5) are satisfied and $|B| \ll |b|$, then the horseshoe Λ_1 will be normally hyperbolic. In other words, any periodic point of such a horseshoe has one unstable multiplier and two stable ones. Besides, one of the stable multipliers is greater, in the absolute value, than the other. Indeed, if $B = 0$ we have the 2D Hénon map whose horseshoe (with γ sufficiently large) has a coefficient of extension (along unstable directions) equal γ in the order and, thus, the contraction is of order b/γ. When $B \neq 0$ and small we have that the coefficient of volume contraction is B and, thus, the contraction along x-direction (orthogonal to (y, z)-plane) will be of order B/b. Thus, the normal hyperbolicity condition means that $|B/b| < |b/\gamma|$, i.e. some condition like $|B| < b^2/\gamma$ defines a domain where horseshoes are completely saddle. We will show that if $|B| \sim b^2/\gamma$ or $|B| > b^2/\gamma$, the horseshoes Λ_1 can be of saddle-focus type. On the other hand, it is not quite clear whether horseshoes Λ_2 at small B can be completely saddle horseshoes: evidently, this question is equivalent to the following one: can snap-back repellers $\Lambda_2|_{B=0}$ have only node type periodic points? Of course, when B is sufficiently large, the horseshoe Λ_2 can be completely of saddle type (since it is inverse to Λ_1 with small Jacobian B^{-1}).

4.1. Saddle-Focus Horseshoes in 3HM1-map

Consider map (2.3) with $\gamma > 0$. This map has always two fixed points O_1 and O_2. The point O_1 is in the origin, i.e. $O_1 = (0, 0, 0)$, the point O_2 has coordinates

$$O_2(x, y, z): \; x = y = z = \frac{\gamma - 1 + B - b}{\gamma}.$$

The characteristic equation for map (2.3) has the form

$$\lambda^3 - \gamma(1 - 2z)\lambda^2 + b\lambda - B = 0, \tag{4.1}$$

Consider first the point O_1. If $B = b = 0$, the equation (4.1) has roots (for $z = 0$) $\lambda_1 = \gamma, \lambda_{2,3} = 0$. Thus, in the domain of hyperbolicity (here $\gamma > 4(1 + |B| + |b|)$), the equation (4.1) with $z = 0$ has always two stable roots (inside the unit circle) and one unstable (outside the unit circle). It means that the point O_1 is a saddle fixed point of type (2,1). However, as far as its "smooth" type is concerned, the point O_1 can be both of saddle type (all multipliers are real) and saddle-focus type (stable multipliers are complex conjugate) depending of values of the parameters. The boundary between these two general cases corresponds, evidently, to the critical case where the characteristic equation has a double root. For such a root λ, we have the following system:

$$\lambda^3 - \gamma\lambda^2 + b\lambda - B = 0, \quad 3\lambda^2 - 2\gamma\lambda + b = 0. \tag{4.2}$$

Since $|\lambda| < 1$, we obtain from the second equation of (4.2) the following relation: $3\lambda = \gamma - \sqrt{\gamma^2 - 3b}$. Then we find from the first equation that

$$B = g_1(b,\gamma) \equiv \frac{1}{27}\left(\gamma - \sqrt{\gamma^2 - 3b}\right)\left(6b - \gamma(\gamma - \sqrt{\gamma^2 - 3b})\right). \tag{4.3}$$

This is the equation of the surface B^1_{sf} (in the (γ, B, b)-parameter space) corresponding to those values of the parameters when the point O_1 has a double (stable) multiplier. The equation (4.3) defines a parabola-like cylinder which contains the axis γ and has a quadratic tangency with the plane $B = 0$; see Fig. 2 (a). Note that, for small B and b, the surface (4.3) can be written as $B = \dfrac{b^2}{4\gamma}\left(1 + O(\dfrac{b}{\gamma})\right)$. The surface B^1_{sf} is positioned in the half-space $B > 0$ and divides the parameter space into two regions: $B > g_1(b,\gamma)$ which we denote as SF^+, where the point O_1 is the saddle-focus, and $B < g_1(b,\gamma)$, where the point O_1 is the saddle (except for the plane $B = 0$).

Note that if $B < 0$ the point O_1 can not be a saddle-focus since it has the positive unstable multiplier and, hence, its stable multipliers must be real and of different signs: the product of all multipliers is equal to B. However, the point O_2 can be saddle-focus in the region $B < 0$ (and, otherwise, O_2 can not be a saddle-focus for $B > 0$).

