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# Approximation of Entropy on Hyperbolic Sets for One-dimensional Maps and Their Multidimensional Perturbations

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**Abstract**—We consider piecewise monotone (not necessarily, strictly) piecewise $C^2$ maps on the interval with positive topological entropy. For such a map $f$ we prove that its topological entropy $h_{\rm top}(f)$ can be approximated (with any required accuracy) by restriction on a compact strictly $f$-invariant hyperbolic set disjoint from some neighborhood of prescribed set consisting of periodic attractors, nonhyperbolic intervals and endpoints of monotonicity intervals. By using this result we are able to generalize main theorem from [1] on chaotic behavior of multidimensional perturbations of solutions for difference equations which depend on two variables at nonperturbed value of parameter.

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## 1. INTRODUCTION

In this paper we generalize main result of [1] from piecewise analytic to piecewise $C^2$ maps. Consider a family of difference equations of the form

$$\Phi_\lambda(y_n, y_{n+1}, \dots, y_{n+m}) = 0,\ n \in \mathbb{Z}, \tag{1.1}$$

where $\lambda$ is a parameter from some metric space. We assume that the difference equation at the exceptional (unperturbed) value of the parameter depends on two variables, i.e.,

$$\Phi_{\lambda_0}(x_0, \dots, x_m) = \xi(x_N, x_{N+L}),$$

where $N$ and $N + L$ are two distinct integers between 0 and $m$, and $\xi(x, y)$ is a function such that for the equation $\xi(x, y) = 0$ there is a branch $y = \varphi(x)$ with positive topological entropy, i.e., $\xi(x, \varphi(x)) = 0$ and $h_{\rm top}(\varphi) > 0$. Notice that in the case when $L = 1$, the solutions of difference equation (1.1) with $\lambda = \lambda_0$ contain orbits of the one-dimensional map $x \mapsto \varphi(x)$. On the other hand, if $L > 1$, the solutions of (1.1) with $\lambda = \lambda_0$ contain orbits of a generalized one-dimensional transformation which can be regarded as the "$L$-th root" of $\varphi$. The case when $L < 0$ is also possible, this case corresponds to reversing the time for solutions induced by the maps.

In view of more applications, we allow the functions $\Phi_\lambda$ and the local map $\varphi$ to be not defined in some regions; more precisely, we suppose that $\Phi_\lambda$ and $\varphi$ are defined on domains $Q$ and $Q^{m+1}$

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respectively, where $Q = [s_1, s_2] \setminus V$ for some fixed real numbers $s_1, s_2$ and open set $V$, the latter being the union of finitely many open intervals in $[s_1, s_2]$. Here $s_1$ and $s_2$ can be regarded as some fixed bounds for coordinate projections of orbits we are interested in, while $V$ stays for an escaping region which is never visited by those interesting orbits. The topological entropy for (1.1) (as a quantity to estimate chaotic behavior of solutions) is defined as $h_{\text{top}}(\sigma)$ for the shift map $\sigma$ restricted to the set of bi-infinite solutions $(x_n)_{n=-\infty}^{\infty}$ for (1.1) (with respect to the product topology) satisfying $x_n \in Q$ for all $n \in \mathbb{Z}$. Also, $h_{\text{top}}(\varphi)$ is meant as the topological entropy of $\varphi$ restricted to $\bigcap_{n=0}^{\infty} \varphi^{-n}(Q)$ . Main result in [1] shows that the perturbed multidimensional difference equations are chaotic provided that the one-dimensional map at the unperturbed value of the parameter is piecewise analytic and has enough (with respect to the topological entropy) chaotic orbits which avoid prescribed regions. In the present paper we are able to extend that result to allow piecewise $C^2$ one-dimensional local maps. More precisely, the following holds.

**Theorem 1.** *Consider a family of difference equations of the form*

$$\Phi_\lambda(y_n, y_{n+1}, \ldots, y_{n+m}) = 0, \ n \in \mathbb{Z}, \tag{1.2}$$

*with the function $\Phi_\lambda : Q^{m+1} \to \mathbb{R}$ which is $C^1$ for each $\lambda$ and is continuous in $\lambda$ along with the partial derivatives $\partial_i \Phi_\lambda$, $i = 1, \ldots, m+1$, $Q = [s_1, s_2] \setminus V$ for some (fixed) real numbers $s_1 < s_2$ and $V$ is the union of finitely many open intervals in $[s_1, s_2]$, while parameter $\lambda$ being from some neighborhood of the unperturbed value $\lambda_0$ in some metric space. Assume that at $\lambda_0$, the function $\Phi$ depends on exactly two variables:*

$$\Phi_{\lambda_0}(x_0, x_1, \ldots, x_m) = \xi(x_N, x_{N+L}),$$

$0 \leqslant N, N+L \leqslant m$. *Assume, in addition, that for the equation $\xi(x, y) = 0$ there is a branch $y = \varphi(x)$ with positive topological entropy, where $\varphi : Q \to [s_1, s_2]$ is supposed to be a piecewise $C^2$ function. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for each $\lambda$ in the $\delta$-neighborhood of $\lambda_0$ , there is a closed (in the product topology) $\sigma$-invariant subset $\Gamma_\lambda$ of the set of solutions for (1.1) with $h_{top}(\sigma|_{\Gamma_\lambda}) > h_{top}(\varphi)/|L| - \epsilon$. Moreover, solutions in $\Gamma_\lambda$ depend continuously on $\lambda$ both in the product and the uniform topologies.*

The only difference between this theorem and similar result in [1] is that the local map $\varphi : Q \to [s_1, s_2]$ was supposed there to be a piecewise analytic function. The proof in [1] contained three ingredients: (i) proving a version of the implicit function theorem in Banach spaces with uncountably many branches along with constructing topological conjugacy in terms of these branches; (ii) checking the assumptions of this theorem for multidimensional perturbations of hyperbolic orbits of one-dimensional local maps; and (iii) finding a closed invariant subset for the local one-dimensional map which consists of hyperbolic repelling orbits and approximates the topological entropy of the local map. By using items (i) and (ii), the hyperbolic orbits from item (iii) can be continued in high dimensional space of solutions for perturbed difference equations. Now in the present paper, only item (iii) needs to be generalized as to admit piecewise $C^2$ local maps. The main difference between piecewise analytic and piecewise $C^2$ maps for the mentioned problem on approximation of entropy by hyperbolic orbits, is possible presence of nonhyperbolic subintervals for piecewise $C^2$ maps (because one needs to avoid neighborhoods of such subintervals without lost of entropy).

It is well known that in contrast to higher dimensions, for smooth one-dimensional maps one has commonly Axiom A and hyperbolicity results. A remarkable theorem by Mañé [5] states that for a $C^2$ interval map $f$ whose periodic points are hyperbolic, any compact $f$ -invariant set away from critical points is hyperbolic repelling (see more precise statement in next section). Nevertheless, given a $C^2$ interval map, it is not easy to check the above assumption on whether all of the periodic orbits are hyperbolic. And also, given a compact invariant set, in order to ensure its hyperbolicity, one needs to check that this set is disjoint from some neighborhood of the critical set. On the other hand, if the topological entropy of $f$ is positive, it is reasonable to ask whether there is a compact $f$-invariant hyperbolic set whose topological entropy approximates $h_{\text{top}}(f)$ with the required accuracy (see similar problems in [12] for piecewise monotone piecewise $C^1$ intervals maps without critical points, and in [3] for $C^{1+\varepsilon}$ surface diffeomorphisms; let us mention in this connection that for merely continuous surface homeomorhisms such an approximation need not take place, because M.Rees in [10] has constructed a minimal positive entropy homeomorphism on the 2-torus).

