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PMC 模式下超立方體的條件偵錯能力

The Conditional Diagnosability of Hypercube under the PMC Model

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摘要

在多處理器系統中，偵錯能力是一個判斷其系統可靠度的重要指標。在研究中我們發現，許多Ｎ正則圖形的偵錯能力爲Ｎ，是因爲當某一點的所有鄰居同時皆爲壞點時，則系統將無法正確進行診斷，一般情況下此情形發生機率是很低的。所以在此，我們藉由要求一個系統中的壞點集合不可包含任一點的所有鄰居，定義出條件偵錯能力。在本篇論文中，我們證明了超立方體在ＰＭＣ模式下的條件偵錯能力爲 4(*n*-2)+1。

關鍵字：偵錯能力，條件偵錯能力，超立方體，ＰＭＣ模式

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Abstract

The diagnosability is an important role to value the reliability of an interconnection networks. We introduce a new measure of conditional diagnosability by requiring any faulty set cannot contain all the neighbors of any vertex in the graph. Based on this measure, the conditional connectivity of Hypercube is shown to be $4(n\text{-}2)+1$ under the PMC model.

Keyword: diagnosability, conditional diagnosability, Hypercube, PMC model

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Chapter 1

Introduction

Large-scale multiprocessor systems have begun to be a vital role in computing in recent years. The potential use of such system in safety-critical applications increase the importance of their reliability. A key problem in this area, known as the fault diagnosis problem, is to identify the faulty processors in a system. In this approach, a well-know PMC diagnosis model [4] was first introduced by Preparata et al. In this diagnosis model, two linked processors can test each other. We use this model as faulty diagnosis model in this thesis.

A lot discussions of diagnosability under PMC model have been done over past years. For a instance, Hakimi and Amin had the result for a multiprocessor system, it is t-connected and has at least $2t+1$ vertices implies this system is t-diagnosable [9]. And many diagnosabilities of cubes have been valued, like the Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n are all n-diagnosable [7, 5, 6, 2]. We are interesting in what stops the abilities of identifying faulty vertices of these cubes being $n+1$. We observe that these cubes are almost $(n+1)$-connected and almost $(n+1)$-diagnosable except that all the neighbors of some vertex are faulty simultaneously. for the classical diagnosability, if all the neighbors of some vertex v are faulty at the same time, the system cannot determine whether the vertex v is faulty-free or not. In other words, it is impossible that the diagnosability of a system is larger than the minimum vertex degree. This becomes a interesting problem how to increase the diagnosability with some reasonable restrictions.

In advance, we know the reason of the diagnosability of cubes not being $(n+1)$ is that all the neighbors of some vertex are faulty simultaneously. Therefore, we introduce the conditional diagnosability by restricting that each vertex v in the graph, all the vertices which are directly connected to v do not fail at the same time. The Hypercube structure is a well-know interconnection model for multiprocessor system. As a topology to interconnect processors, it has many attractive properties. The fault-tolerant computing for the Hypercube structure has been the interest of many researchers. Under our condition, we show that the conditional diagnosability of Hypercube Q_n is $4(n-2)+1$, which is about four times larger than the classical diagnosability.

In this dissertation, we purpose to prove the conditional diagnosability of Hypercube is $4(n-2)+1$. First, in Chapter 2, we provide terminology and preliminaries for diagnosing a system, and introduce Hypercube including the definition and some studies of Hypercube. In chapter 3, we define conditional diagnosability and study properties of conditionally diagnosable system. Then, the conditional diagnosability of Hypercube is shown in Chapter 4. Finally, our conclusions are given in Chapter 5.

Chapter 2

Terminology and Preliminaries

2.1 The PMC Diagnosis Model

Usually we use $G = (V, E)$ to represent a graph G, V is the node set (or vertex set) and E is the edge set of G. A graph is connected if there is a path in G between any given pair of vertices, and disconnected otherwise. Through this thesis, we use $|G|$ as $|V(G)|$ to denote the number of vertices of G.

If a graph S is a sub-graph of G, it implies that $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$ with $E(S) \subseteq V(S) \times V(S)$, where the $V(S)$ means the node set of S and $E(S)$ is the edge set of S. Now we introduce an operation on graphs, let S be a sub-graph of G, $G - S$ means to remove the vertices of S form G and delete the edges which have at least one end-node contained by S from G.

A vertex cut of a connected graph G is a set of vertices with the following two properties: first, the removal of all the vertices in this set disconnect G; second, the removal of some (but not all) of the vertices of the vertex set does not disconnect G. The connectivity $\kappa(G)$ of a connected graph G is the smallest number of vertices whose removal disconnects G.

The components of a graph G are its maximal connected sub-graph. If there is no edge contained by a component, this component is trivial; otherwise, it is nontrivial. By this definition we can tell that a nontrivial component has at least two nodes, and there is just one component if G is connected.

A neighbor of a vertex v means it has an edge connected with v. Let $N(v)$ be the neighbor set of vertex v, that is , all the neighbors of v is contained by $N(v)$. A vertex v and a sub-graph S of G, the restricted degree of v in sub-graph S is denoted as $deg_S(v)$, whose definition is $deg_S(v) = |\{u|(u, v) \in E(G), u \in V(C)\}|$.

When we use a graph model a multi-processor system, the vertices of graph represent the processors and the edges are communication links between processors. We hope the system can identify the inner faulty processors itself, and the process of identifying all the faulty vertices is called the diagnosis. The maximum number of faulty nodes that the system can guarantee to identify is the diagnosability of this system. Use $t(G)$ to denote the diagnosability of graph G.

