Quantile Mean: Statistical Inferences and Applications

SUMMARY

The quantile mean being the average of a pair of symmetric type quantiles, $F^{-1}(\alpha)$ and $F^{-1}(1-\alpha)$, is a robust type location parameter playing an alternative to the population mean. We extend this quantile mean to a pair of quantiles where this correponding quantile interval is the one with smallest width among all choices of $1-2\alpha$ quantile intervals. Parametric statistical inferences and nonparametric estimation techniques are all addressed. Moreover, an extension of the quantile mean to a new general L-estimation has also been provided.

1. Introduction

The most popular technique for estimating a location parameter is the least squares estimator. Its popularity mainly reflects its advantages in the theoretical property from the parametric point of view that it is uniformly minimum variance unbiased estimator when the variable follows a normal distribution. However, the least squares estimator is sensitive to departures from normality and to the presence of outliers. Hence, we need to consider robust estimators.

Among the hundreds or more robust estimators for location parameter investigated in the last three decades, the L-estimators, defined in terms of ordinary quantiles, have been an important class (see Hogg (1974) and Huber (1981)). The benefits of using an estimator that is based on quantiles include its easiness in computation and asymptotic efficiency shown in the literature (see Hogg (1974), Jureckova and Sen (1996), and Chen and Chiang (1996)).

Let F^{-1} be the population quantile function. The class of of ordinary L-estimators is the sample version of the following quantile means as

$$\int_0^1 \delta(\alpha)F^{-1}(\alpha)d\alpha \tag{1.1}$$

for some nonnegative function δ which provides a rich class of quantities very popular and interesting in applications and the theoretical study for measuring the center of the underlying distribution. There are many versions of quantile means in terms of function δ and chosen percentages α. However, they are all constructed in a symmetric way in the sense that δ is symmetric at 0.5 and both $F^{-1}(1-\alpha)$ and $F^{-1}(\alpha)$ correspond to the same value of δ. Let's consider the simplest quantile mean defined as

$$\frac{1}{2}[F^{-1}(\alpha)+F^{-1}(1-\alpha)]. \tag{1.2}$$

Our concern is why we should choose the symmetric ones, α and $1-\alpha$, for use. Among all choices of defining a location quantity as an average of two quantiles with coverage probability $1-2\alpha$ of the form

$$cF^{-1}(\alpha_1)+dF^{-1}(1-2\alpha+\alpha_1), 0 \le \alpha_1 \le 2\alpha, c, d \in R. \tag{1.3}$$

Two criteria may be appropriate settings to determine a quantile mean from the class in (1.2). First, as a location paramter, we may expect that it satisfies several desirable equivariant properties. It is known (see Staudte and Sheather (1990)) that the symmetric quantile mean in (1.2) does fulfill this condition. In statistical inferences of constructing a random interval, for example, the confidence interval or the tolerance interval, we often propose the one with shortest width for applications when the confidence coefficient is fixed. In fact, a quantile mean uses two quantiles that represent the two ends of a population interval which covers the random variable with a fixed coverage probability. Then the second criterion for a quantile mean may be set to have width of its corresponding quantile interval as small as possible.

When the distribution of the random varaible is symmetric and has a single mode, the symmetric quantile interval minimizes the width among all available quantile intervals with the same coverage probability. There is a problem raised from this fact. Can we propose a new quantile mean that its corresponding quantile interval does achieve the minimum width. If yes, this quantile interval then coincides with the symmetric quantile interval when the underlying distribution is symmetric. Actually we expect that this proposed quantile mean gains some statistical efficiency such as smaller (asymptotic) variance.

2. Mode Type Quantile Mean

Before defining a more general quantile mean, we consider a set of desired equivariant conditions for the location parameter. The following condition expected for the location parameter to fulfill may be seen in Staudte and Sheather (1990).

Definition 2.1. We say that μ, a real function of r.v. X, is a measure of location if it satisfies
(a). $\mu(X+b)=\mu(X)+b$ for $b\in R$;
(b). $\mu(ax)=a\mu(x)$ for $a>0$;
(c). $\mu(-X)=-\mu(X)$;
(d). if $X\geq 0$, then $\mu(X)\geq 0$.

Intuitively, any member in the following family of quantile means

$$\{cF^{-1}(\gamma+\alpha)+dF^{-1}(\alpha):0<\alpha<1-\gamma\}$$

may serve a γ quantile mean for the distribution. However, not every one in the family satisfies the preceding condition for a measure of location and we know that the symmetric type quantile mean, $\tau=\frac{1}{2}[F^{-1}(\alpha)+F^{-1}(1-\alpha)]$, $0<\alpha<0.5$, is a measure of location. We are interested in a measure of location that is a quantile mean of the following form

$$\mu=cF^{-1}(\alpha^*)+dF^{-1}(\gamma+\alpha^*), \tag{2.1}$$

where

$$\alpha^*=\arg_\alpha\inf_{0\leq\alpha<1-\gamma}\{F^{-1}(\alpha+\gamma)-F^{-1}(\alpha)\}, \tag{2.2}$$

with $c,d\in R$. The setting in (2.2) gurantees that $(F^{-1}(\alpha^*),F^{-1}(\gamma+\alpha^*))$ is the one minimizes the width among all intervals $(F^{-1}(\alpha),F^{-1}(\gamma+\alpha)),0\leq\alpha\leq 1-\gamma$. The following theorem provides the condition that the quantile mean in (2.1) is a measure of location.