In this paper we answer the above question for a class of piecewise monotone piecewise $C^2$ maps (possibly discontinuous and not necessarily strictly monotone on monotonicity intervals). By small change of entropy we are able to get rid of the redundant assumptions (both on the map and on the orbits of points from an invariant set) to ensure hypebolicity for the restriction of $f$ to an appropriate set. More precisely, we prove the following.

**Theorem A.** *Let $f$ be a piecewise monotone piecewise $C^2$ map on a compact interval $I$ and assume that its derivative is piecewise monotone as well. If $h_{\rm top}(f) > 0$ then for any $\delta > 0$ there exists a compact hyperbolic repelling strictly $f$-invariant set $M$ such that $h_{\rm top}(f|_M) > h_{\rm top}(f) - \delta$.*

In order to prove this result in piecewise $C^2$ setting, we first prove a result on approximating the topological entropy for piecewise $C^0$ maps as follows.

**Theorem B.** *Let $f$ be a piecewise monotone piecewise continuous map on $I = [0, 1]$ with a partition $Z = \{(0, d_1), (d_1, d_2), \ldots, (d_{m-1}, 1)\}$. If $h_{\rm top}(f) > 0$ then for any $\delta > 0$ there exist a compact strictly $f$-invariant set $M \subset I$ and an open set $V \supset \{d_1, d_2, \ldots, d_{m-1}\}$ such that $h_{\rm top}(f|_M) > h_{\rm top}(f) - \delta$ and $M \cap V = \emptyset$.*

In the above theorems, the set $M$ is proved to be strictly invariant (i.e., $f(M) = M$). Such a strict invariance is needed for the mentioned item (ii) in our strategy for multidimensional perturbations of one-dimensional maps. Our technique below uses some constructions from entropic theory for piecewise monotone piecewise continuous maps (see [4, 6–9]). Note that even in the situation when the initial map is continuous, we need to use construction of truncations which lead naturally to discontinuous maps.

## 2. PRELIMINARIES AND PROOF OF THEOREM B

The following result by Mañé [5] states that away from critical points, any $C^2$ interval map is hyperbolic.

**Theorem 2 (Mañé [5]).** *Let $I$ be a compact interval of $\mathbb{R}$ and $g : I \to I$ be a $C^2$ map. Let $U$ be a neighborhood of the set of critical points of $g$. Then*

1. *All periodic orbits of $g$ contained in $I\backslash U$ of sufficiently large periods are hyperbolic repelling.*
2. *If all the periodic orbits of $g$ contained in $I\backslash U$ are hyperbolic repelling, then there exist $C > 0$ and $\lambda > 1$ such that $|Dg^n(x)| \geqslant C\lambda^n$, whenever $g^i(x) \in I\backslash(U \cup B_0)$ for all $0 \leqslant i \leqslant n-1$, where $B_0$ is the union of the immediate basins of the attracting periodic orbits of $g$ contained in $I\backslash U$.*

We need also the following corollary from Mañé's theorem.

**Corollary 1 ([2]).** *Let $g : I \to I$ be a $C^2$ map and $K \subset I$ be a compact forward invariant set. If $K$ does not contain critical points, attracting periodic points and non-hyperbolic periodic points of $g$, then it is a hyperbolic repelling set.*

We consider piecewise $C^2$ maps $f$, so there might be discontinuity points for $f$, its first derivative $Df$ and second derivative. We shall remove small neighborhoods of such discontinuity points along with neighborhoods of critical points, attracting and nonhyperbolic periodic points. Nevertheless, the topological entropy after appropriate removing will be proved to approach $h_{\rm top}(f)$.

Let $I = [0, 1]$ and $f : I \to I$ be a piecewise monotone piecewise continuous map and let $\mathcal{Z}$ be its partition, i.e., $I = \bigcup_{Z\in\mathcal{Z}} \bar{Z}$ and $\mathcal{Z}$ consists of finitely many, say $m$, disjoint open intervals, denoted by $(0, d_1), (d_1, d_2), \ldots, (d_{m-1}, 1)$, on each of which the restriction of $f$ is monotone and continuous. Also, $f$ will be referred to as a piecewise monotone piecewise $C^r$ map if $f$ is monotone and of class $C^r$ on each of the intervals $(0, d_1), (d_1, d_2), \ldots, (d_{m-1}, 1)$. The map $f$ need not to be strictly monotone on each interval in $\mathcal{Z}$, and moreover, at the endpoints $d_i$'s, the map $f$ need not to be discontinuous nor to change the type of monotonicity. To define the topological entropy in the case of piecewise continuous piecewise monotone maps, here we use the approach by doubling points construction, as in [7]; see below for details. There are other definitions of topological entropy for these maps (via separated or spanned sets, and also by counting growth number for preturning points) and they are equivalent, as it is shown in [9].

We now state the result on approximating the topological entropy $h_{\text{top}}(f)$ by restricting on compact $f$-invariant sets whose $f$-orbits are disjoint from some neighborhood of $\{d_1, d_2, \ldots, d_{m-1}\}$. In what follows, by an $f$-orbit we mean the forward orbit under $f$; by an $f$-invariant set $M$, we mean a forward invariant one: $f(M) \subset M$, while strict invariance means that $f(M) = M$.

**Theorem 3.** *Let $f$ be a piecewise monotone piecewise continuous map on $I = [0, 1]$ with a partition $\mathcal{Z} = \{(0, d_1), (d_1, d_2), \ldots, (d_{m-1}, 1)\}$. If $h_{top}(f) > 0$ then for any $\delta > 0$ there exist a compact strictly $f$-invariant set $M \subset I$ and an open set $V \supset \{d_1, d_2, \ldots, d_{m-1}\}$ such that $h_{top}(f|_M) > h_{top}(f) - \delta$ and $M \cap V = \emptyset$.*

Before proving Theorem 3, we state its consequence.

**Theorem 4.** *Let $f$ be a piecewise monotone piecewise $C^2$ map on a compact interval $I$ with finitely many critical points, attracting periodic points, and non-hyperbolic periodic points. If $h_{top}(f) > 0$ then for any $\delta > 0$ there exists a compact hyperbolic repelling strictly $f$-invariant set $M$ such that $h_{top}(f|_M) > h_{top}(f) - \delta$.*

*Proof.* Let $D$ be the (finite) set consisting of: (i) endpoints of intervals on each of which $f$ is monotone and $C^2$, (ii) critical points of $f$, (iii) non-hyperbolic periodic points of $f$, and (iv) attracting periodic points of $f$. By considering addition of finitely many points to the endpoints of monotonicity intervals for an (old) partition, we may assume the points of $D$ as endpoints of the new partition. Then given any $\delta > 0$, by using Theorem 3 we get a compact strictly $f$-invariant set $M$ and an open set $V \supset D$ so that $f$-orbits of $M$ are disjoint from $V$, and $h_{\text{top}}(f|_M) > h_{\text{top}}(f) - \delta$. If $d \in D$ is a point at which $f$ is not of class $C^2$, then we can modify $f$ inside a small neighborhood of $d$ contained in $V$ such that (the modified map) $f$ becomes $C^2$ in the closure of this neighborhood. Repeating such a smoothing for each point under consideration, we get a map, say $F_\delta : I \to I$ of class $C^2$. By such a construction, $F_\delta|_M = f|_M$ and since $M$ is strictly $f$-invariant, we have that $M$ is also strictly $F_\delta$-invariant, and $h_{\text{top}}(F_\delta|_M) = h_{\text{top}}(f|_M) > h_{\text{top}}(f) - \delta$. Therefore, by using Corollary 1 for the map $F_\delta$ and the compact invariant set $M$, we complete the proof.

