Regge closed string scattering and its implication on fixed angle closed string scattering

Jen-Chi Lee *, Yi Yang

Department of Electrophysics, National Chiao-Tung University and Physics Division, National Center for Theoretical Sciences, Hsinchu, Taiwan, ROC

A B S T R A C T

We calculate the complete closed string high energy scattering amplitudes (HSA) in the Regge regime for arbitrary mass levels. As an application, we deduce the complete ratios among closed string HSA in the fixed angle regime by using Stirling number identities. These results are in contrast with the incomplete set of closed string HSA in the fixed angle regime calculated previously. The complete forms of the fixed angle amplitudes, and hence the ratios, were not calculable previously without the input of zero-norm state calculation. This is mainly due to the lack of saddle point in the fixed angle closed string calculation.

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1. Introduction

Recently high-energy, fixed angle behavior of string scattering amplitudes [1–3] was intensively reinvestigated [4–12] for string states at arbitrary mass levels. The motivation was to uncover the long-sought hidden stringy spacetime symmetry. A saddle-point method was developed to calculate the general formula for tree-level high-energy open string scattering amplitudes of four arbitrary string states. Remarkably, it was found that there is only one independent component of the amplitudes at each fixed mass level, and ratios among high energy scattering amplitudes of different string states at each mass level can be obtained. However, it was soon realized that [13] the saddle-point method was applicable to (t, u) channel only but not (s, t) channel. It was also pointed out that, through the observation of the KLT formula [14], this difficulty is associated with the lack of saddle-point in the integration regime for the closed string calculation. To calculate the complete high energy closed string scattering amplitudes in the fixed angle regime [13], one had to rely on calculation based on the method of decoupling of zero-norm states [15–17] in the spectrum. With this new input, an infinite number of linear relations among high energy scattering amplitudes of different string states can be derived and the complete ratios among high energy closed string scattering amplitudes at each fixed mass level can be determined. One can now calculate only high energy amplitude corresponding to the highest spin state at each mass level in the spectrum, and the complete closed string scattering amplitudes can then be obtained.

In this Letter, we will use another method to calculate the closed string ratios in the fixed angle regime mentioned above. We will calculate the complete closed string scattering amplitudes in the Regge regime, which have not been considered in the literature so far. It turned out that both the saddle-point method and the method of decoupling of zero-norm states adopted in the calculation of fixed angle regime do not apply to the case of Regge regime. However a direct calculation is manageable. The calculation will be based on the KLT formula and the open string (s, t) channel scattering amplitudes in the Regge regime calculated previously [18]. By using a set of Stirling number identities developed in combinatoric number theory [19], one can then extract the ratios in the fixed angle regime from Regge closed string scattering amplitudes.

2. Fixed angle scattering

We begin with a brief review of high energy string scatterings in the fixed angle regime,

$$s, -t \to \infty, \qquad t/s \approx -\sin^2\frac{\theta}{2} = \text{fixed} \quad (\text{but } \theta \neq 0) \tag{1}$$

where s, t and u are the Mandelstam variables and θ is the CM scattering angle. It was shown [7,8] that for the 26D open bosonic string the only states that will survive the high-energy limit at mass level $M_2^2 = 2(n-1)$ are of the form

$$|n, 2m, q\rangle \equiv \left(\alpha_{-1}^T\right)^{n-2m-2q} \left(\alpha_{-1}^L\right)^{2m} \left(\alpha_{-2}^L\right)^q |0, k\rangle, \tag{2}$$

where the polarizations of the 2nd particle with momentum k_2 on the scattering plane were defined to be $e^P = \frac{1}{M_2}(E_2, \mathrm{k}_2, 0) = \frac{k_2}{M_2}$ as the momentum polarization, $e^L = \frac{1}{M_2}(\mathrm{k}_2, E_2, 0)$ the longitudinal polarization and $e^T = (0, 0, 1)$ the transverse polarization. Note that e^P approaches to e^L in the fixed angle regime. For simplicity, we choose k_1, k_3 and k_4 to be tachyons. It turned out that the (t, u) channel of the scattering amplitudes can be calculated by using the saddle-point method and the final results are [7,8,13]