To value the ability of identifying faulty nodes of a graph, we introduce the PMC model. When there is an edge between two nodes, which means they can test each other.

For example, there is an edge (u, v) implies that u can test v by checking the response send by v, and v also can test u. The result is 0 of u testing v if u evaluates v as fault-free; otherwise, the result is 1. We list the possible outcomes of u test v in Table 2.1.

u	v	result
0	0	0
0	1	1
1	0	0/1
1	1	0/1

Table 2.1: All possible result of u testing v.

A vertex set is a faulty set if it contains all faulty nodes of a graph. The collection of testing outcomes of all vertex pairs of graph G with faulty set F, is called a syndrome produced by F. Notices that one faulty set F maybe not just form only one syndrome. Therefore, let $\sigma(F)$ represent the set of all the syndromes which can be produced by F.

If now we have two faulty sets F_1 and F_2 of graph G. They are distinguishable if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$; otherwise, they are distinguishable if $\sigma(F_1) \cap \sigma(F_2) = \emptyset$. The meaning of two faulty sets F_1 and F_2 being indistinguishable is that, we will confuse which one is the set containing the real all faulty nodes. For instance, we can not tell the faulty set is $\{1\}$ or $\{1, 2, 3\}$, because $\{1\}$ and $\{1, 2, 3\}$ both can form the syndrome shown in Figure 2.1 if they are the faulty sets.

From the previous research, we already know that two faulty sets F_1 and F_2 of graph G are distinguishable under the PMC model, if and only if there exists some edge between

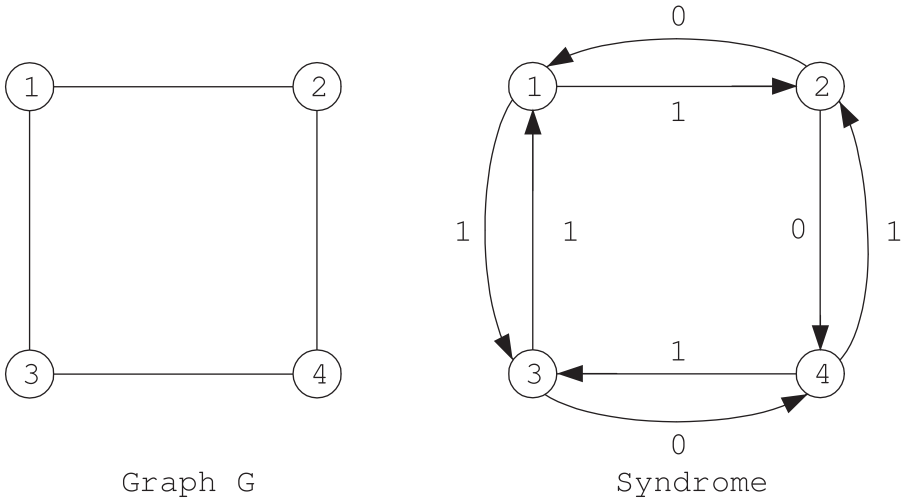

Figure 2.1: A graph and its one syndrome.

$G-(F_1 \cup F_2)$ and $F_1 \triangle F_2$, where $F_1 \triangle F_2$ is $(F1-F2) \cup (F2-F1)$.

Lemma 1 *Let F_1 and F_2 be two distinct faulty sets of graph G. F_1 and F_2 are distinguishable if and only if there exist a vertex $u \in G-(F_1 \cup F_2)$, and a vertex $v \in (F_1-F_2) \cup (F_2-F_1)$, $(u,v) \in E(G)$.*

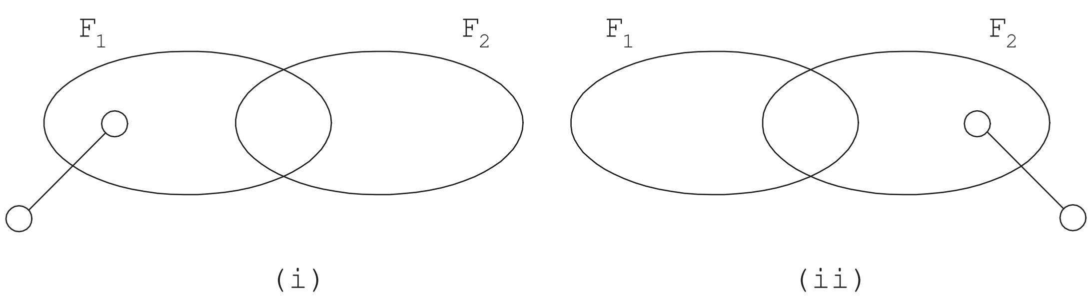

Figure 2.2: Illustrations of a distinguishable faulty set pair (F_1, F_2).

The definition of t-diagnosable system is listed as Definition 1.

Definition 1 *[4] A system is t-diagnosable if all faulty nodes can be identified provided that the number of faulty nodes does not exceed t.*

From the above definitions, we obtain the following lemma.

Lemma 2 *A system is t-diagnosable if and only if any two distinct faulty sets F_1 and F_2 of this system, $|F_1| \leq t$ and $|F_2| \leq t$, are distinguishable.*

And the following lemma 3 is equivalent to lemma 2.

