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Transient solution for radial two-zone flow in unconfined aquifers under constant-head tests

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Abstract:

The constant-head test (CHT) is commonly employed to determine the aquifer parameters. This test is also applied to many environmental problems such as recovering light nonaqueous phase liquids (LNAPL) and controlling off-site migration of contaminated groundwater in low-transmissivity aquifers. A well skin near the wellbore may be produced due to the well construction or development and its formation properties are significantly different from the original ones. A more appropriate description for the skin effect on the aquifer system should treat the well skin as a different formation instead of using a skin factor. Thus, the aquifer system becomes a two-zone formation including the skin and formation zones. This study presents a mathematical model developed for analyzing a two-zone unconfined aquifer system and the associated solution for CHTs at partially penetrating wells under transient state. The solution of the model may be used either to identify the *in situ* aquifer parameters or to investigate the effects of the wellbore storage and well skin on the head changes in unconfined aquifers under constant-head pumping. Copyright © 2010 John Wiley & Sons, Ltd.

KEY WORDS constant-head test; unconfined aquifer; well skin; analytical solution; partially penetrating well

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INTRODUCTION

A constant-head test (CHT) is carried out to maintain a fixed water depth inside the well and measure the transient flow rate at the wellbore. Different from constant-rate test (CRT), CHT is mainly applied to low-transmissivity aquifers for determining the hydraulic parameters. Rice (1998) mentioned some merits in CHTs. The early time drawdown behaviour at pumping well and observation wells is commonly influenced by the wellbore storage at the pumping well in CRT. On the other hand, the wellbore storage effect is reduced in CHT and a part of transient data allowing characterization of the aquifer occurs earlier than that in CRT (Renard, 2005). The wellbore storage can be neglected under field conditions if a constant head is established in a short period time. This situation can be achieved if the aquifer has low transmissivity and the well radius is small (Chen and Chang, 2003). In recent years, owing to some important issues in the low-permeability aquifers, there has been an increasing interest in the study of CHTs. Jones *et al*. (1992) and Jones (1993) discussed the practicality of CHTs on wells completed in low-conductivity glacial till deposits. Mishra and Guyonnet (1992) indicated that the operational benefit of CHTs in situations where the total available drawdown is limited by well construction and aquifer characteristics. They developed a method for analyzing observation-well response to a CHT. For other environmental applications, light non-aqueous phase liquids (LNAPL) are recovered by wells held at a constant drawdown (Abdul, 1992; Murdoch and Franco, 1994) and a constant-head pumping is also used to control off-site migration of contaminated groundwater (Hiller and Levy, 1994). Up to now, the theories of well hydraulics for CHT have been limited to radial flow problems in confined or leaky aquifers. The basic transient model for CHT is the well-known Jacob and Lohman's (1952) solution. They presented the solution of flow rate at the well and used their solution to estimate the storage coefficient and transmissivity. Hantush (1964) provided the drawdown solution for confined and leaky aquifers under a CHT. Issues involving CHT can be found in literatures [e.g. Wilkinson, 1968; Uraiet and Raghaven, 1980; Hiller and Levy, 1994; Chen and Chang, 2002; Singh, 2007].

A finite thickness of well skin may be produced due to the well construction as a result of drilling through a mud or extensive well development. The skin thickness may range from a few millimeters to several meters and thus should be considered in the aquifer tests (Novakowski, 1989). The effect of skin on the response of pumping tests has been recognized for a long time in the petroleum industry. In the past, the skin effect was investigated for two types of skin, i.e. infinitesimal and finite-thickness skins. van Everdingen (1953) defined a skin factor caused by the additional resistance around the well resulting from the drilling and completion and then presented a method to compute the pressure drop due to the reduction of the permeability of the formation near the well. Hawkins (1956) further

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defined a skin effect which was related to external and altered permeability. Streltsova and McKinley (1984) considered an infinitesimal skin and used a skin factor to represent the skin effect.

As in reality the hydrogeologic properties of the skin zone may significantly differ from those of the formation zone, a more appropriate description for such a formation system should treat the skin zone as a different formation instead of using a skin factor. Novakowski (1993) derived a Laplace-domain solution for dimensionless flow rate at the wellbore. Curves of dimensionless flow rate versus dimensionless time illustrated the influences of finite-thickness skin. Markle *et al.* (1995) developed a model with a finite-thickness skin for CHT conducted in vertically fractured media. Their results indicate that the finite-thickness can have influence on transient flow rate along the wellbore. Yang and Yeh (2002) and Yeh *et al.* (2003) provided closed-form solutions for transient flow rate across the wellbore in a confined aquifer with a finite-thickness skin and a fully penetrating well in CHT and in CRT, respectively. Yang and Yeh (2005) presented a Laplace-domain solution for the hydraulic head in a radially confined aquifer considering a finite-thickness skin and a partially penetrating well in CHT. Perina and Lee (2006) presented a general well function for groundwater flow with a partially penetrating well in a confined, leaky or unconfined aquifer. They took a non-uniform radial flux along the screen and a finite-thickness well skin into account. Pasandi *et al.* (2008) presented an analytical solution for CRT conducted in a partially penetrating well in an unconfined aquifer. A finite thickness of well skin is considered in their solution and the influence of the skin is also investigated.