Theorem 2.1. For given $c,d\in R$, μ in (2.1) is a measure of location if $c=d=\frac{1}{2}$.
Proof. Let's redenote $\mu=\mu(X,\gamma),F_x^{-1}(\alpha)=F^{-1}(\alpha)$, and $\alpha^*=\alpha^*(X)$. We know that the quantile function F^{-1} satisfies $F_{X+b}^{-1}(\alpha)=F_X^{-1}(\alpha)+b$ for $b\in R$ and $F_{aX}^{-1}(\alpha)=aF_X^{-1}(\alpha)$ if $a>0$ and $aF_X^{-1}(1-\alpha)$ if $a\leq 0$.
(a). Let $b\in R$. It is easy to see that $\alpha^*(X+b)=\alpha^*(X)$. Then

$$\begin{aligned}\mu(X+b,\gamma)&=cF_{X+b}^{-1}(\alpha^*(X+b))+dF_{X+b}^{-1}(\gamma+\alpha^*(X+b))\\&=cF_{X+b}^{-1}(\alpha^*(X))+dF_{X+b}^{-1}(\gamma+\alpha^*(X))+(c+d)b\\&=cF_X^{-1}(\alpha^*(X))+dF_X^{-1}(\gamma+\alpha^*(X))+(c+d)b.\\&=\mu(X,\gamma)+(c+d)b.\end{aligned}$$

Therefore, $c + d = 1$.
(b). Let $a > 0$. We also see that $\alpha^*(aX) = \arg_\alpha \inf_{0\le\alpha\le 1-\gamma}\{a[F_X^{-1}(\gamma+\alpha) - F_X^{-1}(\alpha)]\} = \alpha^*(X)$. Then we have

$$\begin{aligned}
\mu(aX,\gamma) &= cF_{aX}^{-1}(\alpha^*(aX)) + (1-c)F_{aX}^{-1}(\gamma + \alpha^*(aX)) \\
&= cF_{aX}^{-1}(\alpha^*(X)) + (1-c)F_{aX}^{-1}(\gamma + \alpha^*(X)) \\
&= a\{cF_X^{-1}(\alpha^*(X)) + (1-c)F_X^{-1}(\gamma + \alpha^*(X))\} \\
&= a\mu(X,\gamma).
\end{aligned}$$

(c). Consider the transformation of multiplying X by negative value -1. We see that

$$\begin{aligned}
\alpha^*(-X) &= \arg_\alpha \inf_{0\le\alpha\le 1-\gamma}\{F_{-X}^{-1}(\gamma+\alpha) - F_{-X}^{-1}(\alpha)\} \\
&= \arg_\alpha \inf_{0\le\alpha\le 1-\gamma}\{-F_X^{-1}(1-(\gamma+\alpha)) + F_X^{-1}(1-\alpha)\} \\
&= \arg_\alpha \inf_{0\le\alpha\le 1-\gamma}\{F_X^{-1}(1-\alpha) - F_X^{-1}(1-(\gamma+\alpha))\} \\
&= \arg_\alpha \inf_{0\le 1-(\alpha+\gamma)\le 1-\gamma}\{F_X^{-1}(\gamma+(1-(\alpha+\gamma))) - F_X^{-1}(1-(\gamma+\alpha))\} \\
&= \arg_\alpha \inf_{0\le\beta\le 1-\gamma}\{F_X^{-1}(\gamma+\beta) - F_X^{-1}(\beta)\}.
\end{aligned}$$

This implies that $1 - [\gamma + \alpha^*(-X)] = \alpha^*(X)$ and the we derive $\alpha^*(-X) = 1 - [\gamma + \alpha^*(X)]$. Now we have

$$\begin{aligned}
\mu(-X,\gamma) &= cF_{-X}^{-1}(\alpha^*(-X)) + (1-c)F_{-X}^{-1}(\gamma + \alpha^*(-X)) \\
&= cF_{-X}^{-1}(1-(\gamma+\alpha^*(X))) + (1-c)F_{-X}^{-1}(1-\alpha^*(X)) \\
&= -[cF_X^{-1}(\gamma+\alpha^*(X)) + (1-c)F_X^{-1}(\alpha^*(X))].
\end{aligned}$$

Since $-\mu(X,\gamma) = -[cF_X^{-1}(\alpha^*(X)) + (1-c)F_X^{-1}(\gamma+\alpha^*(X))]$, we know that $\mu(-X,\gamma) = -\mu(X,\gamma)$ if $c = \frac{1}{2}$. That is $\mu(X,\gamma) = \frac{1}{2}\{F_X^{-1}(\alpha^*(X)) + F_X^{-1}(\gamma+\alpha^*(X))\}$.
(d). We also see that $\mu(X,\gamma) = \frac{1}{2}[F_X^{-1}(\alpha^*(X)) + F_X^{-1}(\gamma+\alpha^*(X))] \ge 0$ if $x \ge 0$. $\square$

We note that the interval $(F^{-1}(\alpha^*), F^{-1}(\gamma+\alpha^*))$ converges to the single mode point when γ converges to zero. We combine this fact and the result in the preceding theorem to define a general quantile mean that its corresponding quatile interval plays a generalization of the mode point to the interval.

Definition 2.2. For $0 < \gamma < 1$, we define the γ-mode type quantile mean as $\tau_{mod} = \frac{1}{2}[F^{-1}(\alpha^*) + F^{-1}(\gamma+\alpha^*)]$.

On the other hand, the symmetric quantile interval $(F^{-1}(\alpha), F^{-1}(1-\alpha))$ shrink to the median point when percentage α approaches to 0.5. We then recall the symmetric quantile mean τ the median type quantile mean by denoting $\tau_{med} = 0.5[F^{-1}(\alpha) + F^{-1}(1-\alpha)]$.

In the following, we provide the mode type quantile mean for several distributions.

(a). Let X be a random variable having a p.d.f. ∼of the form $f(x;\theta) = exp[-(x-\theta)]I(\theta < x < \infty)$ with parameter space R. We see that $F^{-1}(\alpha) = \theta - ln(1-\alpha)$ and for any γ, α^* equals to 0. Then $\tau_{mod} = \theta - \frac{1}{2}ln(1-\gamma)$ which depends on unkwown parameter θ. Not every distribution has an explicit form for α^* in terms of parameter θ.

(b). Let X be the r.v.∼ with the Weibull distribution having the p.d.f.∼ $f(x, \theta_1, \theta_2) = \theta_1\theta_2 x^{\theta_2-1}exp(-\theta_1 x^{\theta_2})$, $x > 0$ and $\theta_1, \theta_2 > 0$. We can see that $F^{-1}(\alpha) = \theta_1^{-1/\theta 2}[-ln(1-\alpha)]^{\frac{1}{\theta_2}}$ and $\tau_{mod} = \frac{1}{2}\theta_1^{-1/\theta_2}\{[-ln(1-\gamma-\alpha^*)]^{\frac{1}{\theta_2}} + [-ln(1-\alpha^*)]^{\frac{1}{\theta_2}}\}$, where $\alpha^* = \arg_\alpha$

$\inf_{0\leq\alpha\leq 1-\gamma}\{[-ln(1-(\gamma+\alpha))]^{1/\theta_2} - [-ln(1-\alpha)]^{1/\theta_2}\}$. In this example, α^* is implicitly formulated in terms of θ.