Lemma 3 *A system is t-diagnosable if and only if any indistinguishable faulty sets F_1 and F_2 implies that $|F_1| > t$ or $|F_2| > t$.*

2.2 Hypercubes

In this section, we start to introduce a well-know interconnection network system, Hypercube. An n-dimensional Hypercube is denoted as Q_n, which has 2^n vertices usually represented by n-bits binary strings. Use $\{0,1\}^n$ to denote the set $b_{n-1}b_{n-2}...b_0|b_i \in 0,1 for 0 \leq i \leq n$ and $h(u,v)$ is the hamming distance between vertices u and v. We can define n-dimensional Hypercube as following:

Definition 2 *[11] A n-dimensional Hypercube $Q_n = (V,E)$, $V = \{0,1\}^n$, and $E = \{(u,v)|u, v are vertices of V and h(u,v) = 1\}$.*

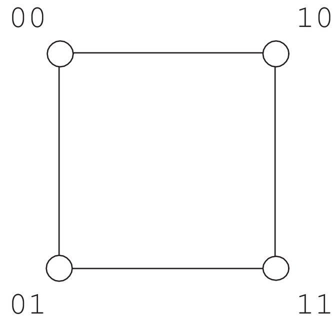

Figure 2.3: A 2-dimensional Hypercube.

There is another way to recursive structure a Hypercube. First, we define Q_1 is a complete graph K_2 with nodes denoted by 0 and 1. And a perfect matching with the definition: $\otimes(0\alpha) = 1\alpha$ with α is a binary string. Now if we want to build a Q_2 graph, we reproduce two Q_1, and add 0 to the left of binary stings of nodes of one Q_1, denoted the result as Q_1^0, and add 1 to the left of binary stings of nodes another, denoted as Q_1^1. Then adding the perfect matching between Q_1^0 and Q_1^1, we can see the outcome is a Q_2 graph.

Definition 3 *Let Q_1 is a complete graph with two nodes labelled by 0 and 1, respectively. For $n \geq 2$, Q_n is obtained by taking two copies of Q_{n-1}, denoted by Q_{n-1}^0 and Q_{n-1}^1. For each $v \in V(Q_n)$, insert a 0 to the front of $(n-1)$-bit binary string for v in Q_{n-1}^0 and a 1 to the front of $(n-1)$-bit binary string for v in Q_{n-1}^1. Let $V(Q_{n-1}^0) = \{0_un-2u_{n-3}...u_0 : u_i = 0or1\}$ and $V(Q_{n-1}^1) = \{1v_{n-2}v_{n-3}...v_0 : v_i = 0or1\}$, where $0 \leq i \leq n-2$. A node $u = 0u_{n-2}u_{n-3}...u_0$ of $V(Q_{n-1}^0)$ is joined to a node $v = 1v_{n-2}v_{n-3}...v_0$ of $V(Q_{n-1}^1)$ if and*

Figure 2.4: Structure Q_2 by recursive way.

only if $u_i = v_i$ *for* $0 \leq i \leq n-2$.

Hypercube is well-know because its good properties. First, an n-dimensional Hypercube is a n-regular graph, which means any vertex of Q_n, its degree is n. Secondly, the connectivity of Q_n is n, and the diagnosability is also n under the PMC model. And Hypercube is vertex symmetric [8], that is, any vertex of Q_n can map to $\{0\}^n$.

Lemma 4 *The connectivity of* Q_n*, denoted as* $\kappa(Q_n)$*, is* n*.*

Lemma 5 *[7] The diagnosability of* Q_n*, denoted as* $t(Q_n)$*, is* t*.*

Chapter 3

Conditionally Diagnosable System

Consider the diagnosability of Hypercube Q_n under the PMC model, the value of Q_n is n by the traditional definition. But system Q_n has $2n$ vertices and can just detect n faulty nodes, this makes the ration of faulty nodes which can be detected to the vertices which Q_n has is decreasing quickly with the value of n increasing.

For improving this defect of traditional definition of diagnosability, we study what on earth limits the ability of faulty-node detection of Hypercube. Then we observe that the only case, which stops the diagnosability of Q_n being $n+1$, is that there exits some vertex whose neighbors are all contained by a faulty set.

Therefore, we now introduce a better measure of diagnosability, the conditional diagnosability, to ameliorate the ability of faulty-node detection. The concept of conditional diagnosability is to avoid all neighbors of some vertex being contained by the faulty set. To make a system with faulty nodes satisfy our claim, we request that any faulty set cannot contain all neighbors of any vertices, and such faulty set is called conditional fault-set. By extending the above definition, we can continue defining the indistinguishable conditional-pair, conditional t-diagnosable system, and the conditional diagnosability, listed below.

Definition 4 *Let vertex set F be a faulty set of system G. Then F is a conditional fault-set if $N(v) \nsubseteq F$ for any vertex $v \in V(G)$.*

Definition 5 *Let F_1 and F_2 be two distinct conditional fault-sets of system G. We call (F_1, F_2) as a conditional-pair. If F_1 and F_2 are distinguishable, (F_1, F_2) is a distinguishable conditional-pair of system G; otherwise, (F_1, F_2) is an indistinguishable conditional-pair of system G.*

Definition 6 *For any conditional-pair (F_1, F_2) of system G, $|F_1| \leq t$ and $|F_2| \leq t$, if F_1 and F_2 are distinguishable, G is conditional t-diagnosable, and the maximum value of t is the conditional diagnosability of G, denoted as $t_c(G) = t$.*

And it is clearly that $t_c(G) \geq t(G)$.