Some literatures have addressed the issues of the effect of partially penetrating well (e.g. Rudd and Kabala, 1997; Cassiani and Kabala, 1998; Cassiani *et al.*, 1999; Chang and Chen, 2003; Yang and Yeh, 2005). Cassiani and Kabala (1998) and Cassiani *et al.* (1999) have developed the Laplace-domain solutions for CRT and CHT in confined aquifers, respectively, where the tests are conducted at partially penetrating wells and based on the infinite aquifer thickness assumption. Yang *et al.* (2006) presented an analytical solution for constant-flux pumping in a well under partial penetration condition in confined aquifers. Chang and Yeh (2009) developed a new solution to the CHT performed at a partially penetrating well in a confined aquifer with a finite thickness. Instead of using numerical approach, the mathematical model with the mixed boundary condition specified along the wellbore is directly solved via the methods of dual series equations and perturbation method. For interpreting the pumping tests in unconfined aquifers, models commonly used in groundwater were based on the assumptions of Dagan (1967) and Neuman (1972, 1974) that the drainage above the water table occurs instantaneously. Contrary to Neuman's assumption, some literatures indicated that the drainage above the water table in unconfined aquifer is transient and both the radial and vertical flow components occur in saturated thickness (e.g. Akindunni and Gillham, 1992; Nwankwor *et al.*, 1992; Narasimhan and Zhu, 1993; Moench, 1994, 2004, 2008). The solutions for flow to a well in unconfined aquifers with considering the effect of unsaturated zone are also presented (e.g. Mathias and Butler, 2006; Tartakovsky and Neuman, 2007). For mathematical simplicity, Chen and Chang (2003) developed a well hydraulic theory for CHT in unconfined aquifer with fully penetrating well based on Neuman's approach. However, solutions for hydraulic head for CHT performed at partially penetrating wells in unconfined aquifers are barely developed.

This study extends the work of Yang and Yeh (2005) to develop a mathematical model for an unconfined aquifer system with treating the skin as a finite thickness zone and the associated solution for the CHT at a partially penetrating well. A numerical approach including roots finding and Shanks' transform is used to evaluate the newly derived solution. The solution may be used either to identify the *in situ* aquifer parameters or to investigate the effects of the wellbore storage and well skin on the head changes in unconfined aquifers with partially penetrating wells.

MATHEMATICAL MODEL

Figure 1 shows a cross-sectional view of an unconfined aquifer system with a partially penetrating well and a finite thickness of well skin under study. The governing equations for hydraulic head in the skin and formation zones can respectively be written as (Novakowski, 1989)

$$K_{r1}\left(\frac{\partial^2 h_1}{\partial r^2} + \frac{1}{r}\frac{\partial h_1}{\partial r}\right) + K_{z1}\frac{\partial^2 h_1}{\partial z^2} = S_{s1}\frac{\partial h_1}{\partial t},$$
$$r_w \le r \le r_s \quad (1)$$

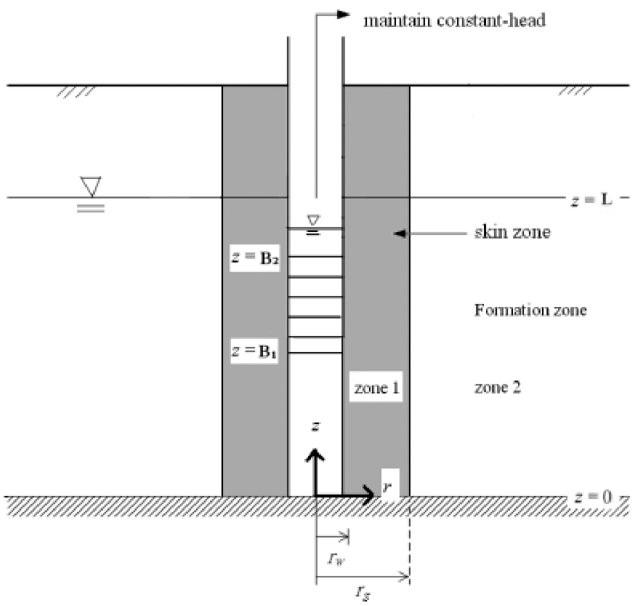

Figure 1. The representation of cross-sectional configuration of the well and aquifer

and

$$K_{r2}\left(\frac{\partial^2 h_2}{\partial r^2}+\frac{1}{r}\frac{\partial h_2}{\partial r}\right)+K_{z2}\frac{\partial^2 h_2}{\partial z^2}=S_{s2}\frac{\partial h_2}{\partial t},\quad r_s \le r < \infty \tag{2}$$

The subscripts 1 and 2 denote the skin and formation zones, respectively. The hydraulic head at a distance r from the centre of the well and a distance z from the bottom of the aquifer at time t is denoted as $h(r, z, t)$. The well screen extends from $z = B_1$ to $z = B_2$ with a length of l. The aquifer has the hydraulic parameters of horizontal hydraulic conductivity K_r, vertical hydraulic conductivity K_z, and specific storage S_s. An effective well radius and the radial distance from the well centreline to the outer skin envelope are denoted by r_w and r_s, respectively, in this system.

