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Cordial labeling of mK_n

David Kuo[a,1], Gerard J. Chang[a,1], Y.H. Harris Kwong[b,*]

[a] *Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan*

[b] *Department of Mathematics and Computer Science, SUNY College at Fredonia, Fredonia, NY 14063, USA*

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Abstract

Suppose $G = (V,E)$ is a graph with vertex set V and edge set E. A vertex labeling f: $V \rightarrow \{0,1\}$ induces an edge labeling f^*: $E \rightarrow \{0,1\}$ defined by $f^*(xy) = |f(x) - f(y)|$. For $i \in \{0,1\}$, let $v_f(i)$ and $e_f(i)$ be the number of vertices v and edges e with $f(v) = i$ and $f^*(e) = i$, respectively. A graph G is cordial if there exists a vertex labeling f such that $|v_f(0) - v_f(1)| \leqslant 1$ and $|e_f(0) - e_f(1)| \leqslant 1$. This paper determines all m and n for which mK_n is cordial.

Keywords: Cordial labeling, graceful labeling, Diophantine equation

1. Introduction

Suppose $G = (V,E)$ is a graph with vertex set V and edge set E. A vertex labeling $f: V \rightarrow \{0,1\}$ *induces* an edge labeling $f^*: E \rightarrow \{0,1\}$ defined by $f^*(xy) = |f(x) - f(y)|$. For a vertex labeling f and $i \in \{0,1\}$, a vertex v is an *i-vertex* if $f(v) = i$ and an edge is an *i-edge* if $f^*(e) = i$. Denote the numbers of 0-vertices, 1-vertices, 0-edges, and 1-edges of G under f by $v_f(0)$, $v_f(1)$, $e_f(0)$, and $e_f(1)$, respectively. A vertex labeling f is *cordial* if $|v_f(0) - v_f(1)| \leqslant 1$ and $|e_f(0) - e_f(1)| \leqslant 1$. A graph is *cordial* if it admits a cordial labeling. A graph is *perfectly cordial* if there exists a vertex labeling f such that $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$. For even m, mK_n has an even number of vertices and an even number of edges, so cordiality is equivalent to perfect cordiality when m is even.

The notion of a cordial labeling was first introduced by Cahit [3] as a weaker version of graceful labeling. See [5,10,13,14,16,17] for related results and [4,6,11] for generalizations. Cordial labelings of various families of graphs were studied in

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* Corresponding author. E-mail: kwong@cs.fredonia.edu.

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[1,8,9,12,15]. This paper completely determines the cordiality of mK_n, the disjoint union of m copies of the complete graph K_n of n vertices.

To do this, we first transform the problem into Diophantine equations with boundary conditions. Results from number theory are then used to solve the problem.

More precisely, given a vertex labeling f of mK_n, let x_i be the number of 1-vertices in the ith copy of K_n. Restricting f to this K_n, we have

$$v_f(0) - v_f(1) = n - 2x_i\,,$$
$$e_f(0) - e_f(1) = ((n - 2x_i)^2 - n)/2\,.$$

Setting $y_i = n - 2x_i$, we obtain the following theorem.

Theorem 1. *mK_n is cordial if and only if the following system has an integral solution* $(y_1, y_2, \ldots, y_m)$:

$$|y_i| \leqslant n \quad \text{for } 1 \leqslant i \leqslant m, \tag{C1}$$
$$y_1 \equiv y_2 \equiv \cdots \equiv y_m \equiv n \pmod 2, \tag{C2}$$
$$|y_1 + y_2 + \cdots + y_m| \leqslant 1, \tag{C3}$$
$$|y_1^2 + y_2^2 + \cdots + y_m^2 - mn| \leqslant 2. \tag{C4}$$

Remark. Note that if (C4) holds and $n \geqslant m$, then $y_i^2 \leqslant mn + 2 \leqslant n^2 + 2 < (n+1)^2$ for each i, $1 \leqslant i \leqslant m$, so (C1) holds. Therefore, when checking whether $(y_1, y_2, \ldots, y_m)$ satisfies (C1)–(C4), it is not necessary to check (C1) if $n \geqslant m$.