Lemma 6 *In a system G, $t_c(G) \geq t(G)$.*

Before discussing the conditional diagnosable system, we first name some sets to simplify the writing through this section. In a system $G = (V, E)$, there is a conditional-pair (F_1, F_2). We use X to represent the vertex set $V - (F_1 \cup F_2)$, $F_1 \triangle F_2$ as $(F_1 - F_2) \cup (F_2 - F_1)$, and S as $F_1 \cap F_2$. We will continue using these symbols in the following discussion. See Figure 3.1.

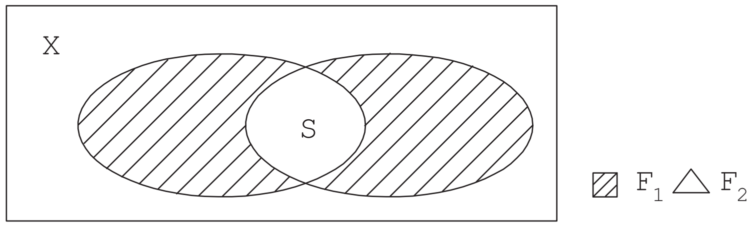

Figure 3.1: Symbols in graph G with a conditional-pair (F_1, F_2).

Let (F_1, F_2) be an indistinguishable conditional-pair of system G. We obverse two phenomena in this system as follows: first of all, there is no edge between X and $F_1 \triangle F_2$ because of F_1 and F_2 being indistinguishable. This means that for any vertex u of $F_1 - F_2$ or $F_2 - F_1$, all neighbors of u must be contained in $F_1 \cup F_2$, but the neighbors of u cannot be all in the set F_1 or F_2 because of F_1 and F_2 being the conditional fault-sets of system G. Therefore, there is at least one neighbor of u belonging to F_1, and another one belonging to F_2. Second, we know that for any vertex v of X, the neighbors of v cannot all belong to $F_1 \cap F_2$, because F_1 and F_2 cannot contain all neighbors of a vertex. Therefore, v has at least one neighbor in X for any vertex v of X. We state these observations in Lemma 7.

Lemma 7 *For any indistinguishable conditional-pair (F_1, F_2) of system $G = (V, E)$, the following two properties hold:*

1. $|N(u) \cap (F_1 - F_2)| \geq 1$ *and* $|N(u) \cap (F_2 - F_1)| \geq 1$*, for each vertex* $u \in F_1 \triangle F_2$*.*

2. $|N(v) \cap X| \geq 1$*, for each vertex* $v \in X = V - (F_1 \cup F_2)$*.*

Proof. (1) Without loss of generality, assume $u \in F_1 - F_2$, then we know that $N(u) \cap X = \emptyset$ because of F_1 and F_2 being indistinguishable, therefore $N(u) \subseteq (F_1 \cup F_2)$. Besides, F_1 and F_2 are conditional fault-sets, which means $N(u) \nsubseteq F_1$ and $N(u) \nsubseteq F_2$, so we can tell that $|N(u) \cap (F_1 - F_2)| \geq 1$ and $|N(u) \cap (F_2 - F_1)| \geq 1$ from $N(u) \subseteq (F_1 \cup F_2)$ and $N(u) \nsubseteq F_1$, $N(u) \nsubseteq F_2$.

(2) Because F_1 and F_2 are indistinguishable, $N(v) \cap (F_1 \triangle F_2) = \emptyset$. F_1 and F_2 are conditional fault-sets, so $N(v) \nsubseteq S$. therefore there is at least one neighbor of v in X, that is, $|N(v) \cap X| \geq 1$. □

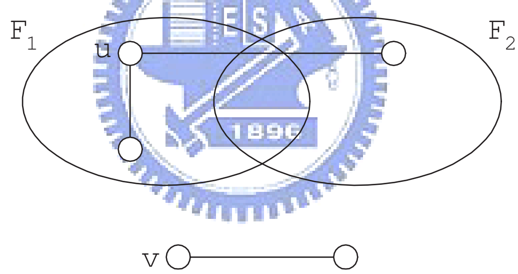

Figure 3.2: Illustration of Lemma 7.

From Lemma 7 we can discover some facts, if a system G has an indistinguishable conditional-pair (F_1, F_2), $S = F_1 \cap F_2$, then $G - S$ is disconnected and every component of $G - S$ must be nontrivial graph. More exactly, if there exists a component C, $C \cap (F_1 \cup F_2) = \emptyset$, then every vertex of C must have at least one neighbor in C; if there exists another component C', $C' \cap (F1 \cup F2) \neq \emptyset$, then every vertex of C' must have at least two neighbors in C'.

Chapter 4

Conditional Diagnosability of Hypercube

In this section, we start to discuss the conditional diagnosability of Hypercube. To simplify the following discussion, we list a lemma to present a property about the relation of common neighbors between two vertices in a Hypercube. This lemma explains that any two nodes have at most two common neighbors in Hypercube.

Lemma 8 *Let u and v be two different nodes of Hypercube $Q_n = (V, E)$, then the following two properties hold:*

1. *If $(u, v) \in E$, u and v have no common neighbor.*

2. *If $(u, v) \notin E$, u and v have at most two common neighbors.*

Proof. (1) From the recursive way of structuring Hypercube, it is clearly that there is no triangle structure in Hypercube. Therefore, u and v have no common neighbor, if $(u, v) \in E$.

(2) We proof this property by using induction on n of Q_n. When $n = 2$, we can see from Figure 4.1 that any (u, v)-pair nodes in Q_2 have at most two common neighbors.

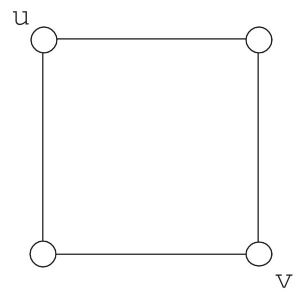

Figure 4.1: An (u, v)-pair of Hypercube Q_2.

