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Defect indices of powers of a contraction

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In memory of our colleague and friend Sen-Yen Shaw (1945–2009)

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ABSTRACT

Let A be a contraction on a Hilbert space H. The defect index d_A of A is, by definition, the dimension of the closure of the range of $I - A^*A$. We prove that (1) $d_{A^n} \leqslant nd_A$ for all $n \geqslant 0$, (2) if, in addition, A^n converges to 0 in the strong operator topology and $d_A = 1$, then $d_{A^n} = n$ for all finite n, $0 \leqslant n \leqslant \dim H$, and (3) $d_A = d_{A^*}$ implies $d_{A^n} = d_{A^{n*}}$ for all $n \geqslant 0$. The norm-one index k_A of A is defined as $\sup\{n \geqslant 0 : \|A^n\| = 1\}$. When $\dim H = m < \infty$, a lower bound for k_A was obtained before: $k_A \geqslant (m/d_A) - 1$. We show that the equality holds if and only if either A is unitary or the eigenvalues of A are all in the open unit disc, d_A divides m and $d_{A^n} = nd_A$ for all n, $1 \leqslant n \leqslant m/d_A$. We also consider the defect index of $f(A)$ for a finite Blaschke product f and show that $d_{f(A)} = d_{A^n}$, where n is the number of zeros of f.

0. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$, and let A be a contraction ($\|A\| \equiv \sup\{\|Ax\| : x \in H, \|x\| = 1\} \leqslant 1$) on H. The *defect index* of A is, by definition, rank $(I - A^*A)$, that is, the dimension of the closure of the range $\overline{\text{ran}\,(I - A^*A)}$ of $I - A^*A$. It is a measure of how far A is from the isometries, and plays a prominent role in the Sz.-Nagy–Foiaş theory of canonical model for contractions [8].

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In this paper, we are concerned with the defect indices of powers of a contraction. We show that, for a contraction A, d_{A^n} is at most nd_A for any $n \geqslant 0$. They are in general not equal. The equality does hold in certain cases. For example, if A^n converges to 0 in the strong operator topology and $d_A = 1$, then $d_{A^n} = n$ for all finite n, $0 \leqslant n \leqslant \dim H$. The equality (for some n's) also arises in another situation, namely, in relation to the norm-one index. Recall that the *norm-one index* k_A of a contraction A is defined as $\sup\{n \geqslant 0 : \|A^n\| = 1\}$. It was proven in [3, Theorem 2.4] that if A acts on an m-dimensional space, then $k_A \geqslant (m/d_A) - 1$. Here we complement this result by characterizing all the m-dimensional A with $k_A = (m/d_A) - 1$: this is the case if and only if either A is unitary or the eigenvalues of A are all in the open unit disc $\mathbb{D}(\equiv \{z \in \mathbb{C} : |z| < 1\})$, d_A divides m and $d_{A^n} = nd_A$ for all n, $1 \leqslant n \leqslant m/d_A$. These will be given in Sections 1 and 2 below, respectively. In Section 3, we consider contractive analytic functions of a contraction, instead of just its powers. Among other things, we show that if f is a Blaschke product with n zeros, then $d_{f(A)} = d_{A^n}$.

1. Powers of a contraction

We start with some basic properties for the defect indices of powers of a contraction. These include a "triangle inequality" and their increasingness.

Theorem 1.1. *Let A be a contraction on H.*

(a) *The inequality $d_{A^{m+n}} \leqslant d_{A^m} + d_{A^n}$ holds for any $m, n \geqslant 0$. In particular, $d_{A^n} \leqslant nd_A$ for $n \geqslant 0$.*
(b) *The sequence $\{d_{A^n}\}_{n=0}^{\infty}$ is increasing in n. Moreover, if $d_{A^n} = d_{A^{n+1}} < \infty$ for some n, $0 \leqslant n \leqslant \dim H$, then $d_{A^k} = d_{A^n}$ for all $k \geqslant n$.*

The proof depends on the following more general lemma.

Lemma 1.2. *Let $A = BC$, where B and C are contractions. Then $d_C \leqslant d_A \leqslant d_B + d_C$. If B and C commute, then we also have $d_B \leqslant d_A$.*

Note that $d_B \leqslant d_A$ may not hold without the commutativity of B and C. For example, if $A = I$, $B = S^*$ and $C = S$, where S denotes the (simple) unilateral shift, then $A = BC$, $d_A = 0$ and $d_B = 1$.

Proof of Lemma 1.2. Since

$$I - A^*A = I - C^*B^*BC \geqslant I - C^*C \geqslant 0,$$

where we used $C^*B^*BC \leqslant C^*C$ because $B^*B \leqslant I$, we obtain $\overline{\operatorname{ran}(I - A^*A)} \supseteq \overline{\operatorname{ran}(I - C^*C)}$ and thus $d_A \geqslant d_C$. If B and C commute, then $A = CB$ and, therefore, $d_B \leqslant d_A$ follows from above.