2. Cordiality of mK_n for $m \leqslant 4$

In this section, we determine the cordiality of mK_n for $m \leqslant 4$. The main results are contained in Theorems 2, 3, 6 and 8.

Theorem 2 (Cahit [3]). *K_n is cordial if and only if $n \in \{1, 2, 3\}$.*

Proof. If K_n is cordial, then by Theorem 1, there exists an integer y_1 such that $|y_1| \leqslant 1$ and $|y_1^2 - n| \leqslant 2$. We conclude that $n \in \{1, 2, 3\}$.

Conversely, for $n \in \{1, 2, 3\}$, $y_1 = (n \bmod 2)$ is a solution to (C1)–(C4), so K_n is cordial. □

Theorem 3. *$2K_n$ is cordial if and only if $n = k^2$ for some integer k.*

Proof. Suppose $2K_n$ is cordial. Let (y_1, y_2) be a solution to (C1)–(C4). By (C2), y_1 and y_2 are of the same parity, i.e., $y_1 + y_2$ is even. This together with (C3) imply

$y_1 + y_2 = 0$. (C4) then gives $|y_1^2 - n| \leqslant 1$. As y_1 and n are of the same parity, we have $n = y_1^2$.

Conversely, if $n = k^2$ for some integer k, then $(y_1, y_2) = (-k, k)$ is a solution to (C1)–(C4). Hence $2K_n$ is cordial. □

Theorem 4. (a) *If n is even, then $3K_n$ is cordial if and only if $n \equiv 0$ or $2 \pmod 8$ and $x^2 + 3y^2 = a/2$ has an integral solution, where a is the only number in $\{3n, 3n+2\}$ satisfying $a \equiv 0 \pmod 8$.*

(b) *If n is odd, then $3K_n$ is cordial if and only if $n \equiv 1, 3$ or $7 \pmod 8$ and $x^2+3y^2 = 6a-2$ has an integral solution, where a is the only number in $\{3n-2, 3n, 3n+2\}$ satisfying $a \equiv 3 \pmod 8$.*

Proof. (a) Suppose n is even. If $3K_n$ is cordial, then (C1)–(C4) has a solution (y_1, y_2, y_3), where y_1, y_2, and y_3 are even. So (C3) implies $y_1 + y_2 + y_3 = 0$. Substituting $y_3 = -y_1 - y_2$ into (C4), we obtain

$$2y_1^2 + 2y_2^2 + 2y_1y_2 - 3n = -2, 0 \text{ or } 2.$$

However, $2y_1^2 + 2y_2^2 + 2y_1y_2 \equiv 0$ or $2 \pmod 3$ only. Thus

$$a = 2y_1^2 + 2y_2^2 + 2y_1y_2 = 3n \text{ or } 3n+2. \tag{2.1}$$

Since $2y_1^2 + 2y_2^2 + 2y_1y_2 \equiv 0 \pmod 8$, $n \equiv 0$ or $2 \pmod 8$, and a is the unique integer amongst $3n$ and $3n+2$ satisfying $a \equiv 0 \pmod 8$. Consequently, $x^2 + 3y^2 = a/2$ has an integral solution because $(y_1 + y_2/2)^2 + 3(y_2/2)^2 = a/2$.

Conversely, suppose $x^2 + 3y^2 = a/2$ has an integral solution (x, y). Let $(y_1, y_2, y_3) = (x - y, 2y, -x - y)$. To verify (C1), by the remark after Theorem 1, we only need to consider the case of $n = 2$. In this case, $a = 8$ and $(x, y) = (\pm 1, \pm 1)$ or $(\pm 2, 0)$; thus (C1) holds. Since $a \equiv 0 \pmod 8$, x and y are of the same parity, hence (C2) holds. (C3) and (C4) follow from a straightforward calculation. Therefore $3K_n$ is cordial.