$$\frac{A^{(n,2m,q)}(t, u)}{A^{(n,0,0)}(t, u)} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!! \tag{3}$$

with

$$A^{(n,0,0)}(t, u) \simeq \sqrt{\pi}(-1)^{n-1} 2^{-n} E^{-1-2n} \left(-2E^3 \sin\theta\right)^n \left(\sin\frac{\theta}{2}\right)^{-3} \left(\cos\frac{\theta}{2}\right)^{5-2n} \exp\left(-\frac{t\ln t + u\ln u - (t+u)\ln(t+u)}{2}\right). \tag{4}$$

To calculate the high energy, fixed angle closed string scattering amplitudes, one encountered the well-known difficulty of the lack of saddle-point in the integration regime. In fact, it was demonstrated [13] by three evidences that the standard saddle-point calculation for high energy closed string scattering amplitudes was not reliable. It was also pointed out [13] that this difficulty is associated with the lack of saddle-point in the integration regime for the calculation of (s, t) channel high energy open string scattering amplitudes. This can be seen from a formula by Kawai, Lewellen and Tye (KLT), which expresses the relation between tree amplitudes of closed and open string ($\alpha'_{\text{closed}} = 4\alpha'_{\text{open}} = 2$) [14]

$$A^{(4)}_{\text{closed}}(s, t, u) = \sin(\pi k_2 \cdot k_3) A^{(4)}_{\text{open}}(s, t) \bar{A}^{(4)}_{\text{open}}(t, u). \tag{5}$$

Note that Eq. (5) is valid for all energies. On the other hand, a direct calculation instead of the saddle point method was not successful either. This is mainly because the true leading order amplitudes for states with $m \neq 0$ drop from energy order E^{4m} to E^{2m} [4–6], and one needs to calculate the complicated subleading order contraction terms. For this reason, the complete forms of the fixed angle closed string and (s, t) channel open string scattering amplitudes were not calculable. However, a simple case of the (s, t) channel scattering amplitude, which is calculable for all energies, with k_2 the highest spin state $V_2 = \alpha_{-1}^{\mu_1} \alpha_{-1}^{\mu_2} \cdots \alpha_{-1}^{\mu_n} |0, k\rangle$ at mass level $M_2^2 = 2(n-1)$ and three tachyons $k_{1,3,4}$ is [6]

$$A_n^{\mu_1\mu_2\cdots\mu_n}(s, t) = \sum_{l=0}^{n} (-)^l \binom{n}{l} B\left(-\frac{s}{2} - 1 + l, -\frac{t}{2} - 1 + n - l\right) k_1^{(\mu_1} \cdots k_1^{\mu_{n-l}} k_3^{\mu_{n-l+1}} \cdots k_3^{\mu_n)}. \tag{6}$$

The high energy limit of Eq. (6) can then be calculated to be [13]

$$A^{(n,0,0)}(s, t) = (-)^n \frac{\sin(\pi u/2)}{\sin(\pi s/2)} A^{(n,0,0)}(t, u). \tag{7}$$

The factor $\frac{\sin(\pi u/2)}{\sin(\pi s/2)}$ which was missing in the literature [1,20] has important physical interpretations. The presence of poles give infinite number of resonances in the string spectrum and zeros give the coherence of string scatterings. These poles and zeros survive in the high energy limit and cannot be dropped out. Presumably, the factor triggers the failure of saddle point calculation mentioned above.

To calculate the complete high energy closed string scattering amplitudes, one had to rely on calculation based on the method of decoupling of zero-norm states, or stringy Ward identities, in the spectrum. With this new input, an infinite number of linear relations among high energy scattering amplitudes of different string states can be derived, and the complete ratios among high energy closed string scattering amplitudes at each fixed mass level can be shown to be the tensor product of two sets of (t, u) channel open string ratios in Eq. (3). The complete high energy closed string and (s, t) channel open string scattering amplitudes can then be obtained by Eqs. (5) and (7). An explicit calculation for the lowest mass level case was presented in [13]. Another independent method to obtain the closed string ratios is to calculate high energy string scattering amplitudes in the Regge regime, which we will discuss in the next section.

