國 立 交 通 大 學

統計學研究所

碩 士 論 文

GARCH 選擇權定價模型在臺灣市場的實證表現

The Empirical Performance of the GARCH Option Pricing Model in Taiwan Market

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ABSTRACT

A central hypothesis of the Black-Scholes model is that the return on the underlying asset distributed log-normally with constant volatility. However, it has been widely recognized that financial asset return processes possess heavy-tailed marginal distributions and volatility clustering. These features are interpreted as the evidence of the stochastic volatility of financial assets, and estimating the term structure of volatility has become an important issue in finance engineering.

We introduced the GARCH option pricing model of Duan (1995), using the LRNVR change measure to price options by Monte Carlo simulation runs and evaluate the empirical performance of different option pricing models on Taiwan Stock Exchange Capitalization Weighted Stock Index Options. We considered the improved and constant volatility (non-update) Black-Scholes models, and the update and non-update GARCH option pricing models. We then compare their pricing performance according to several criteria.

GARCH 選擇權訂價模型在台灣市場的實證表現

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中文摘要

資產的對數報酬服從常態分布以及波動率為一常數是 Black-Scholes 選擇權估價模型的重要假設。然而，資產報酬有著較常態分布大的尾端機率及波動率叢聚現象。這些現象被解讀為財務資產的波動隨機結構，而這也成為財務工程的重要議題。

我們介紹由段錦泉教授在1995年所提出的 GARCH 選擇權訂價模型。在局部風險中立測度下以蒙地卡羅模擬法計算台灣加權股價指數選擇權價格。我們會呈現數個不同的評價準則以比較原始的、修正的 Black-Scholes 模型，與隨時間更新、不隨時間更新的 GARCH 選擇權模型評價表現。

致謝

望向美麗的交大校園，已到了要離去的時刻！快樂、興奮、悵然不捨的情緒混雜交錯。對於兩年前至交大統計所口試的忐忑心情仍是記憶猶新，然而晃眼，悠悠兩年轉眼已過，心中感慨萬千。

本論文得以順利完成，首先要感謝的就是我的指導教授 李昭勝博士，老師豐富的學識涵養、嚴謹的處事態度、以及努力不懈的治學精神，皆令學生獲益匪淺。除此之外，博士班牛維方學長在生活態度與論文研究上給予諸多教導與指正，以學長本身豐富的經驗提供許多的思考方向。而口試之際承蒙逢甲大學陳婉淑教授、東海大學林宗儀教授，及本所所長盧鴻興教授，不吝於百忙之中撥冗審閱論文，在口試時更是給予寶貴意見並改正論文錯誤，於此致上最深的謝忱。

兩年的交大生涯，所有的回憶湧上心頭。在盧所長、郭姐和所有學長同學學弟妹們的協助下一起舉辦統研盃是一個特別的回憶。兩年中的風風雨雨，是我踏過這段歲月的痕跡。感謝陪伴我度過研究所時光的所有好友，特別是阿甘和超毅在生活上的扶持，同門的華勝、坤民、宇青、姿羽與賴董在課業上的切磋與砥礪。感謝阿賓、小強、阿昌與阿結在精神上之支持與鼓勵，感謝同窗忠庭、崢珮、怡均、寶文、蘇董與巧慧帶來的歡樂與笑聲，使生活更加精彩。我僅以誠摯的心，向諸位友人表達我無盡的謝意。

最後，尤其應當致謝的是我最最摯愛的家人與華宣。感謝爸媽及惠英、慧如的關懷與支持，讓我能無憂無慮的完成研究所學業，朝自己的理想邁進。另外，更是感謝華宣多年來的照顧與陪伴，適時地給予我建議、給予我溫暖，讓我能提起精神面對生命中的諸多挑戰，因為有你，讓我一路上有所依靠，也才能造就今日的我。