(b) Suppose n is odd. If $3K_n$ is cordial, then (C1)–(C4) has a solution (y_1, y_2, y_3) where y_1, y_2, and y_3 are odd, $y_1 + y_2 + y_3 = \pm 1$ and

$$y_1^2 + y_2^2 + y_3^2 - 3n = -2, 0 \text{ or } 2.$$

However, $a = y_1^2 + y_2^2 + y_3^2 \equiv 3 \pmod 8$. Hence $n \equiv 1, 3$ or $7 \pmod 8$ and a is the unique integer in $\{0, 3n \pm 2\}$ satisfying $a \equiv 3 \pmod 8$. Using the solution $(-y_1, -y_2, -y_3)$ if necessary, we may further assume $y_1 + y_2 + y_3 = -1$, which leads to $a = 2y_1^2 + 2y_1y_2 + 2y_2^2 + 2y_1 + 2y_2 + 1$, or equivalently, $(3y_1+1)^2 + 3(y_1+2y_2+1)^2 = 6a-2$. Thus $x^2 + 3y^2 = 6a - 2$ has an integral solution.

Conversely, suppose $x^2 + 3y^2 = 6a - 2$ has an integral solution (x, y). Then $x^2 \equiv 1 \pmod 3$. Since $(-x, y)$ is also a solution, we may assume $x \equiv 1 \pmod 3$. Also, $a \equiv 3 \pmod 8$ implies $6a - 2 \equiv 0 \pmod 8$ and $x \equiv y \equiv 0$ or $2 \pmod 4$. Let $y_1 = (x-1)/3$, $y_2 = (y - 1 - y_1)/2$, and $y_3 = -1 - y_1 - y_2 = (-y - 1 - y_1)/2$. For (C1), we only need to check the case of $n = 1$. In this case, $a = 3$ and $(x, y) = (4, 0)$ or $(-2, \pm 2)$.

Thus each $|y_i| \leqslant 1$, hence (C1) holds. Next, $y_1 \equiv (x-1)/3 \equiv 1 - x \pmod 4$ and $2y_2 = y - 1 - y_1 \equiv x + y - 2 \equiv 2 \pmod 4$, so both y_1 and y_2 are odd; consequently, so is y_3. (C3) and (C4) follow from a straightforward calculation. □

In order to use Theorem 4 effectively, we need a result from number theory.

Theorem 5 (Bolker [2, p. 122]). *The equation $x^2+3y^2 = n$ has an integral solution if and only if $n = 3^{a_0} p_1^{a_1} \cdots p_r^{a_r} 2^{2b_0} q_1^{2b_1} \cdots q_s^{2b_s}$ for some nonnegative integers a_i and b_i, where p_i and q_i are primes satisfying $p_i \equiv 1 \pmod 6$ and $q_i \equiv 5 \pmod 6$, respectively.*

The cordiality of $3K_n$ is summarized in the next theorem.

Theorem 6. *Define $b = b(n)$ as*

$$b = \begin{cases} 3n/2 & \text{if } n \equiv 0 \pmod 8, \\ 3n/2 + 1 & \text{if } n \equiv 2 \pmod 8, \\ 18n - 2 & \text{if } n \equiv 1 \pmod 8, \\ 18n + 10 & \text{if } n \equiv 3 \pmod 8, \\ 18n - 14 & \text{if } n \equiv 7 \pmod 8, \end{cases}$$

and let $v_p(z)$ denote the highest power of the prime p that divides z. Then $3K_n$ is cordial if and only if $n \equiv 0,1,2,3,7 \pmod 8$ and $v_2(b)$ and $v_q(b)$ are even for all prime divisors q of b with $q \equiv 5 \pmod 6$.

We now turn our attention to $4K_n$.

Theorem 7 (Grosswald [7, p. 24]). *The equation $x^2 + y^2 + z^2 = n$ has an integral solution (x,y,z) if and only if n is not of the form $4^a(8b+7)$ for any nonnegative integers a and b.*

Theorem 8. *$4K_n$ is cordial if and only if n is not of the form $4^a(8b+7)$ for any nonnegative integers a and b.*

Proof. Suppose $4K_n$ is cordial and (y_1,y_2,y_3,y_4) is a solution to (C1)–(C4). By (C2), y_1, y_2, y_3, and y_4 are of the same parity. (C3) then implies $y_1+y_2+y_3+y_4 = 0$. (C4) implies $y_1^2+y_2^2+y_3^2+y_4^2 = 4n$. Hence $4n = y_1^2+y_2^2+y_3^2+(-y_1-y_2-y_3)^2 = (y_1+y_2)^2+(y_2+y_3)^2+(y_3+y_1)^2$. By Theorem 7, $4n$, and hence n, are not of the form $4^a(8b+7)$ for any nonnegative integers a and b.