3. Regge scattering

Another high energy regime of string scattering amplitudes, which contains complementary information of the theory, is the fixed momentum transfer or Regge regime. That is in the kinematic regime

$$s \to \infty, \qquad \sqrt{-t} = \text{fixed} \quad (\text{but } \sqrt{-t} \neq \infty). \tag{8}$$

It was found [18] that the number of high energy scattering amplitudes for each fixed mass level in this regime is much more numerous than that of fixed angle regime calculated previously. On the other hand, it seems that both the saddle-point method and the method of decoupling of zero-norm states adopted in the calculation of fixed angle regime do not apply to the case of Regge regime. However the calculation is still manageable, and the general formula for the high energy (s, t) channel open string scattering amplitudes at each fixed mass level can be written down explicitly.

It was shown that the most general high energy open string states in the Regge regime at each fixed mass level $n = \sum_{n,m} lk_n + mq_m$ are

$$|k_l, q_m\rangle = \prod_{l>0} (\alpha_{-l}^T)^{k_l} \prod_{m>0} (\alpha_{-m}^L)^{q_m} |0, k\rangle. \tag{9}$$

For our purpose here, however, we will only calculate scattering amplitudes corresponding to the vertex in Eq. (2). The relevant kinematics are

$$e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \qquad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}; \tag{10}$$

$$e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \qquad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2}; \tag{11}$$

and

$$e^T \cdot k_1 = 0, \qquad e^T \cdot k_3 \simeq -\sqrt{-t}. \tag{12}$$

The Regge scattering amplitude for the (s, t) channel was calculated to be [18]

$$R^{(n,2m,q)}(s, t) = B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \sqrt{-t}^{\,n-2m-2q} \left(\frac{1}{2M_2}\right)^{2m+q} 2^{2m} (\tilde{t}')^q U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2}\right). \tag{13}$$

In Eq. (13) U is the Kummer function of the second kind and is defined to be

$$U(a, c, x) = \frac{\pi}{\sin \pi c} \left[\frac{M(a, c, x)}{(a - c)!(c - 1)!} - \frac{x^{1-c} M(a + 1 - c, 2 - c, x)}{(a - 1)!(1 - c)!}\right] \quad (c \neq 2, 3, 4, \ldots) \tag{14}$$

where $M(a, c, x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j} \frac{x^j}{j!}$ is the Kummer function of the first kind. Note that the second argument of Kummer function $c = \frac{t}{2} + 2 - 2m$, and is not a constant as in the usual case.

We now proceed to calculate the Regge (t, u) channel scattering amplitude. The high energy limit of the amplitude can be written as

$$R^{(n,2m,q)}(t, u) = \int_1^{\infty} dx\, x^{k_1 \cdot k_2} (1 - x)^{k_2 \cdot k_3} \left[\frac{e^T \cdot k_3}{1 - x}\right]^{n-2m-2q} \left[\frac{e^L \cdot k_1}{-x} + \frac{e^L \cdot k_3}{1 - x}\right]^{2m} \left[\frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1 - x)^2}\right]^q$$

$$\simeq (\sqrt{-t})^{n-2m-2q} \left(\frac{\tilde{t}'}{2M_2}\right)^{2m+q} \sum_{j=0}^{2m} \binom{2m}{j} (-)^j \left(\frac{s}{\tilde{t}'}\right)^j \int_1^{\infty} dx\, x^{k_1 \cdot k_2 - j} (1 - x)^{k_2 \cdot k_3 + j - n}. \tag{15}$$