On the other hand, since

$$I - A^*A = I - C^*B^*BC = (I - C^*C) + C^*(I - B^*B)C,$$

we have

$$\operatorname{ran}(I - A^*A) \subseteq \operatorname{ran}(I - C^*C) + \operatorname{ran} C^*(I - B^*B)C.$$

Thus

$$\begin{aligned} d_A &\leqslant d_C + \operatorname{rank} C^*(I - B^*B)C \\ &\leqslant d_C + \operatorname{rank}(I - B^*B)C \\ &\leqslant d_C + d_B, \end{aligned}$$

completing the proof. □

We now prove Theorem 1.1. For any contraction A, let $H_n = \overline{\text{ran}\,(I - A^{n*}A^n)}$ for $n \geqslant 0$ and $H_\infty = \vee_{n=0}^{\infty} H_n$. In the following, we will frequently use the fact that, for a contraction A, x is in $\ker(I - A^*A)$ if and only if $\|Ax\| = \|x\|$.

Proof of Theorem 1.1. (a) and the increasingness of the d_{A^n}'s in (b) follow immediately from Lemma 1.2. To prove the remaining part of (b), we check that $H_n = \vee_{k=0}^{n-1} A^{k*}H_1$ for $n \geqslant 1$. Indeed, if $x = (I - A^{n*}A^n)y$ for some y in H, then $x = \sum_{k=0}^{n-1} A^{k*}(I - A^*A)A^k y$, which shows that x is in $\vee_{k=0}^{n-1} A^{k*}H_1$. For the converse containment, note that A maps $\ker(I - A^{k+1*}A^{k+1})$ to $\ker(I - A^{k*}A^k)$ isometrically for each $k \geqslant 0$. Indeed, if x is in the former, then

$$\|x\| = \|A^{k+1}x\| \leqslant \|Ax\| \leqslant \|x\|.$$

Hence we have the equalities throughout and, in particular, $\|A^k(Ax)\| = \|Ax\|$ and $\|Ax\| = \|x\|$. The former implies that $Ax \in \ker(I - A^{k*}A^k)$. Together with the latter, this proves our assertion. Therefore, A^* maps H_k to H_{k+1} for $k \geqslant 0$. By iteration, we have that A^{k*} maps H_1 to H_{k+1} for all $k \geqslant 1$. Arguing as above, we also obtain $\ker(I - A^{k+1*}A^{k+1}) \subseteq \ker(I - A^{k*}A^k)$ and thus $H_k \subseteq H_{k+1}$ for $k \geqslant 0$. Therefore, A^{k*} maps H_1 to H_n for all k, $0 \leqslant k \leqslant n - 1$. This proves $\vee_{k=0}^{n-1} A^{k*}H_1 \subseteq H_n$ and hence our assertion on their equality.

If $d_{A^n} = d_{A^{n+1}} < \infty$ for some n, then $H_n = H_{n+1}$. Hence

$$\begin{aligned} H_{n+2} &= \vee_{k=0}^{n+1} A^{k*}H_1 = (\vee_{k=0}^{n} A^{k*}H_1) \vee (A^{n+1*}H_1) \\ &\subseteq H_{n+1} \vee (A^*H_{n+1}) = H_{n+1} \vee (A^*H_n) \\ &\subseteq H_{n+1} \vee H_{n+1} = H_{n+1} \subseteq H_{n+2}. \end{aligned}$$

Therefore, we have equalities throughout. This implies that $d_{n+1} = d_{n+2}$. Repeating this argument gives us $d_{A^k} = d_{A^n}$ for all $k \geqslant n$. □

Note that, in Theorem 1.1 (a), $d_{A^{m+n}} < d_{A^m} + d_{A^n}$ may happen even for $m = n = 1$. For example, if

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then $d_A = 2$ and $d_{A^2} = 3$. Thus $d_{A^2} < d_A + d_A$.

The following corollary is an easy consequence of Theorem 1.1 (b).

Corollary 1.3. *If A is a contraction with A^n isometric (resp., unitary), then A itself is isometric (resp., unitary).*

The next theorem says that the equalities $d_{A^n} = nd_A$, $n \geqslant 0$, do hold for certain contractions A. It generalizes [3, Theorem 3.1] and [4, Theorem 3.4].