The proof of Theorem 3 contains several lemmas below. We will need the so called doubling points construction (see e.g., [7]). Let $I = [0, 1]$ and $D := \{d_1, d_2, \ldots, d_{m-1}\}$. Let us emphasize that we do not care about values of $f$ at the points from $D$ because only one sided limits of $f$ at these points are of use (see also [9], where the authors proved that the values of the map at the endpoints of intervals of continuity are irrelavent for calculation of topological entropy). Define the set

$$W := \left(\bigcup_{i=0}^{\infty} f^{-i}(D)\right) \setminus \{0, 1\}$$

(which could be thought of as "the set of preturning points"). Now consider the following set $\hat{I}$ which contains "doubling preturning points" rather than single ones:

$$\hat{I} := (I \setminus W) \cup \{w^-, w^+ : w \in W\}.$$

The set $\hat{I} = \hat{I}(f)$ is endowed with the natural (total) ordering so that if $y < w < z$ in $I$ and $w \in W$, then $y < w^- < w^+ < z$. Then it is supposed that $\hat{I}$ is endowed with the order topology (note that $\hat{I}$ is a totally disconnected space provided $f$ has no homtervals). We will call $\hat{I}$ the *doubling construction space* for $f$.

Let $\pi : \hat{I} \to I$ denote the map by

$$\pi(y) = w \quad \text{for } y \in \{w^-, w^+\} \text{ with } w \in W, \text{and } \pi(y) = y \text{ for } y \in I \setminus W. \tag{2.1}$$

For a subset $A \subset I$, let $clos_{\hat{I}} A$ denote the closure of $\pi^{-1}(A \setminus W)$ in $\hat{I}$. Let $\hat{\mathcal{Z}} = \{clos_{\hat{I}} A : A \in \mathcal{Z}\}$. The restriction $f|_{I \setminus W}$ can be uniquely extended to a continuous piecewise monotone map $\hat{f} : \hat{I} \to \hat{I}$. We will call $\hat{f}$ the *doubling extension* of $f$.

For $x, y \in \hat{I}$, let $\ell(x,y)$ be the minimal nonnegative integer $\ell$ such that $\hat{f}^\ell(x)$ and $\hat{f}^\ell(y)$ belong to different elements of $\hat{\mathcal{Z}}$; and set $\ell(x,y) = +\infty$ if for any $n$, $\hat{f}^n(x)$ and $\hat{f}^n(y)$ belong to the same element (depending on $n$) of $\hat{\mathcal{Z}}$. Then the order topology on $\hat{I}$ is induced by the metric $\hat{\rho}$ on $\hat{I}$ defined by the formula

$$\hat{\rho}(x,y) := \frac{1}{\ell(x,y)+1} + |\pi(x) - \pi(y)|. \tag{2.2}$$

We remark that $\hat{\rho}(d^-, d^+) = 1$ for any $d \in D$ (even in the case when $f$ is continuous at $d$). The map $\pi$ is continuous on $\hat{I}$ and it is a semiconjugacy from $\hat{f}$ to $f$ in the following sense: $f \circ \pi\,(x) = \pi \circ \hat{f}(x)$ for all $x \in \hat{I} \setminus \{d^-, d^+ : d \in D\}$, and if $x = d^+$ (resp. $x = d^-$) for some $d \in D$, then $\pi \circ \hat{f}(x) = \lim_{y \searrow d} f \circ \pi\,(y)$ (resp. $\pi \circ \hat{f}(x) = \lim_{y \nearrow d} f \circ \pi\,(y)$). So, if $P$ is an $\hat{f}$-invariant set disjoint from $\{w^-, w^+ : w \in W\}$ then $\pi$ conjugates $\hat{f}|_P$ to $f|_{\pi(P)}$.

According to the total ordering in $\hat{I}$, we can consider intervals in $\hat{I}$ of the form $(a,b)$, $(a,b]$, $[a,b)$, or $[a,b]$ (the latter possibly with $a = b$), $a, b \in \hat{I}$. Let $c \in \hat{I}$, $\epsilon > 0$ and let $U_\epsilon(c)$ denote the open ball of radius $\epsilon$ centered at $c$ : $U_\epsilon(c) = \{x \in \hat{I} : \hat{\rho}(x,c) < \epsilon\}$. Notice that $U_\epsilon(c)$ is an interval in $\hat{I}$ which might have any of the four above forms, and $\pi(U_\epsilon(c))$ is an interval in $I$ which contains $\pi(c)$, but $\pi(c)$ need not to be the middle point of this interval). Nevertheless, it is easily seen that $\pi(U_\epsilon(c))$ tends to $\{c\}$ as $\epsilon \to 0$.

Following [7], we define the topological entropy, $h_{\rm top}(f)$, of the initial map $f$ to be equal to $h_{\rm top}(\hat{f})$, the usual topological entropy of the continuous map $\hat{f}$ on the compact space $\hat{I}$.

Given an $\epsilon > 0$ and $a \in \hat{I}$, let $M_{\epsilon,a}$ denote the set

$$M_{\epsilon,a} := \hat{I} \setminus \bigcup\nolimits_{n=0}^{\infty} \hat{f}^{-n}(U_\epsilon(a)). \tag{2.3}$$

So $M_{\epsilon,a}$ is a compact $\hat{f}$-invariant subset of $\hat{I}$ (in the order topology) such that $\hat{f}$-orbits of points from $M_{\epsilon,a}$ never visit $U_\epsilon(a)$. Let us fix (for a while) an $\epsilon$ with

$$0 < \epsilon < \min(1, \min\{|d - d'|/2 : d, d' \in D, d \neq d'\}) \tag{2.4}$$

and take a point $a \in \{d^-, d^+\}$ for some $d \in D$. We now consider, without loss of generality, the case when $a = d^-$. Then, because of (2.4), the subinterval $U_\epsilon(a)$ of $\hat{I}$ is of the form either $(y, d^-]$ with $y \in I \setminus W$ or $[y, d^-]$ with $y = w^+$ for some $w \in W$. We denote such a left end point $y$ by $d_\epsilon^-$. Notice that (2.4) also implies that these $2(m-1)$ subintervals $\{U_\epsilon(d^-), U_\epsilon(d^+) : d \in D\}$ are disjoint (because the distance $\hat{\rho}$ between points on $\hat{I}$ is bigger than or equal to the distance on $I$ between the $\pi$-images of these points).