Assume that this property holds when $n = k-1$, then now we show this property also holds when $n = k$.

Divide Q_k to two $Q_{k-1}s$, denoted as Q^L_{k-1} and Q^R_{k-1}, we have two cases to discuss, case 1: u, v are in the same sub-Hypercube (Figure 4.2); case 2: u, v are in different sub-Hypercubes (Figure 4.3).

Case 1: u and v are in the same sub-Hypercube.

Without loss of generality, assume u and v are in the Q^L_{k-1}. We can tell that u, v have no common neighbor in another sub-Hypercube, because there is a perfect-matching relation between Q^L_{k-1} and Q^R_{k-1}, u and v can not match to a same node in Q^R_{k-1}. And u and v have at most two common neighbors in Q^L_{k-1} by the induction hypothesis, so u

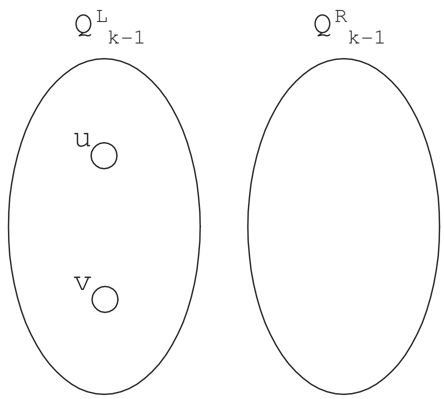

Figure 4.2: u and v are in the same sub-Hypercube.

and v have at most two common neighbors in Q_k.

Case 2: u and v are in different sub-Hypercubes.

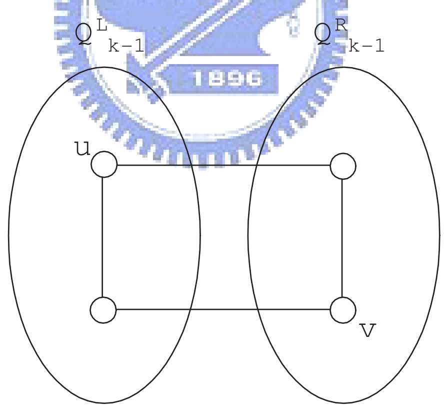

Figure 4.3: u and v are in different sub-Hypercubes.

Without loss of generality, assume u is in the Q_{k-1}^L and v is in the Q_{k-1}^R. Then u and v have no common neighbor without the edges between Q_{k-1}^L and Q_{k-1}^R. Because $u(v)$ can just match to one node of $Q^R_{k-1}(Q^L_{k-1})$, we can easily know that u and v at most have one common neighbor in Q^L_{k-1}, and at most one in Q^R_{k-1}, u and v have at most two common neighbors in Q_k. □

Now, we show the limit of conditional diagnosability of Hypercube by citing a instance, which explains the $t_c(Q_n)$ will not be grater than $4(n-2)+1$.

This example is as shown in Figure 4.4; we take a Q_2 structure from Q_n which has four nodes. Let $\{x, y, v, u\}$ be the four consecutive vertices of this Q_2 graph, which means $x-y-v-u$ is a four-length cycle. Now we make vertex set $S = N(x) \cup N(y) \cup N(v) \cup N(u)$, and $F_1 = \{x\} \cup \{y\} \cup S$, $F_2 = \{u\} \cup \{v\} \cup S$. By Lemma 8, there is no common neighbor of x, y, u and v contained in S, therefore, we can know that $|S| = 4(n-2)$. In this case, if we take F_1 and F_2 to be the conditional fault-sets of Q_n, then (F_1, F_2) will be an indistinguishable conditional-pair because $Q_n - S$ is disconnected. $|F_1|$ and $|F_2|$ in this case are all $4(n-2)+2$, that is, the conditional diagnosability of Hypercube Q_n is less than $4(n-2)+2$. We state this fact in Lemma 9.

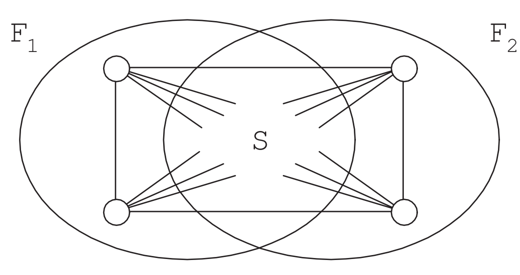

Figure 4.4: Illustration of $t_c(Q_n) \leq 4(n-2)+1$.

Lemma 9 $t_c(Q_n) \leq 4(n-2)+1, n \geq 2$.

Let S be a vertex cut of Hypercube Q_n. Then there are some components in $Q_n - S$. Let C be one of components of $Q_n - S$. We observe that it is important to know some result on the cardinalities of S and C under some conditions. We will state our observations of the cardinalities of S and C in Lemma 10 and Lemma 11.

In the proof of the following two lemmas, we divide Q_n into two $Q_{n-1}s$, denoted as Q^L_{n-1} and Q^R_{n-1}, and use some symbols for simplifying our explanation. We list these definitions as following: $C \cap Q^L_{n-1} = C_L$, $C \cap Q^R_{n-1} = C_R$, $S \cap Q^L_{n-1} = S_L$, $S \cap Q^R_{n-1} = S_R$, S is the cut set of Q_n and C is one of component of $Q_n - S$. We illustrate these symbols with Figure 4.5.