Theorem 1.4. *If A is a contraction on H with A^n converging to 0 in the strong operator topology and $d_A = 1$, then $d_{A^n} = n$ for all finite n, $0 \leqslant n \leqslant \dim H$.*

Proof. Under our assumption that $d_A = 1$, we have $d_{A^n} \leqslant n$ for all $n \geqslant 0$ by Theorem 1.1 (a). Assume that $d_{A^{n_0}} < n_0$ for some finite n_0, $1 < n_0 \leqslant \dim H$. Since d_{A^n} increases in n, the pigeonhole principle and Theorem 1.1 (b) yield that $d_{A^{n_0-1}} = d_{A^{n_0}} = d_{A^n} < n_0 < \infty$ for all $n \geqslant n_0$. Hence

$$\ker(I - A^{n_0*}A^{n_0}) = \overline{\text{ran}\,(I - A^{n_0*}A^{n_0})}^{\perp} = \overline{\text{ran}\,(I - A^{n*}A^{n})}^{\perp} = \ker(I - A^{n*}A^n)$$

for $n \geqslant n_0$. Let K denote this common subspace. For x in K, we have $\|A^n x\| = \|x\|$ for all $n \geqslant n_0$. On the other hand, the assumption that $A^n \to 0$ in the strong operator topology yields that $\|A^n x\| \to 0$ as $n \to \infty$. From these, we conclude that $x = 0$ and hence $K = \{0\}$. This is the same as $\ker(I - A^{n_0*}A^{n_0}) = \{0\}$

or $\overline{\operatorname{ran}(I - A^{n_0*}A^{n_0})} = H$. Thus $\dim H = d_{A^{n_0}} < n_0$, which is a contradiction. Therefore, we must have $d_{A^n} = n$ for all finite n, $0 \leqslant n \leqslant \dim H$. $\square$

Let A be a contraction on H. Since A^* maps H_n to H_{n+1} for $n \geqslant 0$ as shown in the proof of Theorem 1.1 (b), we have $A^*H_\infty \subseteq H_\infty$. Hence

$$A = \begin{bmatrix} A' & 0 \\ B & V \end{bmatrix} \quad \text{on } H = H_\infty \oplus H_\infty^\perp.$$

Note that, for any x in $H_\infty^\perp = \cap_{n=0}^\infty \ker(I - A^{n*}A^n)$, we have $A^*Ax = x$, which implies that $\|Vx\| = \|Ax\| = \|x\|$. Thus V is isometric on $H_\infty^\perp$. Recall that a contraction is *completely nonunitary* (*c.n.u.*) if it has no nontrivial reducing subspace on which it is unitary. A can be uniquely decomposed as $A_1 \oplus U$ on $K \oplus K^\perp$, where A_1 is c.n.u. on K and U is unitary on $K^\perp = \cap_{n=0}^\infty(\ker(I - A^{n*}A^n) \cap \ker(I - A^nA^{n*}))$ (cf. [8, Theorem I.3.2]). Thus the above decomposition can be further refined as

$$A = \begin{bmatrix} A' & 0 & 0 \\ B_1 & S_m & 0 \\ 0 & 0 & U \end{bmatrix},$$

where S_m denotes the unilateral shift with multiplicity $m (0 \leqslant m \leqslant \infty)$, $A_1 = \begin{bmatrix} A' & 0 \\ B_1 & S_m \end{bmatrix}$ is c.n.u., and $V = S_m \oplus U$ corresponds to the Wold decomposition of V (cf. [8, Theorem I.1.1]).

Corollary 1.5. *If A is a contraction on a finite-dimensional space with $d_A = 1$, then*

$$d_{A^n} = \begin{cases} n & \text{if } 0 \leqslant n \leqslant n_0, \\ n_0 & \text{if } n > n_0, \end{cases}$$

where $n_0 = \dim H_\infty$.

Proof. On a finite-dimensional space, the above representation of A becomes $A = A' \oplus V$ on $H = H_\infty \oplus H_\infty^\perp$ with V unitary. It is easily seen that A' has no eigenvalue of modulus one. Hence A'^n converges to 0 in norm (cf. [6, Problem 88]). Our assertion on d_{A^n} then follows from Theorems 1.4 and 1.1 (b). $\square$

The next theorem characterizes those contractions A for which $d_{A^n} = n$ for finitely many n's or for all $n \geqslant 0$. It generalizes Corollary 1.5.

Recall that an operator A on an n-dimensional space is said to be *of class $\mathcal{S}_n$* if A is a contraction, its eigenvalues are all in $\mathbb{D}$ and $d_A = 1$. The n-by-n *Jordan block*

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

is one example. Such operators and their infinite-dimensional analogues $S(\phi)$ (ϕ an inner function) are first studied by Sarason [7]. They play the role of the building blocks of the Jordan model for C_0 contractions [1,8].

Theorem 1.6. *Let A be a contraction on H.*

(a) *Let n_0 be a nonnegative integer. Then*

$$d_{A^n} = \begin{cases} n & \text{if } 1 \leqslant n \leqslant n_0, \\ n_0 & \text{if } n > n_0 \end{cases}$$

if and only if $P_{H_\infty}A|H_\infty$, the compression of A to H_∞, is of class $\mathcal{S}_{n_0}$. In this case, $\dim H_\infty = n_0$.