Define the following map $\tilde{f}_{\epsilon,d^-}$ on $\hat{I}$ by

$$\tilde{f}_{\epsilon,d^-}(x) = \begin{cases} \hat{f}(x), & \text{if } x \notin U_\epsilon(d^-), \\ \hat{f}(d_\epsilon^-), & \text{otherwise,} \end{cases} \tag{2.5}$$

and call it the *left $\epsilon$-truncation of $\hat{f}$ at $d$.*

**Lemma 1.** *The map $\tilde{f}_{\epsilon,d^-} : \hat{I} \to \hat{I}$ is continuous and $h_{top}(\tilde{f}_{\epsilon,d^-}) = h_{top}(\tilde{f}_{\epsilon,d^-}|_{M_{\epsilon,d^-}}) = h_{top}(\hat{f}|_{M_{\epsilon,d^-}})$.*

*Proof.* The map $\tilde{f}_{\epsilon,d^-}$ differs from $\hat{f}$ only on the interval $U_\epsilon(d^-)$, which equals either $(d_\epsilon^-, d^-]$ or $[d_\epsilon^-, d^-]$. Since $\hat{f}$ is continuous, it follows that in both cases $\tilde{f}_{\epsilon,d^-}$ is continuous at $d_\epsilon^-$ (because of the definition by (2.5)). Moreover, $\tilde{f}_{\epsilon,d^-}$ is continuous at $d^-$ because $d^-$ is isolated in $\hat{I}$ from the right.

Let $\Omega(\tilde{f}_{\epsilon,d^-})$ be the nonwandering set of $\tilde{f}_{\epsilon,d^-}$. If $\hat{f}^n(d_\epsilon^-) \cap U_\epsilon(d^-) = \varnothing$ for every $n \geqslant 1$, then $\Omega(\tilde{f}_{\epsilon,d^-}) \subset M_{\epsilon,d^-}$ because $U_\epsilon(d^-)$ consists of wandering points for $\tilde{f}_{\epsilon,d^-}$, and if we suppose, by contrary, that there is a point in $\Omega(\tilde{f}_{\epsilon,d^-}) \setminus M_{\epsilon,d^-}$, then we would have a contradiction to the fact that the nonwandering set is invariant. Therefore,

$$h_{\text{top}}(\tilde{f}_{\epsilon,d^-}) = h_{\text{top}}(\tilde{f}_{\epsilon,d^-}|_{\Omega(\tilde{f}_{\epsilon,d^-})}) = h_{\text{top}}(\tilde{f}_{\epsilon,d^-}|_{M_{\epsilon,d^-}}) = h_{\text{top}}(\hat{f}|_{M_{\epsilon,d^-}}), \tag{2.6}$$

where the last equality holds because the restriction of $\hat{f}$ to $M_{\epsilon,d^-}$ coincides with $\tilde{f}_{\epsilon,d^-}$. In the case when $\hat{f}^{n_0}(d_\epsilon^-) \in U_\epsilon(d^-)$ for some $n_0 \geqslant 1$, it is easily seen that the set $\Omega(\tilde{f}_{\epsilon,d^-}) \setminus M_{\epsilon,d^-}$ consists precisely of one periodic orbit of period $n_0$. Thus we have as before, $h_{\text{top}}(\tilde{f}_{\epsilon,d^-}|_{\Omega(\tilde{f}_{\epsilon,d^-})}) = h_{\text{top}}(\tilde{f}_{\epsilon,d^-}|_{M_{\epsilon,d^-}})$ because the topological entropy on a finite set is zero. So in this case the equalities in (2.6) are true.

Similar to the left $\epsilon$-truncation, we can consider the right $\epsilon$-truncation, $\tilde{f}_{\epsilon,d^+}$, and get $h_{\text{top}}(\hat{f}|_{M_{\epsilon,d^+}}) = h_{\text{top}}(\tilde{f}_{\epsilon,d^+}|_{M_{\epsilon,d^+}})$. Furthermore, we consider the $\epsilon$-truncations for all $d \in D$ simultaneously. To do this, we define the map $\tilde{f}_\epsilon$ on $\hat{I}$ by

$$\tilde{f}_\epsilon(x) = \begin{cases} \hat{f}(x), & \text{if } x \notin U_\epsilon(d^-) \cup U_\epsilon(d^+) \text{ for all } d \in D, \\ \hat{f}(d_\epsilon^-), & \text{if } x \in U_\epsilon(d^-) \text{ with } d \in D, \\ \hat{f}(d_\epsilon^+), & \text{if } x \in U_\epsilon(d^+) \text{ with } d \in D. \end{cases} \tag{2.7}$$

Also let $\tilde{M}_\epsilon := \bigcap_{d\in D}(M_{\epsilon,d^-} \cap M_{\epsilon,d^+})$. Then $\tilde{M}_\epsilon$ is a compact $\hat{f}$-invariant subset of $\hat{I}$ whose $\hat{f}$-orbits never visit $\epsilon$-neighborhood of the set $\bigcup_{d\in D}\{d^-, d^+\}$. Since $D$ is a finite set, the result similar to Lemma 1 readily follows.

**Lemma 2.** *The map $\tilde{f}_\epsilon : \hat{I} \to \hat{I}$ is continuous and $h_{top}(\tilde{f}_\epsilon) = h_{top}(\tilde{f}_\epsilon|_{\tilde{M}_\epsilon}) = h_{top}(\hat{f}|_{\tilde{M}_\epsilon})$.*

In order to relate the above properties of continuous maps on $\hat{I}$ to properties of piecewise continuous maps on $I$ (in other words, "to project" the constructed truncations to maps on $I$), we need to introduce an intermediate space. To do this, we identify those pairs of points $\{w^-, w^+\}, w \in W$, which under some iterate of $\hat{f}$ belong to the same $\epsilon$-neighborhood of either $d^-$ or $d^+$ for some $d \in D$. More precisely, consider the following equivalence relation $\sim$ on $\hat{I}$:

$$x \sim y \iff \begin{array}{l} x = y \text{ or } \{x, y\} = \{w^-, w^+\} \text{ with } w \in W, \text{ and there exist} \\ n \geqslant 0 \text{ and } \tilde{d} \in \bigcup_{d\in D}\{d^-, d^+\} \text{ such that } \{\hat{f}^n(x), \hat{f}^n(y)\} \subset U_\epsilon(\tilde{d}). \end{array}$$

Notice that by the above definition, the relation $\sim$ is preserved by $\tilde{f}_\epsilon$, i.e., if $x \sim y$ then $\tilde{f}_\epsilon(x) \sim \tilde{f}_\epsilon(y)$. Let $\hat{I}_\epsilon$ be the quotient space with respect to this relation $\sim$ and let $\hat{\pi}_\epsilon : \hat{I} \to \hat{I}_\epsilon$ be the corresponding quotient map. Clearly, $\hat{\pi}_\epsilon$ is at most two-to-one, order preserving, and is continuous with respect to the order topologies on $\hat{I}$ and $\hat{I}_\epsilon$. If two points $w^-, w^+$ are collapsed by $\hat{\pi}_\epsilon$ (i.e., $w^- \sim w^+$), we will denote their common image simply by $w$. Let $W_\epsilon$ denote the subset of $W$ which consists of "non-collapsed" points by $\hat{\pi}_\epsilon$, i.e.,

$$W_\epsilon = \{w \in W : \sharp\{\hat{\pi}_\epsilon^{-1}(w)\} = 1\}.$$

Then $\hat{I}_\epsilon$ can be represented as $\hat{I}_\epsilon = (I \setminus W_\epsilon) \cup \{w^-, w^+ : w \in W_\epsilon\}$. By the definition of $\hat{\pi}_\epsilon$, one easily gets $\pi_\epsilon \circ \hat{\pi}_\epsilon = \pi$, where $\pi_\epsilon : \hat{I}_\epsilon \to I$ is defined just as $\pi$ by (2.1) (for $\pi_\epsilon$ we use the subscript $\epsilon$ in order to mention that it acts on the space different from $\hat{I}$).