The hydraulic heads are assumed to be zero initially within the well as well as in the skin and formation zones, that is

$$h_1(r,z,0)=h_2(r,z,0)=0,\quad r\ge r_w \tag{3}$$

The hydraulic head represents the well water level when $r = r_w$ if the well loss is negligible. The outer boundary condition for the formation zone is

$$h_2(\infty,z,t)=0 \tag{4}$$

The no-flow boundary condition at the bottom of the aquifer can be written as:

$$\left.\frac{\partial h_1(r,z,t)}{\partial z}\right|_{z=0}=\left.\frac{\partial h_2(r,z,t)}{\partial z}\right|_{z=0}=0,\quad r\ge r_w \tag{5}$$

Assume that the drainage process is instantaneous and the drawdown everywhere of the aquifer is small in comparison to the initial saturated thickness L. Based on Neuman's approach that neglecting the second-order terms of the hydraulic gradient in the classic equation describing the free surface for the unconfined aquifer (Batu, 1998, p. 107), the top boundaries of the skin and formation zones are, respectively, described as

$$K_{z1}\left.\frac{\partial h_1(r,z,t)}{\partial z}\right|_{z=L}=-S_{y1}\left.\frac{\partial h_1(r,z,t)}{\partial t}\right|_{z=L},\quad r_w\le r\le r_s \tag{6}$$

and

$$K_{z2}\left.\frac{\partial h_2(r,z,t)}{\partial z}\right|_{z=L}=-S_{y2}\left.\frac{\partial h_2(r,z,t)}{\partial t}\right|_{z=L},\quad r_s\le r<\infty \tag{7}$$

where S_{y1} and S_{y2} represent the specific yields of the skin and formation zones, respectively. At the interface between skin and formation zones, the continuities of the hydraulic head and flow rate must be satisfied:

$$h_1(r_s,z,t)=h_2(r_s,z,t),\quad t>0 \tag{8}$$

Table I. Notations

τ	$K_{r2}t/S_{s2}r_w^2$
r_D	r/r_w
r_{Ds}	r_s/r_w
L_D	L/r_w
H_1	h_1/h_0
H_2	h_2/h_0
z_D	z/r_w
b_1	B_1/r_w
b_2	B_2/r_w
α_1	K_{z1}/K_{r1}
α_2	K_{z2}/K_{r2}
γ	K_{r2}/K_{r1}
α_{y1}	$(S_{y1}K_{r2})/(S_{s2}K_{z1}r_w)$
α_{y2}	$(S_{y2}K_{r2})/(S_{s2}K_{z2}r_w)$
ξ	S_{s1}/S_{s2}
$q_D(\tau)$	$q(t)/[2\pi(B_2-B_1)K_{r1}h_0]$
λ_{m1}	$\sqrt{\gamma\xi p+\alpha_1\omega_{m1}^2}$
λ_{m2}	$\sqrt{p+\alpha_2\omega_{m2}^2}$
λ_1	$\sqrt{\gamma\xi p}$
λ_2	$\sqrt{p}$
β_1	$-\lambda_{m1}K_1(\lambda_{m1}r_{Ds})K_0(\lambda_{m2}r_{Ds})+\gamma\lambda_{m2}K_0(\lambda_{m1}r_{Ds})K_1(\lambda_{m2}r_{Ds})$
β_2	$\lambda_{m1}I_1(\lambda_{m1}r_{Ds})K_0(\lambda_{m2}r_{Ds})+\gamma\lambda_{m2}I_0(\lambda_{m1}r_{Ds})K_1(\lambda_{m2}r_{Ds})$
β_3	$-\lambda_1K_1(\lambda_1r_{Ds})K_0(\lambda_2r_{Ds})+\gamma\lambda_2K_0(\lambda_1r_{Ds})K_1(\lambda_2r_{Ds})$
β_4	$\lambda_1I_1(\lambda_1r_{Ds})K_0(\lambda_2r_{Ds})+\gamma\lambda_2I_0(\lambda_1r_{Ds})K_1(\lambda_2r_{Ds})$
W_{1m}	$\frac{L_D}{2}+\frac{\sin(2\omega_{m1}L_D)}{4\omega_{m1}}$
W_{2m}	$\frac{L_D}{2}+\frac{\sin(2\omega_{m2}L_D)}{4\omega_{m2}}$
φ_0	$\beta_1I_1(\lambda_{m1})+\beta_2K_1(\lambda_{m1})$
ϕ	$(b_2-b_1)/L_D$

and

$$K_{r1}\left.\frac{\partial h_1(r,z,t)}{\partial r}\right|_{r=r_s}=K_{r2}\left.\frac{\partial h_2(r,z,t)}{\partial r}\right|_{r=r_s},\quad t>0 \tag{9}$$

The inner boundary condition specified in the well is

$$h_1(r_w,z,t)=h_0 \tag{10}$$

where h_0 is a constant water level in the well at any time. The flow rate along the screen is assumed to be (Yang and Yeh, 2005)

$$-K_{r1}\frac{\partial h_1(r_w,z,t)}{\partial r}=q(t),\quad B_1\le z\le B_2 \tag{11}$$

and the no-flow boundary condition along the well casing is

$$-K_{r1}\frac{\partial h_1(r_w,z,t)}{\partial r}=0,\quad z<B_1,\quad z<B_2 \tag{12}$$

where $q(t)$ is the flow rate per unit area and assumed to be constant along the well screen. The dimensionless parameters used in this study are defined in Table I and the detailed derivations of dimensionless hydraulic head solutions for skin and formation zones are listed in Appendix. The dimensionless hydraulic head solutions in Laplace domain are