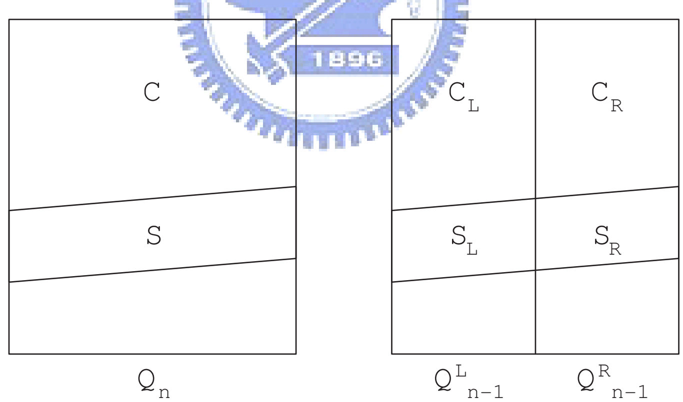

Figure 4.5: Illustrations of symbols used in Lemma 10 and 11.

Lemma 10 *Let Q_n be a Hypercube with $n \geq 3$, and let S be a vertex cut. Then the following two properties hold:*

1. $|S| \geq n$.

2. When $|S| \leq 2(n-1)-1$, $Q_n - S$ has exactly one trivial component and one nontrivial component.

Proof. (1) The property is rue because of $\kappa(Q_n) = n$.

(2) Because $Q_n - S$ is disconnected, we know there are at least two components in $Q_n - S$, and there are three cases we have to discuss, case 1: there are at least two trivial components; case 2: there are at least two nontrivial components; case 3: there are exactly one trivial component and one nontrivial component in $Q_n - S$. If we can show that case 1 and case 2 only hold when $|S| \geq 2(n-1)$, then case 3 holds when $n \leq |S| \leq 2(n-1)-1$.

Case 1: there are at least two trivial components in $Q_n - S$.

Let $\{v_1\}$ and $\{v_2\}$ be the two trivial components in $Q_n - S$, then we know that $N(v_1) \subseteq S$ and $N(v_2) \subseteq S$. By Lemma 8, v_1 and v_2 have at most two common neighbors, so $|N(v_1) \cap N(v_2)| \leq 2$. Therefore, $|S| \geq |N(v_1) \cup N(v_2)| = |N(v_1)| + |N(v_2)| - |N(v_1) \cap N(v_2)| \geq 2n - 2 = 2(n-1)$.

Case 2: there are at least two nontrivial components in $Q_n - S$.

We proof this case by using introduction on n. When $n = 3$, $n \leq |S| \leq 2(n-1) - 1$ implies $|S| = 3$. The only situation is to remove all neighbors of some vertex can make

$Q_3 - S$ being disconnected when $|S| = 3$, and this makes one trivial component and one nontrivial component in $Q_3 - S$. Therefore, $Q_3 - S$ has at least two nontrivial components holds when $|S| \geq 2(n-1)$.

Assume when $n = k-1$, $Q_n - S$ has at least two nontrivial components only at $|S| \geq 2(n-1)$, then we consider when $n = k$:

Let C and C' be the two nontrivial components in $Q_n - S$. Because $|V(C)| \geq 2$, we can divide Q_n into the two disjoint Q_{n-1}s, Q^L_{n-1} and Q^R_{n-1}, such that $|C_L| \geq 1$ and $|C_R| \geq 1$.And there are two components in $Q_n - S$, so at least one of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected.

When both of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are disconnected, $|S_L| \geq (n-1)$ and $|S_R| \geq (n-1)$ because of $\kappa(Q_{n-1}) = n-1$. So $|S| = |S_L| + |S_R| \geq 2(n-1)$.

When exactly one of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected, assume that Q^L_{n-1} is connected and Q^R_{n-1} is disconnected without loss of generality. Then $V(Q^L_{n-1}) = V(C_L) \cup S_L$, C' is totally in Q^R_{n-1}, and the corresponding matched vertices of C' are all in S_L because C and C' are disconnected. That means $C' \subset Q^R_{n-1}$ and $|S_L| \geq |C'| \geq 2$.

When $|S_R| \geq 2(n-2)$, $|S| = |S_R| + |S_L| \geq 2(n-2) + 2 = 2(n-1)$. Otherwise, when $n \leq |S| \leq 2(n-1) - 1$, because Q^R_{n-1} already has one nontrivial component C', we can see that Q^R_{n-1} has one trivial component and one nontrivial component by hypothesis and result of case 1. Then we know that Q_{n-1}^R is composed by one trivial component, one nontrivial component C' and S_R, which means C_R is a trivial component in Q_{n-1}^R. So we can tell $|S_L| \geq |C'| = 2^{n-1} - |S_R| - 1$. Therefore, $|S| = |S_L| + |S_R| \geq 2^{n-1} - |S_R| - 1 + |S_R| \geq 2^{n-1} - 1 \geq 2(n-1)$ when $n \geq 4$.

According to the proof of above two cases, the second property is true and this lemma is proofed. □

Now we introduce another lemma to show the properties about cardinalities of S and C under some conditions, these conditions are formed form Lemma 7. We claim that every component of $Q_n - S$ must be nontrivial, and at least one of these components must satisfy that every vertex v of this component, its restricted degree is greater than 1. We use the same symbols in proof which are defined before Lemma 10.