Conversely, suppose n is not of the form $4^a(8b+7)$ for any nonnegative integers a and b. By Theorem 7, $x^2+y^2+z^2 = n$ has an integer solution (x,y,z). Let

$$y_1 = x+y+z, \qquad y_2 = x-y-z,$$
$$y_3 = -x+y-z, \qquad y_4 = -x-y+z.$$

Note that for each i, $|y_i| \leqslant |x| + |y| + |z| \leqslant x^2 + y^2 + z^2 = n$. Thus (C1) holds. Also, each $y_i \equiv x^2 + y^2 + z^2 \equiv n \pmod 2$, so (C2) holds. (C3) and (C4) follow from a straightforward calculation. Hence $4K_n$ is cordial. □

3. Cordiality of mK_n for $m \geqslant 5$

The cordiality of mK_n is much more uniform when $m \geqslant 5$. There are three cases: $m \equiv 0 \pmod 4$, $m \equiv 2 \pmod 4$, and $m \equiv 1 \pmod 2$. Theorems 12, 14 and 15 contain the corresponding results. Because of the next lemma and its corollary, in each of these three cases, we only have to check the cordiality of mK_n for only a few values of m.

Lemma 9. *If G is perfectly cordial and H is cordial, then $G \cup H$ is cordial.*

Corollary 10. *If mK_n and $m'K_n$ are cordial and m is even, then $(m+m')K_n$ is cordial.*

Proof. The corollary follows from Lemma 9 and the fact that cordiality and perfect cordiality of mK_n are equivalent when m is even. □

To determine the cordiality of mK_n, we need the famous 'four squares theorem' due to Lagrange (see, for example, [7, p. 25]):

Theorem 11. *The equation $z_1^2 + z_2^2 + z_3^2 + z_4^2 = n$ has an integral solution for any nonnegative integer n.*

Theorem 12. *If $m \geqslant 5$ and $m \equiv 0 \pmod 4$, then mK_n is cordial.*

Proof. Because of Corollary 10, we only have to prove the theorem for $m = 8$ and $m = 12$.

Case 1: $m = 8$. By Theorem 11, $z_1^2 + z_2^2 + z_3^2 + z_4^2 = n$ has an integral solution (z_1, z_2, z_3, z_4). Let

$$y_1 = z_1 + z_2 + z_3 + z_4, \qquad y_2 = z_1 - z_2 + z_3 - z_4,$$
$$y_3 = z_1 - z_2 - z_3 + z_4, \qquad y_4 = z_1 + z_2 - z_3 - z_4,$$
$$y_5 = -y_1, \quad y_6 = -y_2, \quad y_7 = -y_3, \quad y_8 = -y_4.$$

Then for each i,

$$|y_i| \leqslant |z_1| + |z_2| + |z_3| + |z_4| \leqslant z_1^2 + z_2^2 + z_3^2 + z_4^2 = n,$$

that is, (C1) holds. Also,

$$y_i \equiv z_1 + z_2 + z_3 + z_4 \equiv z_1^2 + z_2^2 + z_3^2 + z_4^2 \equiv n \pmod 2$$

for each i, so (C2) holds. (C3) is clearly true. (C4) follows a straightforward calculation.

Case 2: $m = 12$ *and* n *is even.* By Theorem 11, $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 3n/2$ has an integral solution (z_1, z_2, z_3, z_4). Let

$$y_i = 2z_i, \quad y_{i+4} = -2z_i \quad \text{and} \quad y_{i+8} = 0 \quad \text{for } 1 \leqslant i \leqslant 4.$$

Then (C2)–(C4) hold. For $n = 2$ or 4, $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 3$ or 6 implies each $|z_i| \leqslant 1$ or 2, respectively, hence (C1) holds. For $n \geqslant 6$, each $|y_i| \leqslant 2|z_j| \leqslant 2\sqrt{3n/2} = \sqrt{6n} \leqslant n$, so (C1) holds again. Thus $12K_n$ is cordial.