We can make a change of variable $y = \frac{x-1}{x}$ to transform the integral of Eq. (15) to

$$R^{(n,2m,q)}(t, u) = (\sqrt{-t})^{n-2m-2q} \left(\frac{\tilde{t}'}{2M_2}\right)^{2m+q} (-)^{k_2 \cdot k_3 - n} \sum_{j=0}^{2m} \binom{2m}{j} \left(\frac{s}{\tilde{t}'}\right)^j \int_0^1 dy\, y^{k_2 \cdot k_3 + j - n} (1 - y)^{n - k_1 \cdot k_2 - k_2 \cdot k_3 - 2}$$

$$= (\sqrt{-t})^{n-2m-2q} \left(\frac{\tilde{t}'}{2M_2}\right)^{2m+q} (-)^{k_2 \cdot k_3 - n} \sum_{j=0}^{2m} \binom{2m}{j} \left(\frac{s}{\tilde{t}'}\right)^j B(k_2 \cdot k_3 + j - n + 1, n - k_1 \cdot k_2 - k_2 \cdot k_3 - 1). \tag{16}$$

In the Regge limit, the beta function can be approximated by

$$B(k_2 \cdot k_3 + j - n + 1, n - k_1 \cdot k_2 - k_2 \cdot k_3 - 1) = B\left(-1 - \frac{t}{2} + j, -1 - \frac{u}{2}\right)$$

$$\simeq B\left(-1 - \frac{t}{2}, -1 - \frac{u}{2}\right) \left(-1 - \frac{t}{2}\right)_j \left(\frac{s}{2}\right)^{-j} \tag{17}$$

where $(a)_j = a(a + 1)(a + 2) \cdots (a + j - 1)$ is the Pochhammer symbol. In the above calculation, we have used $s + t + u = 2n - 8$. Finally, the (t, u) channel amplitude can be written as

$$R^{(n,2m,q)}(t, u) = (-)^{k_2 \cdot k_3 - n} B\left(-1 - \frac{t}{2}, -1 - \frac{u}{2}\right) (\sqrt{-t})^{n-2m-2q} \left(\frac{\tilde{t}'}{2M_2}\right)^{2m+q} 2^{2m} (\tilde{t}')^q U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2}\right). \tag{18}$$

We can now explicitly write down the general formula for high energy closed string scattering amplitude corresponding to the closed string state

$$|n; 2m, 2m'; q, q'\rangle \equiv (\alpha_{-1}^T)^{\frac{n}{2} - 2m - 2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q \otimes (\tilde{\alpha}_{-1}^T)^{\frac{n}{2} - 2m' - 2q'} (\tilde{\alpha}_{-1}^L)^{2m'} (\tilde{\alpha}_{-2}^L)^{q'} |0, k\rangle. \tag{19}$$

By using Eqs. (5), (13) and (18), the amplitude is

$$R_{\text{closed}}^{(n;2m,2m';q,q')}(s,t,u) = (-)^{k_2 \cdot k_3 - n} \sin(\pi k_2 \cdot k_3) B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) B\left(-1 - \frac{t}{2}, -1 - \frac{u}{2}\right)$$

$$\times (\sqrt{-t})^{n-2(m+m')-2(q+q')} \left(\frac{\tilde{t}'}{2M_2}\right)^{2(m+m')+q+q'} 2^{(2m+m')}(\tilde{t}')^{q+q'}$$

$$\times U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2}\right) U\left(-2m', \frac{t}{2} + 2 - 2m', \frac{\tilde{t}'}{2}\right). \tag{20}$$

The Regge scattering amplitudes at each fixed mass level are no longer proportional to each other. The ratios are t dependent functions and can be calculated to be

$$\frac{R^{(n,2m,q)}(s,t)}{R^{(n,0,0)}(s,t)} = (-1)^m \left(-\frac{1}{2M_2}\right)^{2m+q} (\tilde{t}' - 2N)^{-m-q}(\tilde{t}')^{2m+q} \sum_{j=0}^{2m} (-2m)_j \left(-1 + n - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} + O\left\{\left(\frac{1}{t}\right)^{m+1}\right\}. \tag{21}$$

An interesting observation [18] is that the coefficients of the leading power of $\tilde{t}'$ in Eq. (21) can be identified with the ratios in Eqs. (3). To ensure this identification, we need the following identity

$$\sum_{j=0}^{2m} (-2m)_j \left(-1 + n - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} = 0(-\tilde{t}')^0 + 0(-\tilde{t}')^{-1} + \cdots + 0(-\tilde{t}')^{-m+1} + \frac{(2m)!}{m!}(-\tilde{t}')^{-m} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}. \tag{22}$$