論文完成之際，僅以本文獻予所有的好友與摯親，與你們分享我的喜悅與成就。此刻，我將赴往生命中的另一段旅程，冀希未來的路途亦能與你們相伴。

邱政輝　　　　　謹誌於
交通大學統計學研究所
中華民國九十三年六月十九日

Contents

Abstract (in English) …………………….……………… *i*

Abstract (in Chinese) ……………..…………....……… *ii*

Acknowledgement (in Chinese) ………………………….. *iii*

Contents ………………………………………..……...…… *iv*

1. Introduction... 1

2. The Option Pricing Models... 3

2.1 The Black-Scholes Model... 3

2.2 The GARCH Option Pricing Model 6

2.2.1 Introduction of the GARCH model 6

2.2.2 The LRNVR and the GARCH Option Pricing Model 9

3. Description of the Data... 13

4. Empirical Analysis 17

5. Conclusion 22

***References* 23**

***Appendix A. Tables* 25**

***Appendix B. Figures* 29**

1. Introduction

Following the seminal work of Black and Scholes (1973) and Merton (1973), the option literature has developed into an important area of research. A central hypothesis of the Black-Scholes model is that the return on the underlying asset distributed log-normally with constant volatility. However, it has been widely recognized that financial asset return processes possess heavy-tailed marginal distributions and volatility clustering. These features are interpreted as the evidence of the stochastic volatility of financial assets, and estimating the term structure of volatility has become an important issue in finance engineering.

There are basically two types of volatility models: continuous-time stochastic models and discrete-time stochastic generalized autoregressive conditional heteroscedasticity (GARCH) models. On one hand, the continuous-time model can serve as the limit of a certain GARCH model. Duan (1996) argued that most of the existing bivariate diffusion models that had been used to model asset returns and volatility could be represented as limits of a family of GARCH models. The continuous-time stochastic model has an inherent disadvantage that it assumes that volatility is observable, but it is impossible to exactly filter volatility from discrete observations of spot asset prices in a continuous-time stochastic volatility model. Consequently, it is impossible to price an option solely on the basis of the history of asset prices. On the other hand, the GARCH model has an advantage over the continuous-time model in that the volatility is readily observable in the history of asset prices. As a result, the volatility term structure is calculated on the basis of market observations of underlying asset prices.

Although there have been many papers on the comparison of these two models, some of these studies obtain quite different results. Different conclusions are drawn because of the usage of different sample periods, models and estimation methods. Therefore, it is still meaningful to examine the empirical performance of the Black-Scholes model and the GARCH option pricing model by Monte Carlo simulation using a new data set.

In this study, we use the GARCH (1, 1) option pricing model developed by Duan (1995) in a discrete time GARCH (1, 1) environment. Under the local risk-neutral valuation relationship (LRNVR) derived by Duan, the asset return process under the risk-neutralized pricing measure differs from the conventional GARCH process in an interesting way. We perform and compare the option pricing result both of the update and non-update, Black-Scholes and GARCH option pricing model.

The numerical results compare the empirical option pricing performance in different moneyness categories of the update Black-Scholes, non-update Black-Scholes and update and non-update GARCH models. We applied the models to daily closing prices of the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) and its corresponding TAIEX options. We use the TAIEX and its corresponding options based on two considerations. One is that the index and the option prices data for the TAIEX index options are freely available. Furthermore, the TAIEX index option is the most actively traded European-style option in Taiwan.

This thesis is structured as follows: Chapter 2 discusses the famous Black-Scholes option pricing model and the GARCH option pricing model of Duan (1995). The description of the data is provided in Chapter 3. In Chapter 4, we examine the empirical performance of the GARCH option pricing model and the Black-Scholes model in Taiwan's market. Conclusions are offered in Chapter 5.

2. The Option Pricing Models

In this chapter we will introduce the most famous option pricing model, the Black-Scholes model, and the GARCH option pricing model of Duan (1995).

Since for a European put, its price can be derived easily via the following put-call parity relationship:

$$p_t + S_t = C_t + Xe^{-r(T-t)},$$

we will demonstrate only the pricing of the call options in the rest of this thesis.