$$\tilde{h}_1(r_D,z_D,p)=\tilde{q}_D(p)\sum_{m1=1}^{\infty}\frac{[-\beta_1I_0(\lambda_{m1}r_D)+\beta_2K_0(\lambda_{m1}r_D)]\eta}{W_{1m}\lambda_{m1}\varphi_0}\cos(\omega_{m1}z_D),\quad 1\le r_D\le r_{Ds}, \tag{13}$$

and

$$\tilde{h}_2(r_D, z_D, p) = \tilde{q}_D(p) \sum_{m2=1}^{\infty} \frac{K_0(\lambda_{m2} r_D)\eta}{r_{Ds} W_{2m} \lambda_{m2} \varphi_0} \cos(\omega_{m1} z_D),$$

$$r_{Ds} \leq r_D < \infty \tag{14}$$

with

$$\tilde{q}_D(p) = \frac{(b_2 - b_1)}{p} \left\{ \sum_{m1=1}^{\infty} [-\beta_1 I_0(\lambda_{m1}) + \beta_2 K_0(\lambda_{m1})] \frac{\eta^2}{W_{1m} \lambda_{m1} \varphi_0} \right\}^{-1} \tag{15}$$

$$\eta = \int_{b_1}^{b_2} \cos(\omega_{m1} z_D)\, \mathrm{d}z_D \tag{16}$$

$$W_{1m} = \int_0^{L_D} \cos^2(\omega_{m1} z_D)\, \mathrm{d}z_D \tag{17}$$

and

$$W_{2m} = \int_0^{L_D} \cos^2(\omega_{m2} z_D)\, \mathrm{d}z_D \tag{18}$$

where p is the Laplace variable, and variables b_1, b_2, r_D, r_{Ds}, z_D, α_1, α_2, β_1, β_2, λ_{m1}, λ_{m2}, ω_{m1}, ω_{m2}, W_{1m} and W_{2m} are defined in Table I. The functions $I_0(\cdot)$ and $I_1(\cdot)$ are the modified Bessel functions of the first kinds of order zero and one, respectively; and $K_0(\cdot)$ and $K_1(\cdot)$ are the modified Bessel functions of the second kinds of order zero and one, respectively (Abramowitz and Stegun, 1970). The Stehfest method (Stehfest, 1970) is used to evaluate the solutions in time-domain and the Shanks method (Shanks, 1955) is applied to accelerate convergence for the infinite summations in Equations (13) and (14) (Chang and Yeh, 2009). In addition, the Newton method is employed to search the roots in Equations (A38) and (A39). Letting $\alpha_{y1} = \alpha_{y2} = 0$ and $\omega_{m1} = \omega_{m2} = m\pi/L_D$ in Equations (13)–(15), the Laplace-domain solutions for drawdown in skin and formation zones in a partially penetrating well in a confined aquifer are exactly the same as those solutions given in Yang and Yeh (2005, Equations (33) and (34)) after some algebraic manipulations.

RESULTS AND DISCUSSION

In the no skin case ($\gamma = 1$), the temporal distributions of the dimensionless hydraulic head are plotted in Figure 2 to demonstrate the influence of vertical flow from delayed gravity drainage on the hydraulic head at different dimensionless distance r/r_w (r_D). The aquifer parameters used in this figure are as follows: $r_w = 0{\cdot}1$ m, $L = 10$ m, $K_{r1} = K_{r2} = 10^{-4}$ m/s, $K_{z1} = K_{z2} = 10^{-4}$ m/s, $S_{y1} = S_{y2} = 0{\cdot}1$, $S_{s1} = S_{s2} = 10^{-4}$ 1/m, $B_1 = 0$ m, $B_2 = 10$ m and $\phi = 1$ (full penetration). As shown in the figure, the vertical flows at a shorter distance (e.g. $r_D = 10$ or 100) are observed significantly in moderate times but not obviously in early and late times. In addition, the effect of vertical flow vanishes when

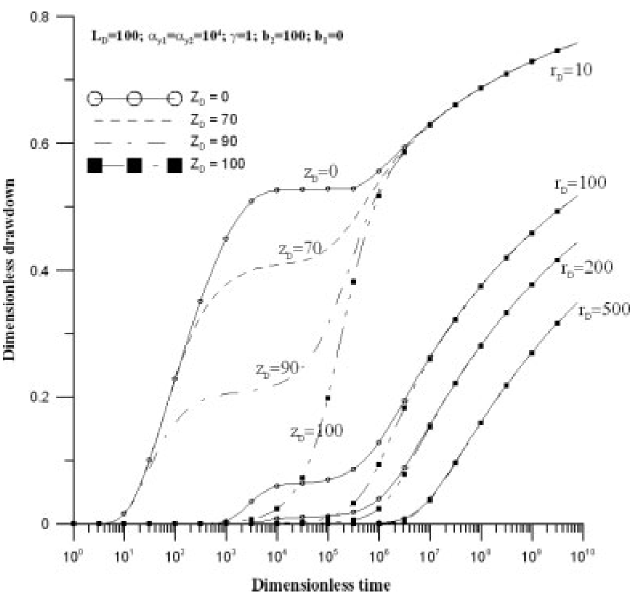

Figure 2. The temporal distributions of dimensionless hydraulic head at $r_D = 10$, 100, 200 and 500 for various dimensionless vertical distance z_D