Case 3: $m = 12$ *and* n *is odd.* By Theorem 11, $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 3(n-1)/2$ has an integral solution. Let

$$y_i = \begin{cases} 1 + 2z_i & \text{if } 1 \leqslant i \leqslant 4, \\ 1 - 2z_{i-6} & \text{if } 7 \leqslant i \leqslant 10, \end{cases}$$

$$y_5 = y_6 = 1, \qquad y_{11} = y_{12} = -1.$$

Then (C2)–(C4) holds. For $n = 1$, 3 or 5, $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$, 3, or 6 implies $|z_i| \leqslant 0$, 1, or 2, respectively, so (C1) holds. For $n \geqslant 7$, each $|y_i| \leqslant 2|z_j| + 1 \leqslant 2\sqrt{3(n-1)/2} + 1 = \sqrt{6(n-1)} + 1 \leqslant (n-1) + 1 = n$, so (C1) still holds. Thus $12K_n$ is cordial. □

Lemma 13. *If* (i) n *is even and* $mn \equiv 4 \pmod 8$, *or* (ii) n *is odd and* $m(n-1) \equiv 4 \pmod 8$, *then* mK_n *is not cordial.*

Proof. Suppose mK_n is cordial and $(y_1, y_2, \ldots, y_n)$ is an integral solution to (C1)–(C4).

Suppose n is even and $mn \equiv 4 \pmod 8$. By (C2), all y_i are even. By (C3), $y_1 + y_2 + \cdots + y_n = 0$. By (C4), $y_1^2 + y_2^2 + \cdots + y_m^2 = mn$. Let $z_i = y_i/2$ for $1 \leqslant i \leqslant m$. Then $z_1 + z_2 + \cdots + z_m = 0$ and $z_1^2 + z_2^2 + \cdots + z_m^2 = mn/4$. Since $mn \equiv 4 \pmod 8$, $mn/4$ is odd. Consequently,

$$0 \equiv z_1 + z_2 + \cdots + z_m \equiv z_1^2 + z_2^2 + \cdots + z_m^2 \equiv mn/4 \equiv 1 \pmod 2,$$

a contradiction.

Suppose n is odd and $m(n-1) \equiv 4 \pmod 8$. By (C2), each y_i is odd, say $y_i = 2z_i + 1$. By (C4), $|4\sum_{i=1}^m z_i(z_i+1) - m(n-1)| \leqslant 2$. This is impossible, since $4\sum_{i=1}^m z_i(z_i+1) \equiv 0 \pmod 8$ and $m(n-1) \equiv 4 \pmod 8$. □

Theorem 14. *For* $m \geqslant 5$ *and* $m \equiv 2 \pmod 4$, mK_n *is cordial if and only if* $n \equiv 0$ *or* 1 $\pmod 4$.

Proof. If mK_n is cordial, then $n \equiv 0$ or $1 \pmod 4$ by Lemma 13. Conversely, suppose $n \equiv 0$ or $1 \pmod 4$. By Corollary 10 and Theorem 12, we only have to prove that mK_n is cordial for $m = 6$ and $m = 10$. Solutions to (C1)–(C4) for $n < m$ are listed below.

$m = 6$ and $n = 1$: $(1,1,1,-1,-1,-1)$.
$m = 6$ and $n = 4$: $(2,2,2,-2,-2,-2)$.
$m = 6$ and $n = 5$: $(3,3,-3,-1,-1,-1)$.
$m = 10$ and $n = 1$: $(1,1,1,1,1,-1,-1,-1,-1,-1)$.
$m = 10$ and $n = 4$: $(2,2,2,2,2,-2,-2,-2,-2,-2)$.
$m = 10$ and $n = 5$: $(5,3,1,-3,-1,-1,-1,-1,-1,-1)$.
$m = 10$ and $n = 8$: $(8,-2,-2,-2,-2,0,0,0,0,0)$.
$m = 10$ and $n = 9$: $(9,-1,-1,-1,-1,-1,-1,-1,-1,-1)$.

Now we may assume $n \geqslant m$, and we only need to check (C2)–(C4) for a solution.