Lemma 11 *Let Q_n be a n-dimensional Hypercube with $n \geq 5$, and let S be a vertex cut of Q_n. Suppose that $Q_n - S$ satisfies that every component of $Q_n - S$ is nontrivial. If there exists a component C of $Q_n - S$ satisfying $deg_C(v) \geq 2$ for every vertex v of C, one of the following two properties holds:*

1. $|S| \geq 4(n-2)$
2. $|C| \geq 4(n-2) - 1$

Proof. Because $deg_C(v) \geq 2$ for every vertex v of C, we can divide Q_n into Q_{n-1}^L and Q_{n-1}^R, where $|C_L| > 0$ and $|C_R| > 0$. If there is a vertex v of C_L, then there is at most one neighbor of v contained in Q^R_{n-1}, so we can tell that there is at least one neighbor of v belonging to C_L because $deg_C(v) \geq 2$. Therefore, C_L is a nontrivial component of Q^L_{n-1}, and by the same way, C_R is also a nontrivial component of Q^R_{n-1}.

Because $Q_n - S$ is disconnected, there is at least one of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected. So there are two cases, case 1: exactly one of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected; case 2: $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are both disconnected.

Case 1: exactly one of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected.

Without loss of generality, assume $Q^L_{n-1} - S_L$ is connected and $Q^R_{n-1} - S_R$ is disconnected. Then $V(Q^L_{n-1}) = V(C_L) \cup S_L$, and there exists another nontrivial component C' contained in Q^R_{n-1}. Because C' and C_R are all nontrivial, we know that $|S_R| \geq 2(n-2)$ by lemma 10. If $|S_L| \geq 2(n-2)$, then $|S| = |S_L| + |S_R| \geq 4(n-2)$, the first property holds. Otherwise, $|S_L| \leq 2(n-2) - 1$, then $|C_L| = 2^{n-1} - |S_L| \geq 2^{n-1} - 2(n-2) + 1$. Besides, $|C_R| \geq 2$, so $|C| = |C_L| + |C_R| \geq 2^{n-1} - 2(n-2) + 3 \geq 4(n-2) - 1$ when $n \geq 4$, the second property holds.

Case 2: $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are both disconnected.

In this case, we have three sub-cases to consider, sub-case 1: $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$; sub-case 2: $|S_L| \leq 2(n-2) - 1$ and $|S_R| \leq 2(n-2) - 1$; sub-case 3: $|S_L| \geq 2(n-2)$ and $|S_R| \leq 2(n-2) - 1$ or $|S_L| \leq 2(n-2)$ and $|S_R| \geq 2(n-2) - 1$.

Sub-case 2.1: $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$.

If $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$, then the first property, $|S| \geq 4(n-2)$, holds.

Sub-case 2.2: $|S_L| \leq 2(n-2)-1$ and $|S_R| \leq 2(n-2)-1$.

By Lemma 10, there is exactly one trivial component and one nontrivial component in $Q^L_{n-1} - S_L$, same as $Q^R_{n-1} - S_R$. C_L and C_R are nontrivial, so there are trivial components $\{v\}$ and $\{u\}$ for some $v \in V(Q^L_{n-1} - S_L)$ and some $u \in V(Q^R_{n-1} - S_R)$, such that $V(Q^L_{n-1}) = V(C_L) \cup S_L \cup \{v\}$ and $V(Q^R_{n-1}) = V(C_R) \cup S_R \cup \{u\}$. Therefore, $|C_L| = 2^{n-1} - |S_L| - 1$, $|C_R| = 2^{n-1} - |S_L| - 1$, and $|C| = |C_L| + |C_R| = 2^n - |S| - 2 \geq 2^n - 4(n-2) \geq 4(n-2) - 1$ with $n \geq 4$. That is, the second property holds.

Sub-case 2.3: $|S_L| \geq 2(n-2)$ and $|S_R| \leq 2(n-2) - 1$ or $|S_L| \leq 2(n-2)$ and $|S_R| \geq 2(n-2)-1$.

With loss of generality, we assume $|S_L| \geq 2(n-2)$ and $|S_R| \leq 2(n-2)-1$. By Lemma 10, $Q^R_{n-1} - S_R$ has one trivial component and one nontrivial component. C_R is nontrivial, so there is a trivial component $\{u\}$ which makes $V(Q^R_{n-1}) = V(C_R) \cup S_R \cup \{u\}$, $|C_R| = 2^{n-1} - |S_L| - 1 \geq 2^{n-1} - 2(n-2)$. $|C| = |C_L| + |C_R| \geq 2 + 2^{n-1} - 2(n-2) \geq 4(n-2) - 1$ with $n \geq 5$, the second property holds.

The proof of Lemma 11 is completed. □

Let Q_n be an n-dimensional Hypercube. By Lemma 9, we know that $t_c(Q_n) \leq 4(n-2)+1$. If we can proof that any indistinguishable conditional-pair (F_1, F_2) of Q_n implies $|F_1| \geq 4(n-2)+2$ or $|F_2| \geq 4(n-2)+2$, then we can tell that $t_c(Q_n) = 4(n-2)+1$.

Lemma 12 *Let Q_n be an n-dimensional Hypercube with $n \geq 5$, $t_c(Q_n) = 4(n-2)+1$.*

Proof. Let (F_1, F_2) be an indistinguishable conditional-pair of $Q-n$ and $S = F_1 \cap F_2$. Then $Q_n - S$ is disconnected and every component of $Q_n - S$ is nontrivial by Lemma 7. Let $F_1 \triangle F_2 = (F_1 - F_2) \cup (F_2 - F_1)$, then by Lemma 7-(1), every vertex v of $F_1 \triangle F_2$ satisfies that $deg_{F_1 \triangle F_2}(v) \geq 2$. So one of $|S| \geq 4(n-2)$ and $|F_1 \triangle F_2| \geq 4(n-2)-1$ will be true by Lemma 11.