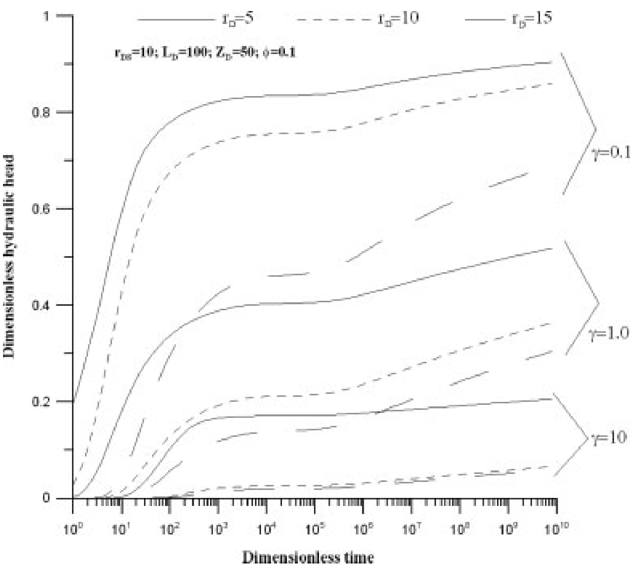

Figure 3. The effect of skin properties on the temporal distribution of dimensionless hydraulic head for $r_D = 5$, 10 and 15

r_D is larger than 500. At the bottom of the aquifer the vertical flow is observed at $\tau = 10^3$ which is the latest time among the various depths when $r_D = 10$. Figure 3 exhibits the behaviour of dimensionless hydraulic head versus dimensionless time τ and illustrates the effect of hydraulic conductivities of skin and formation zones on the hydraulic head response, where the skin thickness r_s is 1 m, the vertical distance $z = 5$ m and $\phi = 0{\cdot}1$ ($B_1 = 4{\cdot}5$ m and $B_2 = 5{\cdot}5$ m) at $r = 0{\cdot}5$, 1 and 1·5 m when K_{r1} ranges from 10^{-5}–10^{-3} m/s (i.e. $\gamma = 0{\cdot}1$, 1·0 and 10). This figure indicates that the dimensionless hydraulic heads increase with decreasing γ. Since $\gamma < 1$ (negative skin) represents the hydraulic conductivity of skin zone larger than that of the formation zone, the skin produces a larger flow rate toward the formation during the well test period and results in a

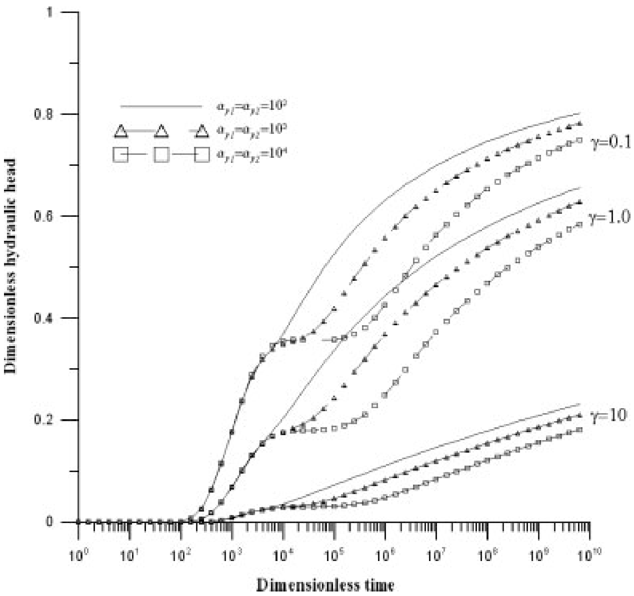

Figure 4. The effect of dimensionless specific yield on the dimensionless hydraulic head at $r_D = 50$ for $\gamma = 0{\cdot}1$, $\gamma = 1$ and $\gamma = 10$

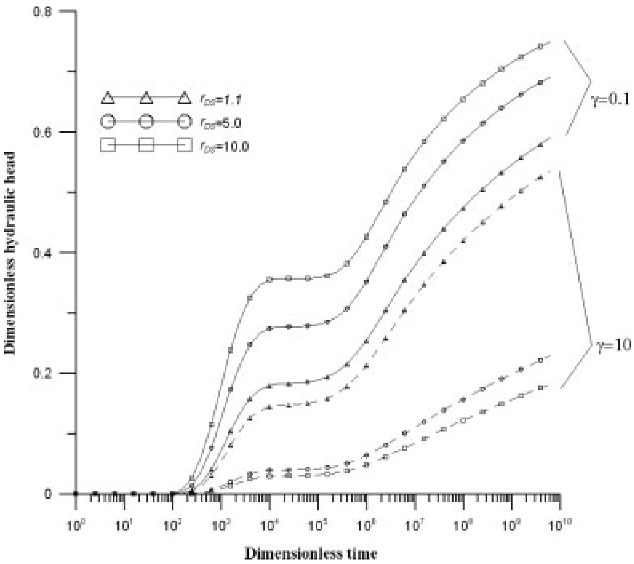

Figure 5. The effect of skin thickness on the dimensionless hydraulic head at $r_D = 50$ for $\gamma = 0{\cdot}1$ and $\gamma = 10$

higher dimensionless hydraulic head. Figure 4 is plotted to examine the effect of specific yields of skin and formation zones on the dimensionless hydraulic head during CHT. This figure displays the response of dimensionless hydraulic head at $r = 5$ m and $z = 0$ m with $r_s = 1$ m for specific yields S_{y1} and S_{y2} ranging from $10^{-1} \sim 10^{-3}$ for $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$ (positive skin). The dimensionless hydraulic heads decrease with increasing dimensionless specific yield. The typical three-stage drawdown patterns are shown in this figure. The early time storage releases from the elastic behaviour of the aquifer formation and water. At the moderate times (also called as delayed yield stage), the vertical flow happened near the water table causes drainage of the porous matrix. Finally the effect of vertical flow decreases with time and flow becomes horizontal at late times. Larger specific