Note that n effects only the sub-leading terms in $O\{(\frac{1}{t'})^{m+1}\}$. Eq. (21) was exactly proved [18] for $n = 0, 1$ by using Stirling number identities developed in combinatoric number theory [19]. For general integer n case, only the identity corresponding to the term $\frac{(2m)!}{m!}(-\tilde{t}')^{-m}$ was rigorously proved [21] but not other "0 identities". We conjecture that Eq. (22) is valid for any *real* number n. We have numerically shown the validity of Eq. (22) for the value of m up to $m = 10$. Here we give only results of $m = 3$ and 4

$$\sum_{j=0}^{6} (-2m)_j \left(-1 + n - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} = \frac{120}{(-\tilde{t}')^3} + \frac{720a^2 + 2640a + 2080}{(-\tilde{t}')^4} + \frac{480a^4 + 4160a^3 + 12000a^2 + 12928a + 3840}{(-\tilde{t}')^5}$$

$$+ \frac{64a^6 + 960a^5 + 5440a^4 + 14400a^3 + 17536a^2 + 7680a}{(-\tilde{t}')^6}, \tag{23}$$

$$\sum_{j=0}^{8} (-2m)_j \left(-1 + n - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!}$$

$$= \frac{1680}{(-\tilde{t}')^4} + \frac{13440a^2 + 67200a + 76160}{(-\tilde{t}')^5}$$

$$+ \frac{13440a^4 + 152320a^3 + 595840a^2 + 930048a + 467712}{(-\tilde{t}')^6}$$

$$+ \frac{3584a^6 + 68096a^5 + 501760a^4 + 1802752a^3 + 3236352a^2 + 2608128a + 645120}{(-\tilde{t}')^7}$$

$$+ \frac{256a^8 + 7168a^7 + 82432a^6 + 501760a^5 + 1732864a^4 + 3361792a^3 + 3345408a^2 + 1290240a}{(-\tilde{t}')^8} \tag{24}$$

where $a = -1 + n$. We can see that a shows up only in the sub-leading order terms as expected. From the form of Eq. (20), we conclude that the high energy closed string ratios in the fixed angle regime can be extracted from Kummer functions and are calculated to be

$$\frac{A_{\text{closed}}^{(n;2m,2m';q,q')}(s,t,u)}{A_{\text{closed}}^{(n;0,0;0,0)}(s,t,u)} = \left(-\frac{1}{M_2}\right)^{2(m+m')+q+q'} \left(\frac{1}{2}\right)^{q+q'} \lim_{t \to \infty} (-t)^{-m-m'} U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{t}{2}\right) U\left(-2m', \frac{t}{2} + 2 - 2m', \frac{t}{2}\right)$$

$$= \left(-\frac{1}{M_2}\right)^{2(m+m')+q+q'} \left(\frac{1}{2}\right)^{m+m'+q+q'} (2m - 1)!!(2m' - 1)!!. \tag{25}$$

This is an alternative method to calculate the high energy closed string ratios other than the method of decoupling of zero norm state adopted previously. In addition to rederiving the ratios calculated previously, one can express the ratios in terms of Kummer functions through the Regge calculation presented in this Letter. This may turn out to be important for the understanding of algebraic structure of stringy symmetry.

In conclusion, a direct calculation of general formula for high energy closed string scattering amplitudes is doable in the Regge regime and is calculated in Eq. (20), but not in the fixed angle regime. The ratios among high energy closed string scattering amplitudes for each fixed mass level in the fixed angle regime, which were calculated previously by the method of decoupling of zero norm states, can be alternatively deduced from general formula of high energy closed string scattering amplitudes in the Regge regime. The result that the ratios can be expressed in terms of Kummer functions in the Regge calculation presented in this Letter may help to understand the algebraic structure of stringy symmetry.

Acknowledgements

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