(b) *$d_{A^n} = n$ for all n, $0 \leqslant n < \infty$, if and only if $d_A = 1$ and $\dim H_\infty = \infty$.*

Proof. (a) Let

$$A = \begin{bmatrix} A' & 0 \\ B & V \end{bmatrix} \quad \text{on } H = H_\infty \oplus H_\infty^\perp,$$

where V is isometric. First assume that the d_{A^n}'s are as asserted. We need to show that $A' = P_{H_\infty}A|H_\infty$ is of class $\mathcal{S}_{n_0}$. Our assumption on d_{A^n} implies that $H_\infty = H_{n_0}$ is of dimension n_0. Moreover, for any $n \geqslant 0$, we have

$$\begin{aligned} I - A^{n*}A^n &= I - \begin{bmatrix} A'^{n*} & B_n^* \\ 0 & V^{n*} \end{bmatrix}\begin{bmatrix} A'^n & 0 \\ B_n & V^n \end{bmatrix} \\ &= \begin{bmatrix} I - A'^{n*}A'^n - B_n^*B_n & -B_n^*V^n \\ -V^{n*}B_n & 0 \end{bmatrix} \\ &= \begin{bmatrix} I - A'^{n*}A'^n - B_n^*B_n & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where the last equality holds because $I - A^{n*}A^n \geqslant 0$. Hence

$$n = d_{A^n} = \text{rank}\,(I - A'^{n*}A'^n - B_n^*B_n) \leqslant \text{rank}\,(I - A'^{n*}A'^n) = d_{A'^n}$$

for $1 \leqslant n \leqslant n_0$. If $n_1 < d_{A'^{n_1}}$ for some n_1, $1 \leqslant n_1 \leqslant n_0$, then the pigeonhole principle and Theorem 1.1 (b) yield that $d_{A'^{n_0-1}} = d_{A'^{n_0}}$. From [3, Lemma 2.3] and the fact that A' has no eigenvalue of modulus one, we conclude that $I - A'^{n_0-1*}A'^{n_0-1}$ is one-to-one and hence $d_{A'^{n_0-1}} = n_0$, contradicting our assumption. Hence $d_{A'^n} = n$ for all n, $1 \leqslant n \leqslant n_0$. [3, Theorem 3.1] implies that A' is of class $\mathcal{S}_{n_0}$. This proves one direction.

For the converse, we derive as above to obtain $I - A^{n*}A^n = (I - A'^{n*}A'^n - B_n^*B_n) \oplus 0$ on $H = H_\infty \oplus H_\infty^\perp$ and

$$d_{A^n} \leqslant d_{A'^n} = \begin{cases} n & \text{if } 1 \leqslant n \leqslant n_0, \\ n_0 & \text{if } n > n_0 \end{cases} \qquad (*)$$

by [3, Theorem 3.1]. Assume that $d_{A^{n_1}} < n_1$ for some n_1, $1 \leqslant n_1 \leqslant n_0$. Then the pigeonhole principle and Theorem 1.1 (b) yield $d_{A^n} = d_{A^{n_0}} < n_0$ for all $n \geqslant n_0$. This implies that $H_n = H_{n_0}$ for all $n \geqslant n_0$. Therefore, $H_\infty = H_{n_0}$ has dimension strictly less than n_0, which contradicts the fact that $\dim H_\infty = d_{A'^{n_0}} = n_0$ (cf. [3, Theorem 3.1]). Hence we have $d_{A^n} = n$ for all n, $1 \leqslant n \leqslant n_0$. If $n > n_0$, then $d_{A^n} \geqslant d_{A^{n_0}} = n_0$ by Theorem 1.1 (b) and what we have just proven. This, together with $(*)$, yields $d_{A^n} = n_0$ for $n > n_0$.

(b) Since $\dim H_\infty \geqslant d_{A^n}$ for all n, the necessity is obvious. Conversely, assume that $d_A = 1$ and $\dim H_\infty = \infty$. Then $d_{A^n} \leqslant n d_A = n$ by Theorem 1.1 (a). If $d_{A^{n_1}} < n_1$ for some $n_1 \geqslant 2$, then an argument analogous to the one for the second half of (a) yields that $H_\infty = H_{n_1}$ is of dimension less than n_1. This contradicts our assumption. Hence we must have $d_{A^n} = n$ for all n. □

We now proceed to consider contractions A with $d_A = d_{A^*}$ and start with the following lemma giving conditions of the equality of d_A and d_{A^*} for an arbitrary operator A. Note that, in this case, the definition of the defect index still makes sense.

Lemma 1.7. *Let A be an operator on H.*

(a) *If $\dim\ker A = \dim\ker A^*$, then $d_A = d_{A^*}$. In particular, if A acts on a finite-dimensional space, then $d_A = d_{A^*}$.*
(b) *If d_A is finite, then the following conditions are equivalent*:

  (1) $d_A = d_{A^*}$;
  (2) $\dim\ker A = \dim\ker A^*$;
  (3) *A^*A and AA^* are unitarily equivalent*;
  (4) *A is the sum of a unitary operator and a finite-rank operator*.