(c). Consider an example of a discrete distribution. Let X be a discrete r.v.∼ with p.m.f.∼ $f(x) = \frac{3-|x-3|}{9}$, x=1, 2, 3, 4, 5. By denoting the corresponding mode type interval as $C_{mod}(\gamma)$, we see that there are multiple choices of γ and α^* that have the same mode type quantile interval and then the same mode quantile mean. This would happen in the same way for the median quantile mean. We display the $C_{mod}(\gamma)$ and τ_{mod} and their corresponding γ and α^* in the following table.

Table 1 Mode Type quantile interval and mean for a discrete distribution

$C_{mod}(\gamma)$	γ	α^*	τ_{mod}
$\{1\}$	$(0, \frac{1}{9})$	$(0, \frac{1}{9} - \gamma]$	1
$\{2\}$	$(0, \frac{2}{9})$	$(\frac{1}{9}, \frac{3}{9} - \gamma]$	2
$\{3\}$	$(0, \frac{3}{9})$	$(\frac{3}{9}, \frac{6}{9} - \gamma]$	3
$\{4\}$	$(0, \frac{2}{9})$	$(\frac{6}{9}, \frac{8}{9} - \gamma]$	4
$\{5\}$	$(0, \frac{1}{9})$	$(\frac{8}{9}, 1 - \gamma]$	5
$\{2,3\}$	$[\frac{3}{9}, \frac{5}{9})$	$(\frac{1}{9}, \frac{6}{9} - \gamma]$	2.5
$\{3,4\}$	$[\frac{3}{9}, \frac{5}{9})$	$(\frac{3}{9}, \frac{8}{9} - \gamma]$	3.5
$\{1,3\}$	$[\frac{5}{9}, \frac{6}{9})$	$(0, \frac{6}{9} - \gamma]$	2
$\{2,4\}$	$[\frac{5}{9}, \frac{7}{9})$	$(\frac{1}{9}, \frac{8}{9} - \gamma]$	3
$\{3,5\}$	$[\frac{5}{9}, \frac{6}{9})$	$(\frac{3}{9}, 1 - \gamma]$	4
$\{1,4\}$	$[\frac{7}{9}, \frac{8}{9})$	$(0, \frac{8}{9} - \gamma]$	2.5
$\{2,5\}$	$[\frac{7}{9}, \frac{8}{9})$	$(\frac{1}{9}, 1 - \gamma]$	3.5
$\{1,5\}$	$[\frac{8}{9}, 1)$	$(0, 1 - \gamma]$	3

For interpretation, for γ in each category and setting α^* being any value in its corresponding interval, they correspond to the same mode quantile interval and quantile mean. For example, let $0 < \gamma < \frac{1}{9}$. There are five choices of $C_{mod}(\gamma)$, any one of $\{i\}, i = 1, ..., 5$. If we choose $C_{mod}(\gamma) = \{3\}$, there are multiple choices of α^* which must be in $(\frac{3}{9}, \frac{6}{9} - \gamma]$. The resulted mode quantile mean is $\tau_{mod} = 3$.

Why should we use the mode type quantiles to construct the quantile mean? Here we interprete one point from the view of quality control. A control chart consider a statistic T, a function of a random sample $X_1, ..., X_n$, and set the control limits as the two ends of the median interval $(F_T^{-1}(\alpha), F_T^{-1}(1-\alpha))$ for some coverage probability $1 - 2\alpha$. In practice, the control limits are replaced by the one with two quantiles replaced by their sample versions and when a new observation of T falls outside the limits, the manaufacturing process may be considered in an out-of-control situation. The mode interval suggests using the two ends of the interval $(F_T^{-1}(\alpha^*), F_T^{-1}(1 - 2\alpha + \alpha^*))$ as the control limits. These two control charts are with the same coverage probability so that the chance making the error of concluding out of control when the process is in control. It is then interesting to see which one has higher chance of concluding out of control when the process is indeed out of control. Let the random sample $X_1, ..., X_n$ be drawn from the expontial distribution with p.d.f. $f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0$. Consider the statistic $T = \sum_{i=1}^{n} X_i$. we may see that the median interval is $C_{med}(\theta) = \frac{\theta}{2}(F_Y^{-1}(\alpha), F_Y^{-1}(1-\alpha))$ and the mode interval is $C_{mod}(\theta) = \frac{\theta}{2}(F_Y^{-1}(\alpha^*), F_Y^{-1}(1 - 2\alpha + \alpha^*))$ where $Y = \frac{2\sum_{i=1}^{n} X_i}{\theta} \sim \chi^2(2n)$. Assume that $\theta = 1.5$

is the case where the process is in control. We compute the following probabilities indicating the abilities of observing out of control when it does be out of control:

$$\begin{aligned}\pi_{mod}(\theta) &= P_\theta(\sum_{i=1}^{n} X_i \notin C_{mod}(1.5)) \\ &= 1 - P_\theta(\sum_{i=1}^{n} X_i \in C_{mod}(1.5)) \\ &= 1 - P_\theta(\frac{1.5}{2} F_Y^{-1}(\alpha^*) \le \sum_{i=1}^{n} X_i \le \frac{1.5}{2} F_Y^{-1}(1-2\alpha+\alpha^*)) \\ &= 1 - P_\theta(\frac{1.5}{\theta} F_Y^{-1}(\alpha^*) \le \frac{2\sum_{i=1}^{n} X_i}{\theta} \le \frac{1.5}{\theta} F_Y^{-1}(1-2\alpha+\alpha^*)) \\ &= 1 - P(\frac{1.5}{\theta} F_Y^{-1}(\alpha^*) \le Y \le \frac{1}{\theta} F_Y^{-1}(1-2\alpha+\alpha^*)), \\ \pi_{med}(\theta) &= 1 - P(\frac{1.5}{\theta} F_Y^{-1}(\alpha) \le Y \le \frac{1.5}{\theta} F_Y^{-1}(1-\alpha)).\end{aligned}$$

The following table provides the results of powers when $n = 2$ and some θ's.