The characteristic equation at the point O_2 is $\lambda^3 + (\gamma - 2 + 2B - 2b)\lambda^2 + b\lambda - B = 0$, This equation has two stable roots and an unstable one, the latter being negative (of order $-\gamma$ when B and b are small). Thus, the stable roots can become double in the case of negative B, so that such a double root λ satisfies the system:

$$\lambda^3 + (\gamma - 2 + 2B - 2b)\lambda^2 + b\lambda - B = 0, \quad 3\lambda^2 + 2(\gamma - 2 + 2B - 2b)\lambda + b = 0.$$

Then we can write the equation of the surface B^2_{sf} corresponding to the existence of double multipliers for the point O_2 in the form

$$B = g_2(b,\gamma) \equiv \frac{1}{9}\left(\sqrt{A^2 - 3b} - A\right)\left[2b + \frac{\sqrt{A^2 - 3b} - A}{3}A\right], \tag{4.4}$$

where $A = \gamma - 2 + 2B - 2b$. This is again a parabola-like cylinder which contains the axis γ and touches the plane $B = 0$; see Fig. 2 (a). For small B and b, the equation (4.4) can be written as $B = -\dfrac{b^2}{4(\gamma - 2)}\left(1 + O(\dfrac{b}{\gamma})\right)$. The surface B^2_{sf} is positioned in the half-space $B < 0$ and divides the parameter space into two regions: $B < g_2(b,\gamma)$ which we denote as SF^-, where the point O_2 is the saddle-focus, and $B > g_2(b,\gamma)$, where the point O_2 is the saddle (for $B \neq 0$). Thus, we have proved the following

Lemma 9. *The surfaces B^1_{sf} and B^2_{sf} divide the region of hyperbolicity* (3.3) *into four open regions SF^+, SF^-, S_1 and S_2 such that the set Λ_1 is a saddle* $(2,1)$*-horseshoe for $(\gamma, B, b) \in S_1 \cup S_2 \backslash \{B = 0\}$ and a saddle-focus* $(2,1)$*-horseshoe for $(\gamma, B, b) \in SF^+ \cup SF^-$. Besides, the point O_1 (resp., O_2) is a saddle-focus for values of the parameters from the region SF^+ (resp., SF^-).*

This lemma gives us Theorem 3 for H_1, if we express γ as a function of M_1 by (3.22).

4.2. Saddle-Focus Horseshoe in 3HM2-map

Consider map (2.4) with $\gamma > 0$. This map has always two fixed points O_1 and O_2. The point O_1 is in the origin, i.e. $O_1 = (0, 0, 0)$ and the point O_2 has coordinates

$$O_2(x, y, z):\ x = y = z = \frac{\gamma - 1 + B + C}{\gamma}$$

The characteristic equation for map (2.4) has the form

$$\lambda^3 - C\lambda^2 - \gamma(1 - 2z)\lambda - B = 0. \tag{4.5}$$

Lemma 10. *The point O_1 is always the saddle (of type $(1, 2)$) in the hyperbolicity domain.*

Proof. Let $B = C = 0$ and γ be sufficiently large (this point is inside the hyperbolicity domain). Then, for $z = 0$, the equation (4.5) has three real roots $0, \pm\sqrt{\gamma}$. When B becomes nonzero the point O_1 is a saddle point of type (1,2) having one stable multiplier and two unstable ones. The latter are real and of opposite signs. It is clear that the appearance of any double root in this situation when parameters (B, C and γ) change is only possible when one of the roots crosses the value $+1$ or -1. It means that we must leave the hyperbolicity domain.

Consider now the point O_2. The characteristic equation (4.5) at O_2 has the form

$$\lambda^3 - C\lambda^2 + (\gamma - 2 + 2B + 2C)\lambda - B = 0.$$

When $B = C = 0$ this equation has the roots $0, \pm i\sqrt{\gamma - 2}$, i.e. point O_2 at small C and $B \neq 0$ is a saddle-focus. One can find boundaries of a domain D^2_{sf} in the (γ, B, C)-parameter space where O_2 is a saddle-focus. The boundaries correspond to the moment when O_2 has a double unstable multiplier λ. For this purpose, we need to solve the system

$$\begin{cases} \lambda^3 - C\lambda^2 + (\gamma - 2 + 2B + 2C)\lambda - B = 0, \\ 3\lambda^2 - 2C\lambda + (\gamma - 2 + 2B + 2C) = 0. \end{cases}$$

The solution defines two boundaries B^+_{sf2} and B^-_{sf2}, where $B^\pm_{sf2}$ has the following equation

$$B = \frac{A^\pm}{3}\left(2(\gamma - 2 + 2B + 2C) - CA^\pm\right),$$

where

$$A^\pm = \frac{C \pm \sqrt{C^2 - 3(\gamma - 2 + 2B + 2C)}}{3}.$$

Then domain D^2_{sf} is defined as $B^-_{sf2} < B < B^+_{sf2}$. It implies the following

Lemma 11. *Let H_2 be the hyperbolicity domain for the 3HM2-map. Then in the domain $SF = H_2 \cap D^2_{sf}$ the horseshoe Λ_2 is of saddle-focus type (the point O_2 is a saddle-focus) when $B \neq 0$. In the complimentary part of H_2, consisting of two disjoint regions S_1 and S_2, the horseshoe Λ_2 is of saddle type: both points O_1 and O_2 are saddles (if $B \neq 0$).*

Theorem 3 for H_2 follows from this lemma, if we express γ as a function of M_1 by the formula $\gamma = (1 - C - B) + \sqrt{4M_1 + (1 - C - B)^2}$.

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