Let $g : \hat{I} \to \hat{I}$ be a continuous map which preserves the relation $\sim$, i.e., $g$ satisfies the assumption that $g(w^-) = g(w^+)$ for every $w \in W$ with $\hat{\pi}_\epsilon(w^-) = \hat{\pi}_\epsilon(w^+)$. Then $g$ projects to a continuous map on $\hat{I}_\epsilon$. Indeed, consider the map $g^\dagger : \hat{I}_\epsilon \to \hat{I}_\epsilon$ defined by

$$g^\dagger(x) = \hat{\pi}_\epsilon \circ g(\hat{\pi}_\epsilon^{-1}(x)),$$

where by $\hat{\pi}_\epsilon^{-1}(x)$ we mean the full preimage of $x$; notice that although $\hat{\pi}_\epsilon^{-1}(x)$ may consist of two points, $g(\hat{\pi}_\epsilon^{-1}(x))$ is a single point because of the above assumption on $g$. By its definition, $g^\dagger$ is continuous and satisfies $g^\dagger \circ \hat{\pi}_\epsilon = \hat{\pi}_\epsilon \circ g$. We may apply the above construction to the map $g = \tilde{f}_\epsilon$ because it is continuous on $\hat{I}$ and preserves the relation $\sim$; hence we have the continuous map $\tilde{f}_\epsilon^\dagger$ on $\hat{I}_\epsilon$.

Now we return to the "initial phase space" $I$. Let $M_\epsilon := \pi(\tilde{M}_\epsilon)$.

**Lemma 3.** *The following statements hold:*

1. *The set $M_\epsilon$ is a compact $f$-invariant subset of $I$;*
2. *There is a neighborhood of $D$ such that for any point $x_0 \in M_\epsilon$, its $f$-orbit is disjoint from this neighborhood.*

*Proof.* Let us recall that $\pi : \hat{I} \to I$ is continuous and the restriction of $\pi$ to any subset of $\hat{I}$ disjoint from $\{w^-, w^+ : w \in W\}$ is a one-to-one map which satisfies $f \circ \pi(x) = \pi \circ \hat{f}(x)$. This implies the first statement of the lemma.

Let

$$\delta_0 = \min_{d \in D} \min(d - \pi(d_\epsilon^-), \pi(d_\epsilon^+) - d).$$

Then the $f$-orbit of any initial point from $M_\epsilon$ is away from $D$ by the distance at least $\delta_0$.

So, in particular, $M_\epsilon$ is disjoint from $W$ and therefore, by notation in (2.1), $M_\epsilon$ is the same as $\tilde{M}_\epsilon$ (they are homeomorphic metric spaces with respect to the usual length on $M_\epsilon$ and the metric $\hat{\rho}$ on $\tilde{M}_\epsilon$). Next, we define the $\epsilon$-truncation map $f_\epsilon : I \to I$ by

$$f_\epsilon(x) = \begin{cases} f(x), & \text{if } x \notin (\pi(d_\epsilon^-), \pi(d_\epsilon^+)), \text{ for all } d \in D, \\ f(\pi(d_\epsilon^-)), & \text{if } x \in (\pi(d_\epsilon^-), d), d \in D, \\ f(\pi(d_\epsilon^+)), & \text{if } x \in (d, \pi(d_\epsilon^+)), d \in D. \end{cases} \tag{2.8}$$

See Figure 1. It is easily seen that $f_\epsilon$ is continuous at the points $\pi(d_\epsilon^-), \pi(d_\epsilon^+)$ for any $d \in D$. Hence, $f_\epsilon$ is a piecewise continuous and piecewise monotone map with the partition $\mathcal{Z}$. Then it is easily checked that the doubling construction space for $f_\epsilon$ is precisely $\hat{I}_\epsilon$( i.e., if we consider $f_\epsilon$ as the initial map then $\hat{I}_{f_\epsilon}$, its doubling construction space, coincides with $\hat{I}_\epsilon$ ), while the doubling extension of $f_\epsilon$ is precisely $\tilde{f}_\epsilon^\dagger$, i.e., $\hat{f}_\epsilon = \tilde{f}_\epsilon^\dagger$. So we have the following commutative diagrams:

$$\begin{array}{ccccc} \hat{I} & \xrightarrow{\hat{\pi}_\epsilon} & \hat{I}_\epsilon & \xrightarrow{\pi_\epsilon} & I \\ \tilde{f}_\epsilon \downarrow & & \tilde{f}_\epsilon^\dagger \downarrow \hat{f}_\epsilon & & \downarrow f_\epsilon \\ \hat{I} & \xrightarrow[\hat{\pi}_\epsilon]{} & \hat{I}_\epsilon & \xrightarrow[\pi_\epsilon]{} & I \end{array} \tag{2.9}$$

For the proof of Theorem 3, we need one general fact on topological entropy as follows.

**Lemma 4.** *Let $X$ be a compact metric space and $\varphi : X \to X$ a continuous map. Then $Y := \cap_{n=0}^{\infty} \varphi^n(X)$ is a compact strictly invariant subset of $X$ and $h_{top}(\varphi|_Y) = h_{top}(\varphi)$.*

*Proof.* The strict $\varphi$-invariance of $Y$ follows from our assumptions on compactness of $X$ and continuity of $\varphi$. The equality $h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi|_{\cap_{n=0}^{\infty}\varphi^n(X)})$ follows from [11], Corollary 8.6.1.

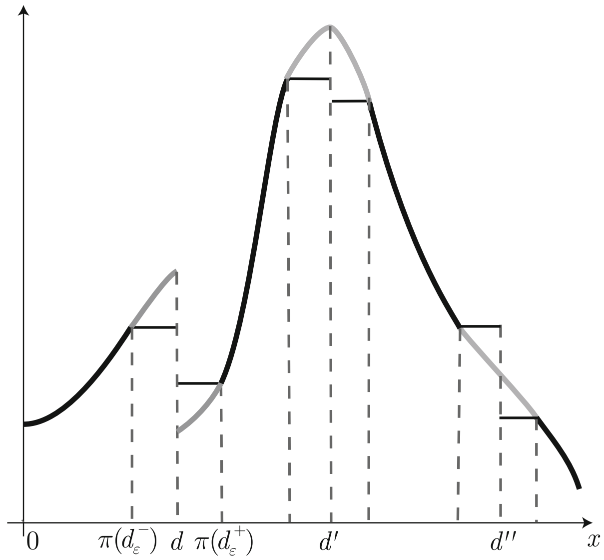


**Fig. 1.** The graph of the $\epsilon$-truncation map $f_\epsilon$.

Now we give the proof of Theorem 3, named as Theorem B in Section 1.