Table 2 Powers control charts based on median and mode intervals

θ	π_{med}	π_{mod}	π_{med}	π_{mod}
	$2\alpha = 0.05$		$2\alpha = 0.01$	
2	0.095	0.130	0.030	0.043
3	0.243	0.315	0.125	0.162
4	0.390	0.469	0.247	0.297
5	0.508	0.584	0.363	0.416
6	0.599	0.667	0.461	0.514
7	0.669	0.729	0.542	0.591
8	0.723	0.776	0.607	0.653
9	0.765	0.812	0.661	0.703
10	0.799	0.840	0.705	0.743

As long as we fixed the probability of the type I error for two quantile intervals, we found that the mode interval generates the control chart with a smaller probability of the type II error (i.e., a larger power when the underlying distribution has been shifted the the right for $\theta > 1$). With more chance in detecting a process change can keep the manufacturing process in control condition. This application provides one evidence in using the mode type interval. We will provide the evidence in its use in the quantile mean.

3. Estimation for Mode Type Quantile Mean

It is assumed that the distribution function F is known which has a p.d.f. $\sim f(x;\theta)$, where parameter θ is unknown. In case that τ_{mod} is a function of the unknown parameter, the criterion established in the statistical inference leads us to develop the m.l.e.$\sim$ or uniformly minimum variance unbiased estimator (UMVUE). Is there any interesting family of distributions that makes the mode type quantile mean easily to display? We show that the continuous type location-scale distributions lead to a simpler form for representing the quantile means which helps us for making statistical inferences.

Theorem 3.1. The family of continuous location-scale distributions with p.d.f.$\sim$ of the form $f(x;\theta_1,\theta_2) = \frac{1}{\theta_2} f_0(\frac{x-\theta_1}{\theta_2})$, $\theta_1 \in R$ and $\theta_2 > 0$, has

$$\tau_{mod} = \theta_1 + \theta_2 \tau_{m0},$$

where τ_{m0} is the mode quantile mean of the distribution with p.d.f. $\sim f_0$.
Proof. It is seen that $F^{-1}(\alpha) = \theta_1 + \theta_2 F_0^{-1}(\alpha)$. Then

$$\begin{aligned}\tau_{mod} &= \theta_1 + \frac{\theta_2}{2}\{F_0^{-1}(\gamma+\alpha^*) + F_0^{-1}(\alpha^*)\} \\ &= \theta_1 + \theta_2 \tau_m(\frac{X-\theta_1}{\theta_2}, \gamma) \\ &= \theta_1 + \theta_2 \tau_{m0}.\end{aligned}$$

Suppose now that we have a random sample $X_1, X_2, \cdots, X_n$ drawn from a distribution with p.d.f. $\sim f(x,\theta)$. Our aim is to estimate the mode quantile mean in (2.1). The mode quantile mean for a location-scale distribution is nothing related to its corresponding location parameter. Consider the mode quantile mean and examples for several distributions.

Normal distribution

Consider the case where the random sample is drawn from the normal distribution $N(\mu,\sigma^2)$ with unknown parameters $\mu \in R$ and $\sigma > 0$. We also denote the r.v.$\sim$ with standard normal distribution $N(0,1)$ by Z. Then we can easily see that $F^{-1}(\alpha) = \mu + \sigma z_{1-\alpha}$ with $1-\alpha = P(Z \geq z_{1-\alpha})$. And since the p.d.f.$\sim$ is symmetric, we have $\alpha^* = \frac{1-\gamma}{2}$ and $\tau_{mod} = \mu$, the population mean.

Example 1

Consider a mechanical engineer who was on the staff of a physical therapy research team made a sampling study to evaluate his new design for an exerciser. The device is intended to strengthen the muscles in persons suffering from chronic lower back pain. The availability of only one test device limited the number of the test subjects that could be accommodated. A random sample of 12 patients is in Lawrence L. Lapin (1997, p.306). The following recovery times (days) were obtained : 15, 23, 32, 18, 16, 22, 41, 29, 25, 27, 30, 18. The mean recovery time is $\bar{X} = \hat{\mu} = 24.7$ days for any γ.

Exponential distribution

Consider the random sample that is drawn from a right skewed exponential distribution with p.d.f.

$$f(x) = \frac{1}{\theta} e^{-\frac{x-k}{\theta}} I(k < x < \infty),$$

where k is a known constant and θ is an unknown parameter. Since the p.d.f.∼ is strictly monotone decreasing which implies $\alpha^* = 0$ with quantile function $F^{-1}(\alpha) = k - \theta ln(1-\alpha)$, we have the mode quantile mean

$$\begin{aligned} \mu_0(X) &= \frac{1}{2}\{F^{-1}(\gamma) + F^{-1}(0)\} \\ &= k - \frac{\theta}{2} ln(1-\gamma). \end{aligned}$$

For this right skewed exponential distribution, with the fact that $E(X) = k + \theta$, the UMVUE of the mode quantile mean is $\hat{\tau}_{mod} = k + \frac{1}{2}(k - \bar{X}) ln(1-\gamma)$.

For comparison, with the fact that $F^{-1}(\alpha) = k - \theta ln(1-\alpha)$, a $1-2\alpha$ median quantile mean is $\tau_{med} = \frac{1}{2}[F^{-1}(\alpha) + F^{-1}(1-\alpha)] = k - \frac{\theta}{2}[ln(\alpha) + ln(1-\alpha)]$. Its UMVUE is $\hat{\tau}_{med} = k - \frac{\bar{X}}{2}[ln(\alpha) + ln(1-\alpha)]$.

Example 2. Consider the 1980 revenues of states of sample size 50 in the United States in tens of billions of US dollars. The data set is in Siegel (1988, p.∼46) and its stem-and-leaf plot shows that the underlying distribution is most likely to be the right skewed exponential distribution with $k = 0$. We have $\bar{x} = 0.396$ and here we list the estimates of the γ median and mode type quantile means in the following table.