Proof. (a) If $\dim\ker A = \dim\ker A^*$, then $A = U(A^*A)^{1/2}$ for some unitary operator U (cf. [6, Problem 135]). Hence $AA^* = U(A^*A)U^*$ is unitarily equivalent to A^*A. Then the same is true for $I - A^*A$ and $I - AA^*$. Thus $d_A = d_{A^*}$.

(b) It was proven in [4, Lemma 1.4] that if $A^*A = A_1 \oplus 0$ (resp., $AA^* = A_2 \oplus 0$) on $H = \overline{\operatorname{ran} A^*} \oplus \ker A$ (resp., $H = \overline{\operatorname{ran} A} \oplus \ker A^*$), then A_1 and A_2 are unitarily equivalent. If $d_A = d_{A^*} < \infty$, then

$$\begin{aligned}\operatorname{rank}(I - A_1) + \dim\ker A = &\ \operatorname{rank}(I - A^*A) = \operatorname{rank}(I - AA^*)\\ = &\ \operatorname{rank}(I - A_2) + \dim\ker A^*\end{aligned}$$

and hence $\dim\ker A = \dim\ker A^*$. This proves that (1) implies (2). If (2) holds, then the unitary equivalence of A_1 and A_2 implies the same for A^*A and AA^*, that is, (2) implies (3). Now assume that (3) holds. Since $\ker A^*A = \ker A$ and $\ker AA^* = \ker A^*$, the unitary equivalence of A^*A and AA^* implies that $\dim\ker A = \dim\ker A^*$. Hence $d_A = d_{A^*}$ by (a), that is, (1) holds. Finally, the equivalence of (1) and (4) was proven in [10, Lemma 3.3]. □

Note that, in the preceding lemma, $d_A = d_{A^*} = \infty$ does not imply $\dim\ker A = \dim\ker A^*$ in general. For example, if $A = \operatorname{diag}(1, 1/2, 1/3, \ldots) \oplus S$, where S is the (simple) unilateral shift, then $d_A = d_{A^*} = \infty$, $\dim\ker A = 0$ and $\dim\ker A^* = 1$.

Theorem 1.8. *Let A be a contraction with $d_A = d_{A^*} < \infty$. Then $\dim H_\infty < \infty$ if and only if the completely nonunitary part of A acts on a finite-dimensional space.*

Proof. Assume that $\dim H_\infty < \infty$ and let

$$A = \begin{bmatrix} A' & 0 & 0\\ B & S_m & 0\\ 0 & 0 & U\end{bmatrix} \quad \text{on } H = H_\infty \oplus K_1 \oplus K_2,$$

where S_m denotes the unilateral shift with multiplicity m, $0 \leqslant m \leqslant \infty$, and U is unitary. We need to show that S_m does not appear in this representation of A or, equivalently, $m = 0$. We first prove that m is finite. Indeed, since

$$I - AA^* = \begin{bmatrix} I - A'A'^* & -A'B^* & 0\\ -BA'^* & I - BB^* - S_mS_m^* & 0\\ 0 & 0 & 0\end{bmatrix},$$

we have

$$\begin{aligned} m = \operatorname{rank}(I - S_mS_m^*) &\leqslant \operatorname{rank}(I - BB^* - S_mS_m^*) + \operatorname{rank} BB^*\\ &\leqslant \operatorname{rank}(I - AA^*) + \operatorname{rank} BB^*\\ &\leqslant d_{A^*} + \dim H_\infty < \infty\end{aligned}$$

as asserted. Now to show that $m = 0$, consider S_m as

$$\begin{bmatrix} 0 & & & \\ I_m & 0 & & \\ & I_m & 0 & \\ & & \ddots & \ddots\end{bmatrix}.$$

Then B is of the form $[B'\ 0\ 0\ \cdots]^T$. Let $\widetilde{A} = \begin{bmatrix} A' & 0\\ B' & 0\end{bmatrix}$. Since $\widetilde{A}$ acts on a finite-dimensional space, we have $d_{\widetilde{A}} = d_{\widetilde{A}^*}$ by Lemma 1.7 (a). Then

$$\begin{aligned} d_{A^*} = &\ \operatorname{rank}(I - AA^*)\\ = &\ \operatorname{rank}\begin{bmatrix} I - A'A'^* & -A'B^*\\ -BA'^* & I - BB^* - S_mS_m^*\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&= d_{\widetilde{A}^*} = d_{\widetilde{A}} = \operatorname{rank} \begin{bmatrix} I - A'^*A' - B'^*B' & 0 \\ 0 & I_m \end{bmatrix} \\
&= m + \operatorname{rank}\,(I - A'^*A' - B'^*B') \\
&= m + \operatorname{rank}\,(I - A'^*A' - B^*B) \\
&= m + \operatorname{rank} \begin{bmatrix} I - A'^*A' - B^*B & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= m + \operatorname{rank}\,(I - A^*A) = m + d_A.
\end{aligned}
$$

We infer from the assumption $d_A = d_{A^*} < \infty$ that $m = 0$. Thus $A = A' \oplus U$, where A' is the c.n.u. part of A acting on the finite-dimensional space H_∞.