*Proof.* Since $\tilde{f}_\epsilon$ is semiconjugate to $\hat{f}_\epsilon$ by the map $\hat{\pi}_\epsilon$, which is at most two-to-one, we have $h_{\mathrm{top}}(\tilde{f}_\epsilon) = h_{\mathrm{top}}(\hat{f}_\epsilon)$. Note that if we consider the restrictions $\tilde{f}_\epsilon|_{M_\epsilon}$, $\hat{f}_\epsilon|_{\hat{\pi}_\epsilon(M_\epsilon)}$ and $f_\epsilon|_{M_\epsilon}$ in diagram (2.9), then the semicongugacies $\hat{\pi}_\epsilon$ and $\pi_\epsilon$ becomes, in fact, conjugacies. Hence $h_{\mathrm{top}}(\tilde{f}_\epsilon|_{M_\epsilon}) = h_{\mathrm{top}}(f_\epsilon|_{M_\epsilon})$. So by using lemma 2, we have

$$h_{\mathrm{top}}(\hat{f}_\epsilon) = h_{\mathrm{top}}(\tilde{f}_\epsilon) = h_{\mathrm{top}}(\tilde{f}_\epsilon|_{M_\epsilon}) = h_{\mathrm{top}}(f_\epsilon|_{M_\epsilon}).$$

Since $\hat{f}_\epsilon$ is the doubling extension of $f_\epsilon$, it follows from the definition of topological entropy for piecewise monotone maps that $h_{\mathrm{top}}(f_\epsilon) = h_{\mathrm{top}}(\hat{f}_\epsilon)$. Now, using the fact that the restrictions to $M_\epsilon$ of the maps $f$ and $f_\epsilon$ coincide, we have

$$h_{\mathrm{top}}(f_\epsilon) = h_{\mathrm{top}}(f_\epsilon|_{M_\epsilon}) = h_{\mathrm{top}}(f|_{M_\epsilon}). \tag{2.10}$$

Since each interval $(\pi(d_\epsilon^-), \pi(d_\epsilon^+))$ tends to $\{d\}$ as $\epsilon \to 0$, we have that the Hausdorff distance between graphs of $f_\epsilon$ and $f$ tends to zero. Thus by the lower semi-continuity property of the topological entropy function on the set of piecewise monotone piecewise continuous maps with the given number of monotone intervals (see [9]), we have $\liminf_{\epsilon\to 0} h_{\mathrm{top}}(f_\epsilon) \geqslant h_{\mathrm{top}}(f)$. On the other hand, it is easily seen that for any $\epsilon > 0$, $h_{\mathrm{top}}(f_\epsilon) \leqslant h_{\mathrm{top}}(f)$. So we get that $\lim_{\epsilon\to 0} h_{\mathrm{top}}(f_\epsilon) = h_{\mathrm{top}}(f)$ and thus, by (2.10), $\lim_{\epsilon\to 0} h_{\mathrm{top}}(f|_{M_\epsilon}) = h_{\mathrm{top}}(f)$. So, given a $\delta > 0$ we can find $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, $h_{\mathrm{top}}(f|_{M_\epsilon}) > h_{\mathrm{top}}(f) - \delta$, and so by using lemma 4 (since $M_\epsilon$ is compact and $f$-invariant) we may set $M := \cap_{n=0}^{\infty} f^n(M_\epsilon)$ and $V := \cup_{d\in D}(\pi(d_\epsilon^-), \pi(d_\epsilon^+))$ according to notations in the statement of Theorem 3.

## 3. PROOF OF THEOREM A

The following result can be regarded in a sense as an extension of Theorem 4.

**Theorem 5.** *Let $f$ be a piecewise monotone piecewise $C^2$ map on a compact interval $I$ and assume that its derivative is piecewise monotone as well. If $h_{top}(f) > 0$ then for any $\delta > 0$ there exists a compact hyperbolic repelling strictly $f$-invariant set $M$ such that $h_{top}(f|_M) > h_{top}(f) - \delta$.*

We will need the following lemma. To clarify the statement, let us assume that for a piecewise monotone piecewise $C^2$ map $g$ with piecewise monotone derivative, its partition is chosen so that on each interval of the partition, $g$ is $C^2$ and both $g$ and $Dg$, the derivative of $g$, are monotone. We will refer to the intervals of such a partition simply as monotonicity intervals.

**Lemma 5.** *Let $g : I \to I$ be a piecewise monotone piecewise $C^2$ map with piecewise monotone derivative and let $U$ be a neighborhood of the set containing critical values of $g$ and endpoints of monotonicity intervals. Then there exists a natural number $K$ such that all $g$-periodic orbits having periods bigger than $K$ and contained in $I \setminus U$ are hyperbolic repelling.*

*Proof.* Since the derivative $Dg$ is piecewise monotone, it follows that $L := \{g(x) :\ Dg(x) = 0\}$, the set of critical values of $g$, is finite. For any $y \in L$, the intersection of $g^{-1}(y)$ and the set of critical points is a union of finitely many closed (possibly trivial) intervals. Take a small neighborhood of each of these intervals so that its $g$-image belongs to $U$. Denote by $V(y)$ the union of such neighborhoods and let $V = \bigcup_{y\in L} V(y)$.

Next we modify the map $g$ inside small neighborhoods containing endpoints of monotonicity intervals and contained in $U$, in order to obtain a $C^2$ map $G : I \to I$. By applying Theorem 2 to $G$, we get that any periodic orbit of $G$ contained in $I \setminus V$ of sufficiently large period is hyperbolic repelling. One the other hand, by the construction of $G$, any $g$-periodic orbit contained in $I \setminus U$ coincides with $G$-periodic orbit and is away from $V$. This completes the proof.

Now we prove Theorem 5 which is named as Theorem A in the Introduction.

*Proof.* Since the set of critical values of $f$ is finite, we may add the points in this set to the endpoints of monotonicity intervals of $f$. Let us denote the obtained set by $D = \{d_1, d_2, \ldots, d_{m-1}\}$ and let $\mathcal{Z}$ be the induced partition (we could regard $\mathcal{Z}$ as the initial partition for $f$). Next we apply Theorem 3 to $f$ with the partition $\mathcal{Z}$ for $\delta/2$ instead of $\delta$ in the statement of Theorem 3. Thus, with the notations in the proof of Theorem 3, for given $\delta > 0$ we have a constant $\epsilon > 0$, an open set $V_\epsilon := \bigcup_{d\in D} (\pi(d_\epsilon^-), \pi(d_\epsilon^+)) \supset D$ and a compact $f$-invariant set $M_\epsilon$ such that $h_{\mathrm{top}}(f|_{M_\epsilon}) > h_{\mathrm{top}}(f) - \delta/2$ and $f^n(M_\epsilon) \cap V_\epsilon = \emptyset$ for any $n \geqslant 0$. We have also constructed the map $f_\epsilon : I \to I$ defined in (2.8) and associated to the set $V_\epsilon$, and for this map we have (see (2.10))

$$h_{\mathrm{top}}(f_\epsilon) = h_{\mathrm{top}}(f|_{M_\epsilon}) > h_{\mathrm{top}}(f) - \delta/2. \tag{3.1}$$

Let us stress that the number $\epsilon$, the map $f_\epsilon$ and the sets $V_\epsilon$, $M_\epsilon$ will be fixed till the end of the proof. Also we note that any $f$-orbit contained in $I \setminus V_\epsilon$ is the same as $f_\epsilon$-orbit.