Table 3 Estimates for $\gamma = 1 - 2\alpha$ median and mode type quantile means for revenue data

γ	$\hat{\tau}_{med}(\gamma)$	$\hat{\tau}_{mod}(\gamma)$
0.5	0.33	0.14
0.6	0.36	0.18
0.7	0.41	0.24
0.8	0.48	0.32
0.9	0.60	0.46
0.9973	1.31	1.17

We have a conclusion drawn from the preceding table: The estimates of the mode quantile mean are significantly smaller than their corresponding estimates of the median quantile mean.

On the other hand, one exponential distribution highly skewed to the left has p.d.f. of the form

$$f(x) = \frac{1}{\theta} e^{\frac{x-k}{\theta}} I(-\infty < x < k). \tag{2.3}$$

We have $F^{-1}(\alpha) = k + \theta ln(\alpha)$ and $\alpha^* = 1 - \gamma$. Then

$$\begin{aligned} \tau_{mod} &= \frac{1}{2}[F^{-1}(1) + F^{-1}(1-\gamma)] \\ &= k + \frac{\theta}{2} ln(1-\gamma). \end{aligned}$$

Its UMVUE is $\hat{\tau}_{mod} = k + \frac{\bar{X}}{2} ln(1-\gamma)$.

A $1-2\alpha$ median quantile mean is $\tau_{med} = k + \frac{\theta}{2}[ln\alpha + ln(1-\alpha)]$ with its UMVUE $\hat{\tau}_{med} = k + \frac{\bar{X}}{2}[ln\alpha + ln(1-\alpha)]$.

Example 3. A sample size $n = 44$ data of midterm examination scores, displayed also in Siegel (1988, p.~47), showed skewed toward low values indicating that the underlying distribution is more likely to be the left skewed exponential distribution with $k = 100$ .We have $\bar{X} = 88.56$. We also list the estimates of median and mode type quantile means for this data set in the following table.

Table 4. Estimates for γ median and mode type quantile means for examination score data

γ	$\hat{\tau}_{med}(\gamma)$	$\hat{\tau}_{mod}(\gamma)$
0.5	25.92	96.04
0.6	18.90	94.76
0.7	8.86	93.11
0.8	-6.53	90.79
0.9	-34.74	86.83
0.9973	-186.5	65.25

For this midterm examination example having variable with a left skewed distribution, the γ measure of location provides the fact that the top 90% students are with scores average 86.83. Lower the value of γ mode quantile mean estimate indicates the worse performance showing by the top student group.

We provide two more distributions that UMVUEs for γ mode quantile mean exist.

Uniform distribution

If the random variable X has the uniform distribution $U(0,\theta)$ with parameter space $(0,\infty)$. Any value α in $(0,1-\gamma)$ is a choice of α^*. This is the case where multiple solutions for α^* exist when the random variable X has a continuous distribution.

Gamma distribution

Consider the random variable X from $Gamma(\frac{k}{2},\theta)$. We denote the distribution function of $\chi^2(k)$ by G. Then $F_X^{-1}(\alpha)=\frac{\theta}{2}G^{-1}(\alpha)$ and $\alpha^*=\arg_\alpha \inf_{0\le\alpha<\le 1-\gamma}\{G^{-1}(\gamma+\alpha)-G^{-1}(\alpha)\}$. Hence

$$\tau_{mod}=\frac{\theta}{4}\{G^{-1}(\gamma+\alpha^*)+G^{-1}(\alpha^*)\}.$$

Suppose that we have a random sample $X_1,...,X_n$ drawn from this gamma distribution. Since the UMVUE of θ is $\frac{2\bar{X}}{k}$, we have the UMVUE of τ_{mod} is $\frac{\bar{X}}{2k}[G^{-1}(\gamma+\alpha^*)+G^{-1}(\alpha^*)]$.

4. Confidence Interval and Hypothesis Testing for Mode Type Quantile Mean

Confidence interval (C.I.) is another useful tool in applications for an unknown parameter. We will develop this for the mode quantile mean τ_{mod}. Basically, we may interprete a $100(1-2\alpha)\%$ C.I. for τ_{mod} by saying that with $100(1-2\alpha)\%$ confidence the measure of location lies between two ends of the C.I. Three types of C.I. for τ_{mod}, two-sided $\{\tau_{mod}:T_1\le\tau_{mod}\le T_2\}$, right-hand-sided $\{\tau_{mod}:\tau_{mod}\ge T\}$, and left-hand-sided $\{\tau_{mod}:\tau_{mod}\le T\}$ for some statistics T_1,T_2, and T, are the most popular choices for C.I. However, the decision for making a choice is determined by the problem we may concern.

C.I. for Normal distribution

For the normal case, a $100(1-2\alpha)\%$ C.I. of $\tau_{mod}=\mu$ is

$$(\bar{X}-S\frac{t_\alpha}{\sqrt{n}},\bar{X}+S\frac{t_\alpha}{\sqrt{n}}),$$

where $\alpha = P(T \geq t_\alpha)$ with $T \sim t(n-1)$. For this symmetric distribution, the $1-2\alpha$ C.I. for the median quantile mean τ_{med} coincides with the preceding random interval.

Example 3. Consider the muscle strengthening exerciser data. With sample standard deviation $S = 7.6$, the 95% C.I. of τ_{mod} is (20.4, 29) for any $\gamma \geq 0$.

C.I. for exponential distribution

Let $X_1, X_2, \cdots, X_n$ be a random sample from the right skewed distribution with p.d.f. $\sim f(x) = \frac{1}{\theta} e^{-\frac{x-k}{\theta}} I(x > k)$. Then $100(1-2\alpha)\%$ left-hand-sided C.I. for τ_{mod} is $(k, k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-2\alpha}} ln(1-\gamma))$ and the two-sided C.I. is $(k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{\alpha}} ln(1-\gamma), k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-\alpha}} ln(1-\gamma))$.