Case 1: $n \equiv 0 \pmod 4$ *and* $mn/4$ *is not of the form* $4^a(8b+7)$. By Theorem 7, $x^2 + y^2 + z^2 = mn/4$ has an integral solution (x,y,z). Let

$$y_1 = x + y + z, \qquad y_2 = x - y - z,$$
$$y_3 = -x + y - z, \qquad y_4 = -x - y + z,$$
$$y_i = 0 \quad \text{for } i \geqslant 5.$$

For instance, when $m = 10$, $n = 12$, we could let $(x,y,z) = (5,2,1)$, from which we obtain the solution $(8,2,-4,-6,0,0,0,0,0,0)$.

In general, each $y_i \equiv x^2 + y^2 + z^2 \equiv mn/4 \equiv 0 \equiv n \pmod 2$, so (C2) holds. (C3) and (C4) follow from a straightforward calculation.

Case 2: $n \equiv 0 \pmod 4$ *and* $mn/4$ *is of the form* $4^a(8b+7)$. Then $mn/2$ is not of the form $4^a(8b+7)$. By Theorem 7, $x^2 + y^2 + z^2 = mn/2$ has an integral solution (x,y,z). Since $mn/2 \equiv 0 \pmod 4$, x, y, z are all even. Let

$$y_1 = x, \qquad y_2 = y, \qquad y_3 = z,$$
$$y_4 = -x, \qquad y_5 = -y, \qquad y_6 = -z;$$

and for $m = 10$, set

$$y_7 = y_8 = y_9 = y_{10} = 0.$$

Then (C2)–(C4) clearly hold.

For example, when $m = 10$, $n = 24$, we could pick $(x,y,z) = (10,4,2)$, which leads to the solution $(10,4,2,-10,-4,-2,0,0,0,0)$.

Case 3: $n \equiv 1 \pmod 4$. Note that $(mn - m + 2)/2 \equiv 1 \pmod 4$, so it is not of the form $4^a(8b+7)$. By Theorem 7, $x^2 + y^2 + z^2 = (mn - m + 2)/2$ has an integral solution (x,y,z), two of them, say x and y, are even and the other one is odd. Let

$$y_1 = 1 + x, \qquad y_2 = -1 + y, \qquad y_3 = z,$$
$$y_4 = 1 - x, \qquad y_5 = -1 - y, \qquad y_6 = -z;$$

and for $m = 10$, set

$$y_7 = y_8 = 1 \quad \text{and} \quad y_9 = y_{10} = -1.$$

Then (C2)–(C4) hold.

An example: when $m = 10$, $n = 13$, selecting $(x, y, z) = (6, 4, 3)$ gives the solution $(7, 3, 3, -5, -5, -3, 1, 1, -1, -1)$.

We have shown that, in all cases, mK_n is cordial. □

Theorem 15. *For odd m satisfying $m \geqslant 5$, mK_n is cordial if and only if $n \not\equiv 4, 5 \pmod 8$.*

Proof. If mK_n is cordial, then $n \not\equiv 4, 5 \pmod 8$ by Lemma 13. Conversely, suppose $n \not\equiv 4, 5 \pmod 8$. By Corollary 10 and Theorem 12, we only need to prove that mK_n is cordial for $m \in \{5, 7, 9, 11\}$.

For $n < m$, the following are solutions to (C1)–(C4).