Next we apply Lemma 5 to the map $f$ and the open set $V_\epsilon$ (denoted there by $g$ and $U$ respectively). Let $K$ be as in the statement of Lemma 5. Then all attracting and nonhyperbolic periodic orbits contained in $I \setminus V_\epsilon$ have periods less than $K$. Therefore the points of these orbits are fixed points for the $K!$-th iterate of $f$, where $K!$ means the factorial of $K$. Note that there are only finitely many attracting periodic points under consideration; indeed, since the derivative $Df$ is piecewise monotone, it follows that on each monotonicity interval for $f^{K!}$, there might exist at most one attracting fixed point because attracting fixed points must alternate with non-attracting ones. As for nonhyperbolic periodic $f$-orbits contained in $I \setminus V_\epsilon$, there are fixed points for the map $f^{K!}$, which coincides with the map $f_\epsilon^{K!}$ on $\bigcup_{n=0}^{K!-1} f^{-n}(V_\epsilon)$. We show now that these points form a union of finitely many closed (possibly, trivial) intervals. Indeed, because of piecewise monotonicity of $Df_\epsilon$, solutions for the equation $|Df_\epsilon^{K!}(x)| = 1$ form a finite union of closed (possibly, trivial) intervals. Let be $J$ be such an interval with $|Df_\epsilon^{K!}(x)| = 1$ on $J$ and $f_\epsilon^{K!}(x_0) = x_0$ for some $x_0 \in J$. In the case when $Df_\epsilon^{K!}(x) = 1$ on $J$, it is clear that $f_\epsilon^{K!}(x) = x$ for any $x \in J$. If $Df_\epsilon^{K!}(x) = -1$ on $J$, then in fact, $J = \{x_0\}$ because for any other point in $J$ one would have that its least period is $2K!$, which contradicts the definition of the number $K$. So the set of nonhyperbolic periodic points under consideration consists of finitely many (possibly none) nontrivial closed intervals, say $J_i = [\alpha_i, \beta_i]$, $i = 1, \ldots, k$, and finitely many single points. We will denote by $E$ the (finite) set consisting of attracting and single nonhyperbolc periodic points (whose orbits are not in $V_\epsilon$).

We need to construct further truncations for $f_\epsilon$ in order to remove from $M_\epsilon$ the nonhyperbolic periodic points and also the points which belong to basins of periodic attractors (whenever they exist). For this we supply the map $f_\epsilon$ with a finer partition: namely, let $D' = D \cup E \cup \{\alpha_i, \beta_i : i = 1, \ldots, k\}$, so we have added to $D$ attracting periodic points, single nonhyperbolic periodic points and endpoints of nonhyperbolic periodic intervals (away from $V_\epsilon$). Let $\mathcal{Z}'$ be the partition for $f_\epsilon$ induced by $D'$. Let us rename $f_\epsilon$ by $g$ (since $\epsilon$ is fixed, it is convenient to regard $g = f_\epsilon$ as the initial map for the doubling construction). It is also convenient to rename for brevity $d_\ell := \pi(d_\epsilon^-)$ and $d_r := \pi(d_\epsilon^+)$. Then for $g$ with the partition $\mathcal{Z}'$, we consider the doubling construction space $\hat{I}' := \hat{I}(g)$ and the doubling extension $\hat{g}$.

Let the map $\pi' : \hat{I}' \to I$ and the metric $\hat{\rho}'$ on $\hat{I}'$ be defined as in formulas (2.1) and (2.2). For $\gamma > 0$ and $a \in \hat{I}'$, we denote by $U'_\gamma(a)$ an open ball in $\hat{I}'$ of radius $\gamma$ centered at $a$ with respect to the metric $\hat{\rho}'$. Then $U'_\gamma(a)$ is actually an interval in $\hat{I}'$. We further denote by $a_\gamma^-$, $a_\gamma^+$ the endpoints of the interval $U'_\gamma(a)$.

Let $\eta$ be a positive number smaller than the half of the minimal pairwise distances between points of the set $D' \cup \{d_\ell, d_r :\ d \in D\}$. Let the left and right $\eta$-truncations for the map $\hat{g} : \hat{I}' \to \hat{I}'$ be constructed at the points $e^-, e^+ \in \hat{I}'$ for each $e \in E$ as in formulas (2.5) and (2.7) for similar truncations for the map $\hat{f}$), and denote them by $\tilde{g}_{\eta,e^-}$ and $\tilde{g}_{\eta,e^+}$, respectively. Then for the obtained truncations $\tilde{g}_{\eta,e^-}$, $\tilde{g}_{\eta,e^+}$ and the corresponding sets $M'_{\eta,e^-}$, $M'_{\eta,e^+}$ (i.e., the sets whose $\hat{g}$-orbits never visit $\eta$-neighborhoods of $e^-$ and $e^-$ respectively; see formula (2.3)), the results similar to those in Lemma 1 clearly hold true.

Furthermore, for given a "periodic" interval $J = [\alpha, \beta]$ in the collection $\{[\alpha_i, \beta_i] : i = 1, \dots, k\}$, we construct the following map $\tilde{g}_{\eta,J} : \hat{I}' \to \hat{I}'$, the $\eta$-truncation associated to $J$ , by

$$\tilde{g}_{\eta,J}(x) = \begin{cases} \hat{g}(x), & \text{if } x \notin U'_\eta(\alpha^-) \cup U'_\eta(\beta^+), \\ \hat{g}(\alpha_\eta^-), & \text{if } x \in U'_\eta(\alpha^-), \\ \hat{g}(\beta_\eta^+), & \text{if } x \in U'_\eta(\beta^+). \end{cases} \tag{3.2}$$

Then $\tilde{g}_{\eta,J}$ is continuous on $\hat{I}'$, and we obtain the results on topological entropy similar to those in Lemma 1. More precisely, let $\hat{J}'$ be the closure of

$$(\pi')^{-1}(J \setminus \Big(\Big(\bigcup_{i=0}^{\infty} g^{-i}(D')\Big) \setminus \{0,1\}\Big)$$

in $\hat{I}'$, and let $M'_{\eta,J}$ denote the set of points in $\hat{I}'$ whose $\hat{g}$-orbits never visit $\eta$-neighborhood of $\hat{J}'$, *i.e.*,

$$M'_{\eta,J} := \hat{I}' \setminus \bigcup_{n=0}^{\infty} \hat{g}^{-n}(U'_\eta(J)), \quad \text{where } U'_\eta(J) = U'_\eta(\alpha^-) \cup \hat{J}' \cup U'_\eta(\beta^+)).$$

Then $M'_{\eta,J}$ is compact and $\hat{g}$-invariant. Next, we claim that $h_{\text{top}}(\tilde{g}_{\eta,J}) = h_{\text{top}}(\hat{g}|_{M'_{\eta,J}})$. Indeed, the only difference from the proof of Lemma 1 is that the nonwandering points in $U'_\eta(\alpha^-) \cup \hat{J}' \cup U'_\eta(\beta^+)$, except for two possible periodic points in $U'_\eta(\alpha^-)$ and $U'_\eta(\beta^+)$, fill the whole periodic interval $\hat{J}'$. Since $\hat{g}^{K!}$ is the identity map on this interval, $h_{\text{top}}(\hat{g}|_{\hat{J}'}) = 0$ and therefore, the claim is true.