Case 1: $|S| \geq 4(n-2)$.

Because every vertex v of $F_1 \triangle F_2$ satisfies that $deg_{F_1 \triangle F_2}(v) \geq 2$, and there is no triangle structure in Hypercube, there are at least four nodes which form a 4-length cycle contained in $F_1 \triangle F_2$, which means $|F_1 \triangle F_2| \geq 4$. Therefore, $|F_1 - F_2| \geq \lceil \frac{4}{2} \rceil = 2$ or $|F_2 - F_1| \geq \lceil \frac{4}{2} \rceil = 2$, that is, $|F_1| = |S| + |F_1 - F_2| \geq 4(n-2)+2$ or $|F_2| = |S| + |F_2 - F_1| \geq 4(n-2)+2$.

Case 2: $|F_1 \triangle F_2| \geq 4(n-2)-1$.

There are at least two components and every component of $Q_n - S$ is nontrivial, by Lemma 10, $|S| \geq 2(n-1)$. $|F_1 \triangle F_2| \geq 4(n-2)+1$, so $|F_1 - F_2| \geq \lceil \frac{4(n-2)-1}{2} \rceil = 2(n-2)$ or

$|F_2 - F_1| \geq \lceil \frac{4(n-2)-1}{2} \rceil = 2(n-2)$, therefore , $|F_1| = |S| + |F_1 - F_2| \geq 2(n-1) + 2(n-2) = 4(n-2) + 2$ or $|F_2| = |S| + |F_2 - F_1| \geq 2(n-1) + 2(n-2) = 4(n-2) + 2$.

According to Lemma 9 and the above proof, $t_c(Q_n) = 4(n-2) + 1$ with $n \geq 5$. □

Now think about Q_3 and Q_4. We can find two indistinguishable conditional-pair examples to show that $t_c(Q_3) \leq 3$ and $t_c(Q_4) \leq 7$ which are shown in Figure 4.6.

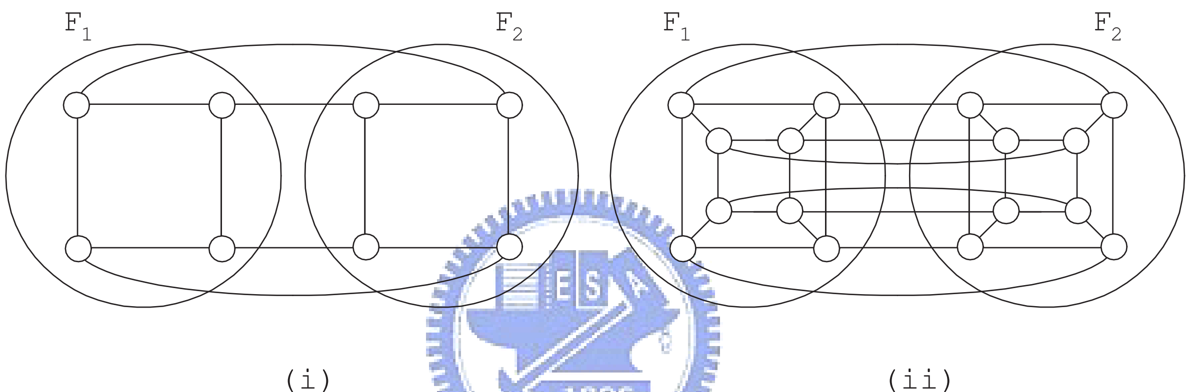

Figure 4.6: Two examples of indistinguishable conditional-pair for Q3 and Q4.

By Lemma 6, $t_c(Q_3) \geq t(Q_3) = 3$. And $t_c(Q_3) \leq 3$, therefore, $t_c(Q_3) = 3$. For Q_4, we proof its conditional diagnosability in Lemma 13.

Lemma 13 $t_c(Q_4) = 7$

Proof. Because we already know that $t_c(Q_4) \leq 7$, we only to proof that any indistinguishable-pair (F_1, F_2) of Q_4 implies that $|F_1| \geq 8$ or $|F_1| \geq 8$. Let (F_1, F_2) be an indistinguishable conditional-pair of Q_4, and $S = F_1 \cap F_2$. Then $|S| \geq 2(n-1) = 6$ by Lemma 7 and Lemma 10. And $|F_1 - F_2| \geq 2$ or $|F_2 - F_1| \geq 2$ by Lemma 7-(1), that is, $|F_1| \geq 8$ or $|F_2| \geq 8$. Therefore, $t_c(Q_4) = 7$. □

By above lemmas, we get Theorem .

Theorem 1 *Let Q_n be a Hypercube. Then $t_c(Q_n) = 4(n-2)+1$ for $n \geq 5$, and $t_c(Q_3) = 3$, $t_c(Q_4) = 7$.*

Chapter 5

Conclusion

The PMC diagnosis model is used generally, and Hypercube is a well-know topology. In this thesis, we define the conditional diagnosability under the PMC model, and show that the conditional diagnosability of Hypercube Q_n is $4(n-2)+1$. There are other cubes and another well-know diagnosis model, the comparison model. Hence, it is interesting to find the conditional diagnosability of other famous cubes, like the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n. And it is also interesting to investigate the conditional diagnosability of a system under the comparison model. Besides, it is a attractive problem how can we increase the diagnosability with other reasonable restrictions.

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