$m = 5$ and $n = 1$: $(1, 1, 1, -1, -1)$.
$m = 5$ and $n = 2$: $(2, -2, 0, 0, 0)$.
$m = 5$ and $n = 3$: $(3, 1, -1, -1, -1)$.
$m = 7$ and $n = 1$: $(1, 1, 1, 1, -1, -1, -1)$.
$m = 7$ and $n = 2$: $(2, 2, -2, -2, 0, 0, 0)$.
$m = 7$ and $n = 3$: $(3, 1, 1, 1, -3, -1, -1)$.
$m = 7$ and $n = 6$: $(4, 2, -4, -2, 0, 0, 0)$.
$m = 9$ and $n = 1$: $(1, 1, 1, 1, 1, -1, -1, -1, -1)$.
$m = 9$ and $n = 2$: $(2, 2, -2, -2, 0, 0, 0, 0, 0)$.
$m = 9$ and $n = 3$: $(3, 3, 1, -1, -1, -1, -1, -1, -1)$.
$m = 9$ and $n = 6$: $(6, -4, -2, 0, 0, 0, 0, 0, 0)$.
$m = 9$ and $n = 7$: $(5, 3, 1, 1, -5, -1, -1, -1, -1)$.
$m = 9$ and $n = 8$: $(6, -6, 0, 0, 0, 0, 0, 0, 0)$.
$m = 11$ and $n = 1$: $(1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1)$.
$m = 11$ and $n = 2$: $(2, 2, 2, -2, -2, -2, 0, 0, 0, 0, 0)$.
$m = 11$ and $n = 3$: $(3, 3, 1, 1, 1, -3, -1, -1, -1, -1, -1)$.
$m = 11$ and $n = 6$: $(6, 2, 2, -2, -2, -2, -2, -2, 0, 0, 0)$.
$m = 11$ and $n = 7$: $(7, 3, 1, -3, -1, -1, -1, -1, -1, -1, -1)$.
$m = 11$ and $n = 8$: $(8, -4, -2, -2, 0, 0, 0, 0, 0, 0, 0)$.
$m = 11$ and $n = 9$: $(5, 5, 3, 1, -5, -3, -1, -1, -1, -1, -1)$.
$m = 11$ and $n = 10$: $(8, 2, -6, -2, -2, 0, 0, 0, 0, 0, 0)$.

So we may assume $n \geqslant m$. All we need now is to find solutions to (C1)–(C4).

Let $n' = (n \bmod 2)$. Since $n - n' \not\equiv 4 \pmod 8$, exactly one of the three numbers $mn - mn' - 2$, $mn - mn'$, $mn - mn' + 2$ is of the form $8t$.

Case 1: *$2t + n'$ is not of the form $4^a(8b + 7)$ for any nonnegative integers a and b.* By Theorem 7, $x^2 + y^2 + z^2 = 2t + n'$ has an integral solution (x, y, z). Let

$$y_1 = x + y + z, \qquad y_2 = x - y - z,$$
$$y_3 = -x + y - z, \qquad y_4 = -x - y + z,$$

and

$$y_i = \begin{cases} n' & \text{if } 5 \leqslant i \leqslant (m+5)/2, \\ -n' & \text{if } (m+7)/2 \leqslant i \leqslant m. \end{cases}$$

Note that $y_i \equiv x^2 + y^2 + z^2 \equiv 2t + n' \equiv n' \equiv n \pmod 2$ for each i, $1 \leqslant i \leqslant 4$. Also, for each $i \geqslant 5$, $y_i \equiv n' \equiv n \pmod 2$. Then (C2) holds. Next, $\sum_{i=1}^{m} y_i = n' = 0$ or 1, thus (C3) holds. Finally, $\sum_{i=1}^{m} y_i^2 = 4(x^2+y^2+z^2)+(m-4)n' = 4(2t+n')+(m-4)n' = 8t + mn' = mn - 2$, mn, or $mn+2$; therefore (C4) also holds.

For example, when $m = 9$, $n = 15$, we have $t = 16$; by letting $(x,y,z) = (4,4,1)$, we obtain the solution $(9,-1,-1,-7,1,1,1,-1,-1)$. When $m = 9$, $n = 22$, we have $t = 25$, and $(x,y,z) = (5,4,3)$ gives rise to the solution $(12,-2,-4,-6,0,0,0,0,0)$.

Case 2: $2t + n'$ *is of the form* $4^a(8b+7)$ *for some nonnegative integers a and b.* Since $128b + 107 - 31n' \equiv 11 + n' \pmod{32}$, it is not of the form $4^{a'}(8b'+7)$ for any nonnegative integers a' and b'. By Theorem 7, $x^2 + y^2 + z^2 = 128b + 107 - 31n'$ has an integral solutions (x,y,z).

For $n' = 0$, x, y, z are all odd, for otherwise $x^2+y^2+z^2 \equiv 0$ or $4 \pmod 8$, whereas $128b + 107 - 31n' \equiv 3 \pmod 8$. We can further assume $x \equiv y \equiv z \equiv 1 \pmod 4$, for if necessary we could replace x or y or z by $-x$ or $-y$ or $-z$, respectively.

For $n' = 1$, we must have $x \equiv y \equiv z \equiv 2 \pmod 4$ in order for $128b + 107 - 31n' \equiv 12 \pmod{16}$. We can further assume $x \equiv y \equiv z - 4 \equiv 2 \pmod 8$, otherwise we could replace x or y or z by $-x$ or $-y$ or $-z$, respectively.