Proof. Since $\frac{2\sum_{i=1}^n (X_i - k)}{\theta} \sim \chi^2(2n)$, then

$$\begin{aligned} 1 - 2\alpha &= P(\chi^2_{1-2\alpha} \leq \frac{2\sum_{i=1}^n (X_i - k)}{\theta} < \infty) \\ &= P(0 < \frac{\theta}{2\sum_{i=1}^n (X_i - k)} < \frac{1}{\chi^2_{1-2\alpha}}) \\ &= P(0 < \theta \leq \frac{2\sum_{i=1}^n (X_i - k)}{\chi^2_{1-2\alpha}}) \end{aligned}$$

By letting $\tau_{mod} = k - \frac{\theta}{2} ln(1-\gamma)$, we have

$$1 - 2\alpha = P(k < \tau_{mod} \leq k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-2\alpha}} ln(1-\gamma))$$

For γ median quantile mean $\tau_{med} = k - \frac{\theta}{2}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2}))$, we may also see that a $100(1-2\alpha)\%$ left sided C.I. for τ_{med} is $(k, k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-2\alpha}}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2})))$ and the two sided C.I. is $(k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{\alpha}}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2})), k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-\alpha}}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2})))$.

On the other hand, let $X_1, X_2, \cdots, X_n$ be a random sample from the left skewed distribution with p.d.f. $f(x) = \frac{1}{\theta} e^{\frac{x-k}{\theta}} I(-\infty < x < k)$. From the fact that $-\frac{2\sum_{i=1}^n (X_i - k)}{\theta} \sim \chi^2(2n)$, a $100(1-2\alpha)\%$ right sided C.I. for $\tau_{mod} = k + \frac{\theta}{2} ln(1-\gamma)$ is $(k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-2\alpha}} ln(1-\gamma), k)$ and a two sided C.I. is $(k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-\alpha}} ln(1-\gamma), k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{\alpha}} ln(1-\gamma))$. For γ median quantile $\tau_{med} = k + \frac{\theta}{2}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2}))$, a $100(1-2\alpha)\%$ a right sided C.I. for it is $(k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-2\alpha}}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2})), k)$ and a two sided C.I. is $(k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{1-\alpha}}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2})), k - \frac{\sum_{i=1}^n (X_i - k)}{\chi^2_{\alpha}}(ln(\frac{1-\gamma}{2}) + ln(\frac{1+\gamma}{2})))$.

Example 4. (a). Consider the revenues data. The 95% one sided C.I. for τ_{mod} with $\gamma = 0.9973$ and $\gamma = 0.9$ respectively are $(0, 0.942)$ and $(0, 0.367)$. With 95% confidence, among the 99.73% higher-incomed people , the sum of the highest and the lowest revenue is 1.884 tens billions of dollars. And among the 90% higher-incomed people, the sum of the highest and the lowest revenue is 0.734 tens of billions of dollars. The fact that the former is 2.56 times of the later is surprising. It means that the 9.73% lowest among the 99.73% higher-incomed people get astonishingly less revenue and then affect the C.I. of τ_{mod} in a dramatic way. And we find that the 95% one sided C.I.'s for τ_{med} with $\gamma = 0.9973$ and $\gamma = 0.9$ are $(0, 1.052)$ and $(0, 0.572)$, respectively.Obviously, the C.I. of τ_{med} is wider than that of τ_{mod} for each γ.
(b).With the midterm data, we have 95% one sided C.I.'s for τ_{mod} and τ_{med} are $(98.989, 100)$ and $(93.527, 100)$, respectively. We may conclude as follows: (i)With 95% confidence, the interval coveraging τ_{mod} with probability 0.2 should have the range with value between 98.989 and 100. It's a quite narrow interval. (ii)On the other hand , with the same condifence, the corresponding interval coveraging τ_{med} wiht probability 95% should have the value between 93.527 and 100. Its width is 6.402 times of the former.
We may say that two observers of evaluating the performance of the class will have more likely results as basing on τ_{mod} than τ_{med}. This statistical inference supports the use of τ_{mod}.

C.I. for Uniform distribution

Consider the uniform distribution $U(0, \theta)$ case. As we know that $Z = \frac{X_{(n)}}{\theta}$ has a distribution with p.d.f. nz^{n-1},$0 < z < 1$. We may choose a, b that satisfy $1 - 2\alpha = P(a < z < b) = b^n - a^n$. Then we can see that $100(1 - 2\alpha)\%$ C.I. for $\tau_{mod} = \frac{1}{2}\{(\gamma + \alpha^*)^{\frac{1}{n}} - (\alpha^*)^{\frac{1}{n}}\}$ which is independent of parameter θ.

C.I. for Gamma distribution

Consider a random sample $X_1, X_2, \cdots, X_n$ from p.d.f. $Gamma(\frac{k}{2}, \theta)$, and we denote $\chi^2(k)$ by G. Then $100(1 - 2\alpha)\%$ C.I. for $\tau_{mod} = \frac{\theta}{4}\{G^{-1}(\gamma + \alpha^*) + G^{-1}(\alpha^*)\}$ can be found form the fact that $\frac{2\sum_{i=1}^n X_i}{\theta} \sim \chi^2(nk)$. Now,

$$1 - 2\alpha = P(\chi^2_{1-\alpha} < \frac{2\sum_{i=1}^n X_i}{\theta} < \chi^2_\alpha)$$
$$= P(\frac{2\sum_{i=1}^n X_i}{\chi^2_\alpha} < \theta < \frac{2\sum_{i=1}^n X_i}{\chi^2_{1-\alpha}})$$

where χ^2_δ satisfies $P(\chi^2(nk) \geq \chi^2_\delta) = \delta$ for $0 < \delta < 1$. Therefore the C.I. is $(\frac{\sum_{i=1}^n X_i}{2\chi^2_\alpha}(G^{-1}(\gamma+\alpha^*)+G^{-1}(\alpha^*)), \frac{\sum_{i=1}^n X_i}{2\chi^2_{1-\alpha}}(G^{-1}(\gamma+\alpha^*)+G^{-1}(\alpha^*))$.

5.Testing Hypothesis for Mode Quantile Mean

Being an unknown parameter, hypothesis testing is also very popular in statistical inferences. Since the rules for hypothesis testing may be reversely operated as we did in C.I., we then simply display the testing rules for τ_{mod} under several distributions.