The converse is trivial. $\square$

The next two results are valid for any operators.

Proposition 1.9. *If A is an operator with $d_A = d_{A^*}$, then $d_{A^n} = d_{A^{n*}}$ for all $n \geqslant 1$.*

Proof. If $d_A = d_{A^*} < \infty$, then $A = U + F$, where U is unitary and F has finite rank, by Lemma 1.7 (b). For any $n \geqslant 1$, we have $A^n = U^n + F_n$, where F_n is some finite-rank operator. By Lemma 1.7 (b) again, this implies that $d_{A^n} = d_{A^{n*}}$. On the other hand, if $d_A = d_{A^*} = \infty$, then $d_{A^n} = d_{A^{n*}} = \infty$ for any $n \geqslant 1$ by Theorem 1.1 (b). This completes the proof. $\square$

Two operators A on H and B on K are said to be *quasi-similar* if there are operators $X : H \to K$ and $Y : K \to H$ which are one-to-one and have dense range such that $XA = BX$ and $YB = AY$.

We conclude this section with the following result on quasi-similar operators.

Proposition 1.10. *Let A and B be quasi-similar operators. If $d_A = d_{A^*} < \infty$, then $d_B = d_{B^*}$.*

Proof. Our assumption of $d_A = d_{A^*} < \infty$ implies, by Lemma 1.7 (b), that $\dim\ker A = \dim\ker A^*$. The quasi-similarity of A and B then yields

$$\dim\ker B = \dim\ker A = \dim\ker A^* = \dim\ker B^*.$$

Then $d_B = d_{B^*}$ by Lemma 1.7 (a). $\square$

Note that the preceding proposition is false if $d_A = d_{A^*} = \infty$.

Example 1.11. Let $\{a_n\}_{n=1}^\infty$ be a sequence of distinct complex numbers in $\mathbb{D}$ with $\sum_n (1 - |a_n|) < \infty$. Let $A = \operatorname{diag}(a_1, a_2, \ldots) \oplus S$, where S denotes the (simple) unilateral shift. Let ϕ be the Blaschke product with zeros a_n:

$$\phi(z) = \prod_{n=1}^{\infty} \frac{\bar{a}_n}{|a_n|} \frac{z - a_n}{1 - \bar{a}_n z}, \quad z \in \mathbb{D},$$

and let $B = S(\phi) \oplus S$, where $S(\phi)$ denotes the compression of the shift

$$S(\phi)f = P(zf(z)), \quad f \in H^2 \ominus \phi H^2,$$

P being the (orthogonal) projection from H^2 onto $H^2 \ominus \phi H^2$. It is known that diag (a_n) is itself a C_0 contraction which is quasi-similar to $S(\phi)$ (cf. [9, Theorem 3]). Thus A is quasi-similar to B. But $d_A = d_{A^*} = \infty$, $d_B = 1$ and $d_{B^*} = 2$.

2. Relation to norm-one index

As defined in [3, p. 364], the *norm-one index* of a contraction A on H is $k_A \equiv \sup\{n \geqslant 0 : \|A^n\| = 1\}$. This number is to measure how far the powers of A remain to have norm one. It is easily seen that (1) $0 \leqslant k_A \leqslant \infty$, (2) $k_A = 0$ if and only if $\|A\| < 1$, and (3) $k_A = \infty$ if and only if $\sigma(A) \cap \partial\mathbb{D} \neq \emptyset$. The main results in [3] say that if $\dim H = m < \infty$, then (4) $0 \leqslant k_A \leqslant m-1$ or $k_A = \infty$ [3, Proposition 2.1 or Theorem 2.2], (5) $k_A = m-1$ if and only if A is of class $\mathcal{S}_m$ [3, Theorem 3.1], and (6) $k_A \geqslant (m/d_A) - 1$ [3, Theorem 2.2]. The purpose of this section is to determine when the equality holds in (6).

Theorem 2.1. *Let A be a contraction on an m-dimensional space. Then $k_A = (m/d_A) - 1$ if and only if one of the following holds*:

(a) *A is unitary,*
(b) *$\sigma(A) \subseteq \mathbb{D}$, d_A divides m, and $d_{A^n} = nd_A$ for all n, $1 \leqslant n \leqslant m/d_A$.*