In either case,

$$x - 1 - n' \equiv y - 1 - n' \equiv z - 1 - n' - 4 \equiv 0 \pmod{4(1+n')}.$$

Let

$$\begin{aligned} y_1 &= 2^a(-x+y+z-1-n')/4, & y_2 &= 2^a(x-y+z-1-n')/4,\\ y_3 &= 2^a(x+y-z-1-n')/4, & y_4 &= 2^a(-x-y-z-1-n')/4,\\ y_5 &= 2^a(1+2n'); \end{aligned}$$

and for $m > 5$, set

$$y_i = \begin{cases} n' & \text{if } 6 \leqslant i \leqslant (m+5)/2,\\ -n' & \text{if } (m+7)/2 \leqslant i \leqslant m. \end{cases}$$

Note that $a \geqslant 1$ when $n' = 0$, and $a = 0$ when $n' = 1$, so $(4^a - 1)n' = 0$.

For the case of $n' = 0$, all $y_i \equiv 0 \equiv n' \equiv n \pmod 2$, so (C2) holds. Also, for the case of $n' = 1$, all $y_i \equiv 1 \equiv n' \equiv n \pmod 2$, so (C2) holds again. Next, $\sum_{i=1}^{m} y_i = 2^a(-1-n') + 2^a(1+2n') = 2^a n' = 0$ or 1, so we also have (C3). Finally,

$$\begin{aligned} \sum_{i=1}^{m} y_i^2 &= 4^a \frac{(x^2+y^2+z^2+1+3n')}{4} + 4^a(1+8n') + (m-5)n'\\ &= 4^a \frac{(128b+107-31n'+1+3n')}{4} + 4^a(1+8n') + (m-5)n'\\ &= 4^a(32b+28) + (4^a+m-5)n'\\ &= 4(2t+n) + (4^a+m-5)n'\\ &= 8t + mn' + (4^a-1)n'\\ &= 8t + mn' \end{aligned}$$

Since $8t + mn' \in \{mn-2, mn, mn+2\}$, (C4) holds.

Take $m = 11$, $n = 22$ as an illustration. We have $t = 30$, $a = 1$, and $b = 1$. We could take $(x,y,z) = (-15,-3,1)$, which leads to the solution $(6,-6,-10,8,2,0,0,0,0,0,0)$.

Consider $m = 9$, $n = 7$ as another example. We have $t = 7$, $a = 0$, and $b = 1$. Consequently, we could let $(x,y,z) = (10,10,-2)$. The solution thus obtained is $(-1,-1,5,-5,3,1,1,-1,-1)$. □

4. Conclusion

This paper completely determines the cordiality of mK_n as follows:

- K_n is cordial if and only if $n \in \{1,2,3\}$.
- $2K_n$ is cordial if and only if $n = k^2$ for some integer k.
- Define $b = b(n)$ as

$$b = \begin{cases} 3n/2 & \text{if } n \equiv 0 \pmod 8, \\ 3n/2+1 & \text{if } n \equiv 2 \pmod 8, \\ 18n-2 & \text{if } n \equiv 1 \pmod 8, \\ 18n+10 & \text{if } n \equiv 3 \pmod 8, \\ 18n-14 & \text{if } n \equiv 7 \pmod 8, \end{cases}$$

  and let $v_p(z)$ denote the highest power of the prime p that divides z. Then $3K_n$ is cordial if and only if $n \equiv 0,1,2,3,7 \pmod 8$ and $v_2(b)$ and $v_q(b)$ are even for all prime divisors q of b with $q \equiv 5 \pmod 6$.
- $4K_n$ is cordial if and only if n is not of the form $4^a(8b+7)$ for any nonnegative integers a and b.
- For $m \geqslant 5$, the following hold.
  - If $m \equiv 0 \pmod 4$, mK_n is cordial for all n.
  - If $m \equiv 2 \pmod 4$, mK_n is cordial if and only if $n \equiv 0$ or $1 \pmod 4$.
  - If m is odd, mK_n is cordial if and only if $n \not\equiv 4,5 \pmod 8$.

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