Test for Normal Distribution

Table 5 Tests for $H_0\colon \tau_{mod} = t_0$ for normal distribution

H_1	Critical Region
$\tau_{mod} > t_0$	$\bar{X} > t_0 + t_{2\alpha}\frac{S}{\sqrt{n}}$
$\tau_{mod} < t_0$	$\bar{X} > t_0 - t_{2\alpha}\frac{S}{\sqrt{n}}$
$\tau_{mod} \neq t_0$	$\bar{X} > t_0 + t_\alpha\frac{S}{\sqrt{n}}$ or $\bar{X} < t_0 - t_\alpha\frac{S}{\sqrt{n}}$

Test for Exponential Distribution

Table 6 Tests for $H_0\colon \tau_{mod} = t$ for exponential distribution

H_1	Distribution	Critical Region
$\tau_{mod} > t$	$f(x) = \frac{1}{\theta}e^{-\frac{x-k}{\theta}}I(x \geq k)$	$k - \frac{\sum_{i=1}^n (x_i - k)}{\chi^2_{1-2\alpha}} ln(1-\gamma) > t$
$\tau_{mod} < t$	$f(x) = \frac{1}{\theta}e^{\frac{x-k}{\theta}}I(x \leq k)$	$k - \frac{\sum_{i=1}^n (x_i - k)}{\chi^2_{1-2\alpha}} ln(1-\gamma) < t$

a

6. Asymptotic Analysis and Monte Carlo Study for Nonparametric Estimation

In this section, we consider comparing the asymptotic variances of two quantile means as a large sample analysis. Suppose that we have a random sample from a distribution with p.d.f. $f(x,\theta)$. Assuming that F_n is the empirical distribution function, we consider nonparametric estimators of the two quantile means as $\hat{\tau}_{med} = \frac{1}{2}(F_n^{-1}(\alpha)+F_n^{-1}(1-\alpha))$ and $\hat{\tau}_{mod} = \frac{1}{2}(F_n^{-1}(\alpha^*)+F_n^{-1}(\gamma+\alpha^*))$ where α^* is determined through the underlying distribution. With the fact that the empirical quantile has the following representation

$$n^{1/2}(F_n^{-1}(\alpha) - F^{-1}(\alpha)) = f^{-1}(F^{-1}(\alpha))n^{-1/2}\sum_{i=1}^n(\alpha - I(X_i \leq F^{-1}(\alpha))) + o_p(1),$$

we may see that $n^{1/2}(\hat{\tau}_{med} - \tau_{med})$ has an asymptotic normal distribution with zero mean and variance as

$$\frac{1}{4}\alpha(1-\alpha)[f^{-2}(F^{-1}(\alpha)) + f^{-2}(F^{-1}(1-\alpha))] + \frac{\alpha^2}{2}f^{-1}(F^{-1}(\alpha))f^{-1}(F^{-1}(1-\alpha))$$

and $n^{1/2}(\hat{\tau}_m - \tau_m)$ has an asymptotic normal distribution with zero mean and variance as

$$\frac{1}{4}[\alpha^*(1-\alpha^*)f^{-2}(F^{-1}(\alpha^*)) + (\gamma+\alpha^*)(1-(\gamma+\alpha^*))f^{-2}(F^{-1}(\gamma+\alpha^*)) + \frac{1}{2}(\alpha^*)^2 f^{-1}(F^{-1}(\gamma+\alpha^*))f^{-1}(F^{-1}(\alpha^*)).$$

For comparison, we let $\gamma = 1 - 2\alpha$.

Table 7 Asymptotic variance anayses for distribution $Gamma(3.5, \beta)$ with $\beta = 0.3$ and 1.0

γ	τ_{med}	τ_{mod}	τ_{med}	τ_{mod}
	$\beta = 0.3$		$\beta = 1.0$	
0.95	1.751	1.144	19.44	5.076
0.9	0.976	0.694	10.85	2.899
0.85	0.711	0.535	7.906	2.144
0.8	0.578	0.454	6.431	1.760
0.75	0.500	0.405	5.560	1.529
0.7	0.450	0.373	5.000	1.376
0.65	0.416	0.351	4.622	1.270
0.6	0.392	0.336	4.364	1.194
0.55	0.377	0.325	4.189	1.138
0.5	0.366	0.318	4.075	1.098
0.45	0.360	0.313	4.009	1.070
0.4	0.358	0.310	3.983	1.051
0.35	0.359	0.309	3.992	1.040
0.3	0.362	0.310	4.031	1.036
0.25	0.369	0.312	4.100	1.038
0.2	0.377	0.315	4.197	1.046
0.15	0.389	0.320	4.323	1.061
0.1	0.403	0.326	4.480	1.081
0.05	0.420	0.333	4.669	1.108

The asymptotic variances for estimator of τ_{mod} are uniformly smaller than those of τ_{med}. For larger values of β in gamma distribution, such as $\beta = 1$, the discrepancies are even large. Although we employ the nonparametric sample quantile function F_n^{-1} for estimating F^{-1}, however, the parameter α^* is computed from known distribution. It is then worth to propose a purely nonparametric method to estimate the quantile means. Here we introduce one.

Let $X_{(1)}, ..., X_{(n)}$ be the order statistics of a random sample $X_1, ..., X_n$ drawn from a distribution F. By letting $h = [n\gamma] + 1$, we denote

$$h^* = \arg_{h,h+1,\ldots,n}\min\{X_{(h)} - X_{(1)}, X_{(h+1)} - X_{(2)}, ..., X_{(n)} - X_{(n-h+1)}\}.$$

Then we define a nonparametric estimator of mode type quantile mean as the width of the shortest γ sample as

$$\hat{\tau}_{mod} = \frac{X_{(h^*-(h-1))} + X_{(h^*)}}{2}.$$

Let also denote the symmetric type quantile mean estimate as

$$\hat{\tau}_{med} = \frac{1}{2}(X_{(m_1)} + X_{(m_2)})$$

where $m_1 = [n\frac{1-\gamma}{2}]$ and $m_2 = [n\frac{1+\gamma}{2}]$. With replication $m = 1000$, we randomly generate a sample of size $n = 30$ from the underlying distribution F and let $\hat{\tau}^j_{mod}$ and $\hat{\tau}^j_{med}$ be, respectively, the estimates of mode type and median type quantile means for the jth random sample. We define the mean squares errors of these two quantile means as

$$MSE_{mod} = \frac{1}{m}\sum_{j=1}^{1000}\frac{(\hat{\tau}^j_{mod} - \tau_{mod})^2}{s^2}$$

and

$$MSE_{med} = \frac{1}{m}\sum_{j=1}^{1000}\frac{(\hat{\tau}^j_{med} - \tau_{med})^2}{s^2}.$$

We did a simulation under the exponential distribution and gamma distribution. The results of MSE's are listed in the following two tables.