Proof. Assume that $k_A = (m/d_A) - 1$. If $\sigma(A) \cap \partial\mathbb{D} \neq \emptyset$, then $(m/d_A) - 1 = k_A = \infty$, which implies that $d_A = 0$ or A is unitary. Hence we may assume that $\sigma(A) \subseteq \mathbb{D}$. Then $k_A < \infty$. From $k_A = (m/d_A) - 1$, we have $d_A|m$. By the pigeonhole principle and Theorem 1.1 (b), there is a smallest integer l, $1 \leqslant l \leqslant m$, such that $d_{A^l} = d_{A^{l+1}}$. Since A has no unitary part, this is equivalent to $I - A^{l*}A^l$ being one-to-one (cf. [3, Lemma 2.3]) or $\|A^l\| < 1$. As l is the smallest such integer, we obtain $k_A = l - 1$. From $k_A = (m/d_A) - 1$, we have $m/d_A = l$. Note that $d_{A^n} \leqslant nd_A$ for $1 \leqslant n \leqslant l$ by Theorem 1.1 (a). If $d_{A^{n_0}} < n_0 d_A$ for some n_0, $1 \leqslant n_0 \leqslant l$, then

$$d_{A^l} \leqslant d_{A^{n_0}} + d_{A^{l-n_0}} < n_0 d_A + (l - n_0)d_A = ld_A = m$$

again by Theorem 1.1 (a). This contradicts the fact that $I - A^{l*}A^l$ is one-to-one. Hence we must have $d_{A^n} = nd_A$ for $1 \leqslant n \leqslant m/d_A$. This proves (b).

Conversely, if (a) holds, that is, if A is unitary, then $k_A = \infty$ and $d_A = 0$. Hence $k_A = (m/d_A) - 1$. Now assume that (b) holds. If $l = m/d_A$, then our assumptions imply that $1 \leqslant d_A < d_{A^2} < \cdots < d_{A^l} = m$. Hence $I - A^{l*}A^l$ is one-to-one, but $I - A^{l-1*}A^{l-1}$ is not. Thus $\|A^l\| < 1$ and $\|A^{l-1}\| = 1$. This yields $k_A = l - 1 = (m/d_A) - 1$ as required. □

On an m-dimensional space, other than unitary operators, $\mathcal{S}_m$-operators and strict contractions (operators with norm strictly less than one), which correspond to $d_A = 0$, 1 and m, respectively, there are other contractions A satisfying $k_A = (m/d_A) - 1$. For example, if $A = \underbrace{J_l \oplus \cdots \oplus J_l}_{m/l}$, where l divides m, then $k_A = l - 1 = (m/d_A) - 1$. The same is true for the more general $B = \underbrace{A_1 \oplus \cdots \oplus A_1}_{m/l}$, where A_1 is an $\mathcal{S}_l$-operator. Another generalization of the contraction A is

$$C = \begin{bmatrix} 0 & a_1 & & \\ & 0 & \ddots & \\ & & \ddots & a_{m-1} \\ & & & 0 \end{bmatrix},$$

where $|a_j| < 1$ for $j = kl$, $1 \leqslant k \leqslant (m/l) - 1$ ($l|m$), and $|a_j| = 1$ for all other j's. In this case, it is easily seen that d_C equals m minus the number of j's for which $|a_j| = 1$ and hence $d_C = m/l$. On the other hand, k_C equals the maximum number of consecutive j's with $|a_j| = 1$, and thus $k_C = l - 1$. Therefore, $k_C = (m/d_C) - 1$ holds.

3. Contractive functions of a contraction

In this section, we consider the defect indices of contractive functions of a contraction, instead of just its powers. The first one is finite Blaschke products:

$$f(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}, \quad z \in \mathbb{D},$$

where $|a_j| < 1$ for all j.

Theorem 3.1. *If A is a contraction on H and f is a Blaschke product with n zeros (counting multiplicity), then $d_{f(A)} = d_{A^n}$.*

Note that if f is as above, then $f(A) = \prod_{j=1}^{n}(A - a_j I)(I - \bar{a}_j A)^{-1}$ is also a contraction (cf. [8, Theorem III.2.1 (b)]).

Proof of Theorem 3.1. Let f be as above and let $f_j(z) = (z - a_j)/(1 - \bar{a}_j z), z \in \mathbb{D}$, for each j. Let $X = \prod_{j=1}^{n}(I - \bar{a}_j A)$, $K_1 = \ker(I - A^{n*}A^n)$ and $K_2 = \ker(I - f(A)^* f(A))$. We first show that $XK_1 \subseteq K_2$. Indeed, if x is in K_1, then $\|A^n x\| = \|x\|$. Applying [3, Lemma 1.2] once (with ϕ_1 there as f_1 and the remaining ϕ_j's given by $\phi_j(z) = z$) yields $\|f_1(A)A^{n-1}(I - \bar{a}_1 A)x\| = \|(I - \bar{a}_1 A)x\|$. We then apply [3, Lemma 1.2] repeatedly to obtain $\|f_1(A) \cdots f_n(A)Xx\| = \|Xx\|$. This means that Xx is in K_2. Hence we have $XK_1 \subseteq K_2$ as asserted. Since X is invertible, if