Now by constructing simultaneously the $\eta$-truncations at $e^-, e^+$ and $J$ for all $e \in E$ and all $J$ in the collection $\mathcal{P} := \{[\alpha_i, \beta_i] : i = 1, \dots, k\}$, we get the $\eta$-truncation map, say $\tilde{g}_\eta$. Let $M'_g := clos_{\hat{I}'} M_\epsilon$. Then the $\hat{g}$-orbits contained $M'_g$ in never visit open intervals $(\pi')^{-1}(d_\ell, d_r)$, $d \in D$. Let

$$\tilde{M}'_\eta := M'_g \cap \bigcap_{e \in E} (M'_{\eta,e^-} \cap M'_{\eta,e^+}) \cap (\bigcap_{J \in \mathcal{P}} M'_{\eta,J}). \tag{3.3}$$

Then $\tilde{M}'_\eta$ is a compact $\hat{g}$-invariant set whose orbits never visit neither the intervals $(d_\ell, d_r)$ for $d \in D$, nor the $\eta$-neighborhoods of the points $e^-, e^+$ for $e \in E$, and of the intervals $\hat{J}'$ for $J \in \mathcal{P}$. Hence, similar to Lemma 2, we have the result that $h_{\text{top}}(\tilde{g}_\eta) = h_{\text{top}}(\hat{g}|_{\tilde{M}'_\eta})$.

Next, back to the "initial phase space" $I$, we consider the set $M'_\eta := \pi'(\tilde{M}'_\eta)$ and the truncation map $g_\eta : I \to I$ which is defined to coincide with $g$ except at the intervals $(\pi'(e_\eta^-), e)$, $(e, \pi'(e_\eta^+))$ and $(\pi'(\alpha_\eta^-), \alpha)$, $(\beta, \pi'(\beta_\eta^+))$, the map $g_\eta$ takes the constant values $g(\pi'(e_\eta^-)$, $g(\pi'(e_\eta^+)$ and $g(\pi'(\alpha_\eta^-)$,

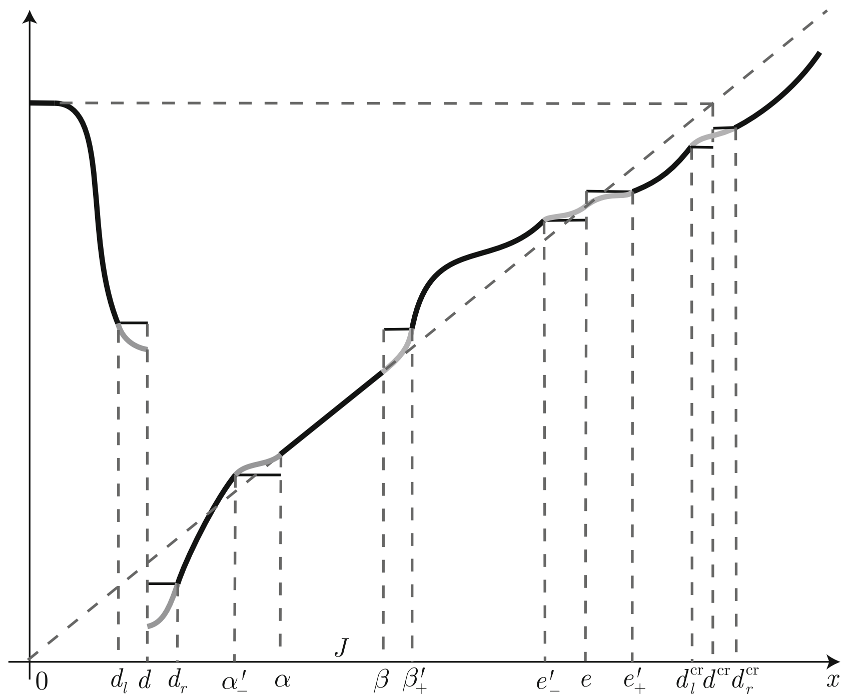


**Fig. 2.** The graph of the $\eta$-truncation map $g_\eta$; here we denote for brevity $\alpha'_- = \pi'(\alpha^-_\eta)$, $\beta'_+ = \pi'(\beta^+_\eta)$, where $J = [\alpha, \beta]$ is a nonhyperbolic periodic interval, and $e'_\pm = \pi'(e^\pm_\eta)$, where $e$ is an attracting periodic point. The points $d$ and $d^{\rm cr}$ belong to the set $D$ (which induces the partition of $f$), where $d^{\rm cr}$ is a critical value of $f$ which equals $f$-image of a nontrivial interval consisting of critical points.

$g(\pi'(\beta^+_\eta))$, respectively (as shown in Fig. 1). As (2.10) in the proof of Theorem 3, we have that $h_{\rm top}(g_\eta) = h_{\rm top}(g|_{M'_\eta})$.

Finally, we consider that $\eta$ approaches zero (while $\epsilon$ is still fixed). Then the Hausdorff distance between graphs of $g_\eta$ and $g$ tends to zero. By the lower semi-continuity property of the topological entropy function due to [9], together with the fact that $h_{\rm top}(g_\eta) \leqslant h_{\rm top}(g)$, we have that $\lim_{\eta\to 0} h_{\rm top}(g_\eta) = h_{\rm top}(g)$. Hence, there exists $\eta > 0$ such that $h_{\rm top}(g|_{M'_\eta}) = h_{\rm top}(g_\eta) > h_{\rm top}(g) - \delta/2$. By the definition of $\tilde{M}'_\eta$ in (3.3), we have $M'_\eta = \pi'(\tilde{M}'_\eta) \subset \pi'(M'_g) \subset M_\epsilon$; hence $f|_{M'_\eta} = f_\epsilon|_{M'_\eta}$. By recalling that $g = f_\epsilon$, we have that

$$h_{\rm top}(f|_{M'_\eta}) = h_{\rm top}(f_\epsilon|_{M'_\eta}) > h_{\rm top}(f_\epsilon) - \delta/2 > h_{\rm top}(f) - \delta,$$

where the last inequality holds due to inequality (3.1).

So, we have obtained for any $\delta > 0$, the compact $f$-invariant set $M'_\eta$ with $h_{\rm top}(f|_{M_\eta}) > h_{\rm top}(f) - \delta$ and the neighborhood

$$V := \bigcup_{e\in E} (\pi'(e^-_\eta), \pi'(e^+_\eta)) \cup \bigcup_{[\alpha,\beta]\in\mathcal{P}} (\pi'(\alpha^-_\eta), \pi'(\beta^+_\eta)) \cup \bigcup_{d\in D} (d_\ell, d_r)$$

of the set consisting of discontinuity points, critical points, attracting periodic points and non-hyperbolic points such that $f$-orbits of $M_\eta$ are disjoint from $V$. Finally, as in Theorem 3, in order to get strictly invariant set with the same entropy on its restriction, we replace $M_\eta$ by $M := \cap_{n=0}^\infty f^n(M_\eta)$.

If $d \in D$ is a point at which $f$ is not of class $C^2$, then we can modify $f$ inside a small neighborhood of $d$ contained in $V$, so that (the modified map) $f$ becomes $C^2$ in the closure of this neighborhood. Repeating such a smoothing for each point under consideration, we get a map, say $F_\delta : I \to I$ of class $C^2$. By this construction, $F_\delta|_M = f|_M$ and since $M$ is strictly $f$-invariant we have that $M$ is also strictly $F_\delta$-invariant, and $h_{\rm top}(F_\delta|_M) = h_{\rm top}(f|_M) > h_{\rm top}(f) - \delta$. Therefore, by applying Corollary 1 to the map $F_\delta$ and the compact invariant set $M$, we complete the proof.

## ACKNOWLEDGMENTS

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