Table 8 MSE's for right skewed exponential distribution

Interval	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$	$\gamma = 0.9973$
$\theta = 2$					
τ_{med}	0.0455	0.0679	0.0868	0.1122	0.5336
τ_{mod}	0.0008	0.0011	0.0018	0.0044	0.1988
$\theta = 5$					
τ_{med}	0.0432	0.0656	0.0894	0.0114	0.5724
τ_{mod}	0.0008	0.0011	0.0019	0.0046	0.2136
$\theta = 25$					
τ_{med}	0.0441	0.0667	0.0887	0.1137	0.5702
τ_{mod}	0.0008	0.0011	0.0020	0.0044	0.2121

Table 9 MSE's for $Gamma(\alpha, \beta)$ distribution

Interval	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$	$\gamma = 0.9973$
$\alpha = 2, \beta = 2$					
τ_{med}	0.0483	0.0612	0.0796	0.1165	1.5782
τ_{mod}	0.0453	0.0351	0.0330	0.0386	0.9316
$\alpha = 2, \beta = 3$					
τ_{med}	0.0485	0.0616	0.0804	0.1179	1.5907
τ_{mod}	0.0453	0.0342	0.0321	0.0386	0.9406
$\alpha = 4, \beta = 3$					
τ_{med}	0.0398	0.0464	0.0576	0.0829	0.9817
τ_{mod}	0.0692	0.0562	0.0484	0.0465	0.5408

In this Monte Carlo study, besides the cases of $\gamma = 0.6$ and 0.7, for distribution $Gamma(4, 3)$, the MSE's for τ_{mod} are all smaller than those corresponding one's for τ_{med}. We may conclude that the nonparametric estimator for mode quantile mean is relatively more efficient than the median quantile mean.

7. Extension of Mode Type Quantile Mean

Although the discussion in the above section dealt all with the mode quantile mean which is an average of two mode type quantiles, this concept may be extended to the average of arbitrary number of mode type quantiles which then plays an alternative choice of the general L-estimator. This generalization is introduced in the following definition.

Definition 7.1. For δ_i and $\gamma_i, i = 1, ..., k$, $0 \le \delta_i \le 0.5, 0 \le \gamma_i < 1$ and $\sum_{i=1}^{k} \delta_i = 0.5$, the mode type L-estimate is defined as

$$L_{mod} = \sum_{i=1}^{k} \delta_i (F^{-1}(\alpha_i^*) + F^{-1}(\gamma_i + \alpha_i^*))$$

where $\alpha_i^* = \arg_\alpha \inf_{0 \le \alpha < 1-\gamma_i}\{F^{-1}(\alpha+\gamma_i) - F^{-1}(\alpha)\}$ and, in case that $\gamma_i = 0$, $F^{-1}(\alpha_i^*)$ is the location point of mode.

This generalizes the following ordinary L-estimate, very popular in application and theoretical study,

$$L_{med} = \sum_{i=1}^{k} \delta_i (F^{-1}(\alpha_i) + F^{-1}(1-\alpha_i))$$

where $0 \le \alpha_i \le 0.5, i = 1, ..., k$. These two L-estimates are identical when the distribution is symmetric and we let $\alpha_i = \frac{1-\gamma_i}{2}$ that implies $1 - \alpha_i = \frac{1+\gamma_i}{2}$.

For this median type L-estimate, one special case proposed by Gastwirth (1966) is the one with $k = 2, \delta_1 = 0.3, \delta_2 = 0.2$ and $\alpha_1 = 0.3, \alpha_2 = 0.5$, as

$$L_{Gmed} = 0.3F^{-1}(0.3) + 0.4F^{-1}(0.5) + 0.3F^{-1}(0.7).$$

The mode type Gastwirth L-estimate then may be set as

$$L_{Gmod} = 0.3F^{-1}(\alpha_{.4}^*) + 0.4F^{-1}(\alpha_{.0}^*) + 0.3F^{-1}(\alpha_{.4}^* + 0.4).$$

For simulation study, we consider to compare the L-estimayes of the versions using two and four quantiles. We denote the followings, for $\gamma_1 < \gamma_2$,

$$L_{med}^1 = 0.5(F^{-1}(\frac{1-\gamma_1}{2}) + F^{-1}(\frac{1+\gamma_1}{2})),$$
$$L_{mod}^1 = 0.5(F^{-1}(\alpha^1) + F^{-1}(\alpha^1 + \gamma_1)),$$
$$L_{med}^2 = 0.25(F^{-1}(\frac{1-\gamma_2}{2}) + F^{-1}(\frac{1-\gamma_1}{2}) + F^{-1}(\frac{1+\gamma_1}{2}) + F^{-1}(\frac{1+\gamma_2}{2}))$$
$$L_{mod}^2 = 0.25(F^{-1}(\alpha^1) + F^{-1}(\alpha^1 + \gamma_1) + F^{-1}(\alpha^2) + F^{-1}(\alpha^2 + \gamma_2)).$$

In this simulation, we set the same design, besides the coverage probabilities γ_1 and γ_2, as we did before. We display the simulation results of MSE's under $\gamma_1 = 0.65$ and 0.8 and several γ_2's in the following table.

Table 10 MSE's for $Gamma(2, 2)$ distribution

γ_1, γ_2	L^1_{med}	L^1_{mod}	L^2_{med}	L^2_{mod}
($\gamma_1 = 0.65$)				
$\gamma_2 = 0.75$	0.0559	0.0400	0.0626	0.0274
$\gamma_2 = 0.80$			0.0641	0.0249
$\gamma_2 = 0.85$			0.0728	0.0228
$\gamma_2 = 0.90$			0.0768	0.0235
$\gamma_2 = 0.95$			0.1084	0.0259
($\gamma_1 = 0.80$)				
$\gamma_2 = 0.85$	0.0816	0.0321	0.0910	0.0271
$\gamma_2 = 0.90$			0.0950	0.0266
$\gamma_2 = 0.95$			0.1285	0.0290

In this design, we find the evidence that the mode L-estimator may improves the efficiencies performed by the mode quantile mean τ_{mod}.

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