$$X = \begin{bmatrix} X_1 & * \\ 0 & X_2 \end{bmatrix} : H = K_1 \oplus K_1^{\perp} \to H = K_2 \oplus K_2^{\perp},$$

then X_2 has dense range. Thus $X_2^* : K_2^{\perp} \to K_1^{\perp}$ is one-to-one. Therefore,

$$d_{f(A)} = \dim K_2^{\perp} \leqslant \dim K_1^{\perp} = d_{A^n}$$

(cf. [6, Problem 56]). In a similar fashion, if $Y = \prod_{j=1}^{n}(I + \bar{a}_j A)$, then successive applications of [3, Lemma 1.2] also yield $YK_2 \subseteq K_1$. We can then infer as above that $d_{A^n} \leqslant d_{f(A)}$. This proves their equality. □

For more general functions, we use the Sz.-Nagy–Foiaş functional calculus for contractions [8, Section III.2]. For any absolutely continuous contraction A (this means that A has no nontrivial reducing subspace on which A is a singular unitary operator) and any function f in H^∞ with $\|f\|_\infty \leqslant 1$, the operator $f(A)$ can be defined and is again a contraction. Note that every function in H^∞ can be factored as the product of an inner and an outer function, and every inner function is the product of a Blaschke product and a singular inner function (cf. [8, Section III.1]).

Theorem 3.2. *Let A be an absolutely continuous contraction on H and f be a function in H^∞ with $\|f\|_\infty \leqslant 1$.*

(a) *If f has an infinite Blaschke product factor, then $d_{f(A)} \geqslant \sup\{d_{A^n} : n \geqslant 0\}$.*
(b) *If f is a (nonconstant) inner function, then $d_{f(A)} \leqslant \sup\{d_{A^n} : n \geqslant 0\}$. In particular, if f is an inner function with an infinite Blaschke product factor, then $d_{f(A)} = \sup\{d_{A^n} : n \geqslant 0\}$.*

Proof. (a) For each $n \geqslant 1$, let $f = f_n g_n$, where f_n is a finite Blaschke product with n zeros and g_n is in H^∞. Then $f(A) = f_n(A)g_n(A)$. Theorem 3.1 and Lemma 1.2 imply that $d_{A^n} = d_{f_n(A)} \leqslant d_{f(A)}$ for all $n \geqslant 1$. Thus $d_{f(A)} \geqslant \sup\{d_{A^n} : n \geqslant 0\}$.

(b) We may assume that $n_0 \equiv \sup\{d_{A^n} : n \geqslant 0\} < \infty$. This means that $\dim H_\infty = n_0$ is finite. Let

$$A = \begin{bmatrix} A' & 0 & 0 \\ B & S_m & 0 \\ 0 & 0 & U \end{bmatrix} \quad \text{on } H = H_\infty \oplus K_1 \oplus K_2,$$

where S_m is the unilateral shift with multiplicity m, $0 \leqslant m \leqslant \infty$, and U is unitary. Then

$$f(A) = \begin{bmatrix} f(A') & 0 & 0 \\ C & f(S_m) & 0 \\ 0 & 0 & f(U) \end{bmatrix}.$$

Note that $f(S_m)$ is itself a unilateral shift, say, $S_l (0 \leqslant l \leqslant \infty)$ (cf. [2,5]) and $f(U)$ is unitary because f is inner. Hence

$$I - f(A)^* f(A) = \begin{bmatrix} I - f(A')^* f(A') - C^*C & -C^*S_l & 0 \\ -S_l^*C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I - f(A')^* f(A') - C^*C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

since $I - f(A)^* f(A) \geqslant 0$. Therefore,

$$d_{f(A)} = \text{rank } (I - f(A')^* f(A') - C^*C) \leqslant \text{rank } (I - f(A')^* f(A'))$$
$$= d_{f(A')} \leqslant n_0.$$

This completes the proof. □

Note that Theorem 3.2 (a) is in general false if f is a finite Blaschke product. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $f(z) = z$, then $d_{f(A)} = d_A = 1$, but $\sup\{d_{A^n} : n \geqslant 0\} = 2$. Theorem 3.2 (b) is also false for general f in H^∞ with $\|f\|_\infty \leqslant 1$. As an example, let A be the (simple) unilateral shift. Then $\sup\{d_{A^n} : n \geqslant 0\} = 0$. On the other hand, $f(A)$ is an analytic Toeplitz operator with symbol f, which is an isometry if and only if f is inner (cf. [2]). Thus $d_{f(A)} = 0$ can happen only when f is inner.

The next corollary generalizes Proposition 1.9.

Corollary 3.3. *If A is an absolutely continuous contraction and f is either a finite Blaschke product or an inner function with an infinite Blaschke product factor, then $d_{f(A)} = d_{f(A)^*}$.*

Proof. Since $f(A)^* = \widetilde{f}(A^*)$, where $\widetilde{f}(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{D}$, the assertion follows easily from Theorems 3.1 and 3.2. □

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