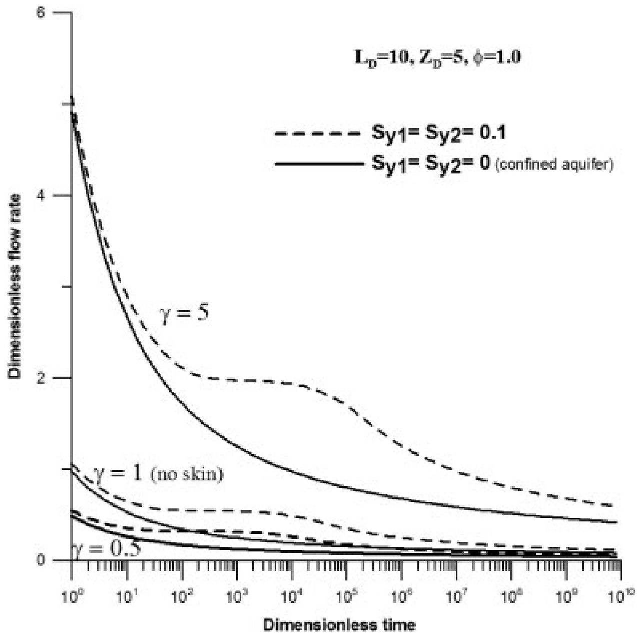

Figure 6. The effect of skin properties on the temporal distribution of dimensionless flow rate for $S_y = 0$ and $S_y = 0{\cdot}1$

yields supply more water from the drainage and thus the period of moderate times increases with dimensionless specific yield for the cases of the positive, no and negative skins. In order to explore the response of dimensionless hydraulic head for various skin thicknesses, Figure 5 is plotted for $r_s = 0{\cdot}11$, $0{\cdot}5$ and 1 m at $r = 5$ m and $z = 0$ m for both negative and positive skins. The dimensionless hydraulic head increases with the thickness for negative skin ($K_{r1} = 10^{-3}$ m/s) and decreases with the increasing thickness for positive skin ($K_{r1} = 10^{-5}$ m/s). This result demonstrates that the skin thickness has significant effect upon the distribution of hydraulic head in CHT. In reality, the behaviour of curves is similar to those of hydraulic head for different skin thicknesses in CHT in Yang and Yeh (2006), except in the delayed yield stage. Figure 6 shows the curves of the dimensionless flow rate at wellbore versus dimensionless time for different values of specific yield. The dimensionless flow rate decreases rapidly with increasing dimensionless time. The dimensionless flow rate increases with γ for both $S_{y1} = S_{y2} = 0$ and $S_{y1} = S_{y2} = 0{\cdot}1$ at the same dimensionless time indicating that the dimensionless flow rate is significantly influenced by the conductivity ratio, γ.

CONCLUDING REMARKS

The solution for transient hydraulic head distribution is developed for CHT performed in a two-zone unconfined aquifer with a partially penetrating well. The transient hydraulic head distribution in Laplace domain for CHT is developed via Laplace transforms and the method of separation of variables. The Stehfest algorithm is used for numerical Laplace inversion and the Shanks method is adopted to accelerate convergence in calculating the infinite summation solutions. In addition, the solutions of this study reduce to the solutions for the cases of fully and partially penetrating well in confined aquifers in CHT.

For the CHT in an unconfined aquifer, the present solution can be used for describing the transient hydraulic head distribution and investigating the effects of vertical flow from delayed gravity drainage, thickness of skin zone and specific yield on the hydraulic head distribution. As the distance from the well increases, the effect of vertical flow is negligible. For negative skin thickness, the hydraulic head increases with decreasing γ; however, the hydraulic head increases with γ for positive skin. An aquifer with a higher specific yield will have longer delayed yield stage in the hydraulic head curve for all the positive- and negative-skin cases. In addition, the skin thickness has great effect on the hydraulic head distribution in both negative- and positive-skin cases. The thicker skin zone gives higher hydraulic head for negative skin and lower hydraulic head for positive skin case.

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APPENDIX

Equations (1) and (12) can be expressed in terms of dimensionless forms as

$$\frac{\partial^2 H_1}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial H_1}{\partial r_D} + \alpha_1 \frac{\partial^2 H_1}{\partial z_D^2} = \gamma\xi \frac{\partial H_1}{\partial \tau}, \quad 1 \le r_D \le r_{Ds} \tag{A1}$$

$$\frac{\partial^2 H_2}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial H_2}{\partial r_D} + \alpha_2 \frac{\partial^2 H_2}{\partial z_D^2} = \frac{\partial H_2}{\partial \tau}, \quad r_{Ds} \le r_D < \infty \tag{A2}$$

$$H_1(r_D, z_D, 0) = H_2(r_D, z_D, 0) = 0, \quad r_D \ge 1 \tag{A3}$$

$$H_2(\infty, z_D, \tau) = 0 \tag{A4}$$

$$\left.\frac{\partial H_1(r_D, z_D, \tau)}{\partial z_D}\right|_{z_D=0} = \left.\frac{\partial H_2(r_D, z_D, \tau)}{\partial z_D}\right|_{z_D=0} = 0 \tag{A5}$$

$$\left.\frac{\partial H_1(r_D, z_D, \tau)}{\partial z_D}\right|_{z_D=L_D} = -\alpha_{y1} \left.\frac{\partial H_1(r_D, z_D, \tau)}{\partial \tau}\right|_{z_D=L_D}, \quad 1 \le r_D \le r_{Ds} \tag{A6}$$

$$\left.\frac{\partial H_2(r_D, z_D, \tau)}{\partial z_D}\right|_{z_D=L_D} = -\alpha_{y2} \left.\frac{\partial H_2(r_D, z_D, \tau)}{\partial \tau}\right|_{z_D=L_D}, \quad r_{Ds} \le r_D < \infty \tag{A7}$$

$$H_1(r_{Ds}, z_D, \tau) = H_2(r_{Ds}, z_D, \tau), \quad \tau > 0 \tag{A8}$$

$$\left.\frac{\partial H_1(r_D, z_D, \tau)}{\partial r_D}\right|_{r=r_s} = \gamma \left.\frac{\partial H_2(r_D, z_D, \tau)}{\partial r_D}\right|_{r=r_s}, \quad \tau > 0 \tag{A9}$$

$$H_1(1, z_D, \tau) = 1 \tag{A10}$$

$$-\frac{\partial H_1(1, z_D, \tau)}{\partial r_D} = q_D(\tau), \quad b_1 \le z_D \le b_2 \tag{A11}$$

and

$$-\frac{\partial H_1(1, z_D, \tau)}{\partial r_D} = 0, \quad z_D < b_1, \quad z_D < b_2 \tag{A12}$$

If the Laplace transforms (Sneddon, 1972) is applied to Equations (A1)–(A12), the results will be

$$\frac{\partial^2 \tilde{H}_1}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \tilde{H}_1}{\partial r_D} + \alpha_1 \frac{\partial^2 \tilde{H}_1}{\partial z_D^2} = \gamma\xi p \tilde{H}_1, \quad 1 \le r_D \le r_{Ds} \tag{A13}$$

$$\frac{\partial^2 \tilde{H}_2}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial \tilde{H}_2}{\partial r_D} + \alpha_2 \frac{\partial^2 \tilde{H}_2}{\partial z_D^2} = p \tilde{H}_2, \quad r_{Ds} \le r_D < \infty \tag{A14}$$

$$\tilde{H}_2(\infty, z_D, p) = 0 \tag{A15}$$

$$\left.\frac{\partial \tilde{H}_1(r_D, z_D, p)}{\partial z_D}\right|_{z_D=0} = \left.\frac{\partial \tilde{H}_2(r_D, z_D, p)}{\partial z_D}\right|_{z_D=0} = 0 \tag{A16}$$

$$\left.\frac{\partial \tilde{H}_1(r_D, z_D, p)}{\partial z_D}\right|_{z_D=L_D} = -p\alpha_{y1}\tilde{H}_1, \quad 1 \le r_D \le r_{Ds} \tag{A17}$$

$$\left.\frac{\partial \tilde{H}_2(r_D, z_D, \tau)}{\partial z_D}\right|_{z_D=L_D} = -p\alpha_{y2}\tilde{H}_2, \quad r_{Ds} \le r_D < \infty \tag{A18}$$

$$\tilde{H}_1(r_{Ds}, z_D, p) = \tilde{H}_2(r_{Ds}, z_D, p) \tag{A19}$$

$$\left.\frac{\partial \tilde{H}_1(r_D, z_D, p)}{\partial r_D}\right|_{r=r_s} = \gamma \left.\frac{\partial \tilde{H}_2(r_D, z_D, p)}{\partial r_D}\right|_{r=r_s} \tag{A20}$$

$$\tilde{H}_1(1, z_D, p) = \frac{1}{p} \tag{A21}$$

$$-\frac{\partial \tilde{H}_1(1, z_D, p)}{\partial r_D} = \tilde{q}_D(p), \quad b_1 \le z_D \le b_2 \tag{A22}$$

and

$$-\frac{\partial H_1(1, z_D, p)}{\partial r_D} = 0, \quad z_D < b_1, \quad z_D < b_2 \tag{A23}$$

Assuming that $\tilde{H}_1$ and $\tilde{H}_2$ are the product of two distinct functions, i.e. $\tilde{H}_1(r_D, z_D, p) = F_1(r_D, p)G_1(z_D, p)$ and $\tilde{H}_2(r_D, z_D, p) = F_2(r_D, p)G_2(z_D, p)$, respectively, one can transform Equations (13) and (14) to

$$G_1\frac{\partial^2 F_1}{\partial r_D^2} + G_1\frac{1}{r_D}\frac{\partial F_1}{\partial r_D} + F_1\alpha_1\frac{\partial^2 G_1}{\partial z_D^2} = \gamma\xi p F_1 G_1, \quad 1 \le r_D \le r_{Ds} \tag{A24}$$

and

$$G_2\frac{\partial^2 F_2}{\partial r_D^2} + G_2\frac{1}{r_D}\frac{\partial F_2}{\partial r_D} + F_2\alpha_2\frac{\partial^2 G_2}{\partial z_D^2} = p F_2 G_2, \quad r_{Ds} \le r_D < \infty \tag{A25}$$

Equations (A24) and (A25) can be separated into the following two systems of equations

$$\alpha_1\frac{\partial^2 G_1}{\partial z_D^2} + \omega_{m1}^2\alpha_1 G_1 = 0 \tag{A26}$$

$$\frac{\partial^2 F_1}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial F_1}{\partial r_D} - \left[\gamma\xi p + \omega_{m1}^2\alpha_1\right] F_1 = 0, \quad 1 \le r_D \le r_{Ds} \tag{A27}$$

and

$$\alpha_2\frac{\partial^2 G_2}{\partial z_D^2} + \omega_{m2}^2\alpha_2 G_2 = 0 \tag{A28}$$

$$\frac{\partial^2 F_2}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial F_2}{\partial r_D} - \left[p + \omega_{m2}^2\alpha_2\right] F_2 = 0, \quad r_{Ds} \le r_D < \infty \tag{A29}$$

The solutions of Equations (A26) and (A28) subject to the boundary in Equation (A16) are respectively

$$G_1(z_D, p) = a_{m1}(p)\cos(\omega_{m1} z_D) \tag{A30}$$

and

$$G_2(z_D, p) = a_{m2}(p)\cos(\omega_{m2} z_D) \tag{A31}$$

where $a_{m1}(p)$ and $a_{m2}(p)$ are constants. In addition, substituting Equation (A30) into Equation (A17) yields the root ω_{m1} of

$$\omega_{m1}\tan(\omega_{m1} L_D) = p\alpha_{y1} \tag{A32}$$

Similarly, substituting Equation (A31) into Equation (A18) results in the root ω_{m2} of

$$\omega_{m2}\tan(\omega_{m2} L_D) = p\alpha_{y2} \tag{A33}$$

The solutions of Equations (A27) and (A29) then are respectively

$$F_1(r_D, p) = b_{m1}(p) I_0(\lambda_{m1} r_D) + c_{m1}(p) K_0(\lambda_{m1} r_D), \quad 1 \le r_D \le r_{Ds} \tag{A34}$$

and

$$F_2(r_D, p) = b_{m2}(p) K_0(\lambda_{m2} r_D) + c_{m2}(p) K_0(\lambda_{m2} r_D), \quad r_{Ds} \le r_D < \infty \tag{A35}$$

where $b_{m1}(p)$, $b_{m2}(p)$ and $c_{m1}(p)$ are constants, and $\lambda_{m2} = \sqrt{p + \omega_{m2}^2 \alpha_2}$. The $c_{m2}(p)$ equals zero when using the boundary of Equation (A15).

The product of Equations (A30) and (A34) for the m 1-root gives

$$\tilde{H}_{m1}(r_D, z_D, p) = A_{m1} I_0(\lambda_{m1} r_D)\cos(\omega_{m1} z_D) + B_{m1} K_0(\lambda_{m1} r_D)\cos(\omega_{m1} z_D), \quad 1 \le r_D \le r_{Ds} \tag{A36}$$

where A_{m1} equals the product of $a_{m1}(p)$ and $b_{m1}(p)$ and B_{m1} denotes the product of $a_{m1}(p)$ and $c_{m1}(p)$. On the other hand, the product of Equations (A31) and (A35) for the $m2$-root yields

$$\tilde{H}_{m2}(r_D, z_D, p) = B_{m2} K_0(\lambda_{m2} r_D)\cos(\omega_{m2} z_D), \quad r_{Ds} \le r_D < \infty \tag{A37}$$

where A_{m2} is the product of $a_{m2}(p)$ and $b_{m2}(p)$. There are infinite number of the roots in Equations (A32) and (A33). Accordingly, based on Equations (A36) and (A37), the complete solution for $\tilde{H}_1$ and $\tilde{H}_2$ can be, respectively, obtained as

$$\tilde{H}_1(r_D, z_D, p) = \sum_{m1=1}^{\infty} \tilde{H}_{m1} = \sum_{m1=1}^{\infty} [A_{m1} I_0(\lambda_{m1} r_D)\cos(\omega_{m1} z_D) + B_{m1} K_0(\lambda_{m1} r_D)\cos(\omega_{m1} z_D)], \quad 1 \le r_D \le r_{Ds} \tag{A38}$$

and

$$\tilde{H}_2(r_D, z_D, p) = \sum_{m2=1}^{\infty} \tilde{H}_{m2} = \sum_{m2=1}^{\infty} B_{m2} K_0(\lambda_{m2} r_D)\cos(\omega_{m2} z_D), \quad r_{Ds} \le r_D < \infty \tag{A39}$$

The coefficients in Equations (A38) and (A39) can be obtained simultaneously from Equations (A19), (A20), (A22) and (A23) as

$$A_{m1} = -\frac{\tilde{q}_D(p)}{W_{1m}} \frac{\beta_1 \eta}{\lambda_{m1} \varphi_0} \tag{A40}$$

$$B_{m1} = \frac{\tilde{q}_D(p)}{W_{1m}} \frac{\beta_2 \eta}{\lambda_{m1} \varphi_0} \tag{A41}$$

$$B_{m2} = \frac{\tilde{q}_D(p)}{W_{2m}} \frac{\eta}{\lambda_{m1} r_{Ds} \varphi_0} \tag{A42}$$

Note that the dimensionless flow rate $\tilde{q}_D(p)$ is still unknown at present stage, and can be obtained by substituting Equation (A38) into Equation (A21) as Equation (24).