2.1 The Black-Scholes Model

Black and Scholes (1973) published an option valuation formula which is known today as the Black-Scholes model.

The Black-Scholes model was established under the following conditions:

1. The short selling of securities with full use of proceeds is permitted.
2. There are no transition casts or taxes.
3. All securities are perfectly divisible.
4. There are no arbitrage opportunities.
5. Security trading is continuous.
6. The risk-free rate of interest, r, is constant and the same for all maturities.

When the underlying asset follows geometric Brownian motion

$$dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dw_t$$

with constants μ and σ, then the Black-Scholes formula calculates the price of a call option to be:

$$C = S \cdot N(d_1) - Ke^{-rT} \cdot N(d_2)$$

where

C = price of a call option,

S = price of the underlying asset,

K = option exercise price,

r = risk-free rate,

T = current time to maturity,

$N(.)$ = standard normal cumulative distribution function ,

$$d_1 = \frac{\ln(S/K) + (r + 0.5\sigma^2)\cdot T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The Black-Scholes model assumes that the volatility of stock prices is a constant. In practice, some analysts use the market option prices to obtain the implied volatility as an estimate of the volatility for the Black-Scholes model. There is no explicit for the implied volatility. However, using some simple numerical method like Newton method, we can easily obtain the implied volatility from the option pricing formula.

Nevertheless, the implied volatility is found to vary, and thus the assumption of constant volatility is violated. The plot of the implied volatility of an option as a function of its strike price is known as a volatility smile. The volatility smile of the TAIEX options in September, 2, 2003 has the form shown in Figure 1. The implied volatility decreases as the strike price increases, which is sometimes referred as volatility skew. In addition to a volatility smile, volatility term structure is also observed. The volatility used to price an option depends on the length of time to maturity of the option. When volatility smile and volatility term structure are combined, they produce a volatility surface. This defines implied volatility as a function of both the strike price and the time to maturity.

We perform both the original (non-update, constant volatility) Black-Scholes option pricing and improved models. In the improved Black-Scholes model, we allow different volatilities for different lengths of time to maturity. Since the only unobserved variable is the implied volatility, we can compute the implied volatility for all the call options on each day by numerical method. In fact, we can also use the information contained in the put option prices. For simplicity, we will just consider the call options here. We then use this implied volatility to value all the call options on the following day.

2.2 The GARCH Option Pricing Model

The objective of this section is to give a brief introduction to the GARCH model and its applications. In 2.2.2, we will introduce the locally risk-neutral valuation relationship, and the GARCH option pricing model proposed by Duan (1995).

2.2.1 Introduction of the GARCH model

The objective of this sub section is to study the generalized auto-regressive conditional heteroscedastic model of Bollerslev (1986).

Define h_t as the variance rate of a market variable on day t which can be estimated at the end of day ($t-1$). The square root of the variance on day t, $\sqrt{h_t}$, is called the volatility. Suppose that the value of the market variable at the end of day t is S_t. The log return during day t (between the end of day $t-1$ and the end of day t), r_t, is defined as

$$r_t = \log\left(\frac{S_t}{S_{t-1}}\right).$$

A log return series, $\{r_t\}$, is said to follow a GARCH (p, q) model, if

$$r_t = \mu_t + \varepsilon_t, \quad \varepsilon_t = \sqrt{h_t} \cdot z_t,$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \beta_i \cdot h_{t-i} + \sum_{j=1}^{p} \alpha_j \cdot \varepsilon_{t-j}^2$$

where $\{z_t\}$ is a sequence of identically and independently distributed random variables with mean zero and variance 1, and it is often assumed to be a standard normal or standardized Student-t distribution, μ_t is the trend of log return at time t. For a stable GARCH (p, q) process, the condition $\alpha_0 > 0$, $\alpha_i, \beta_j \geq 0$, is required to assume the weight of the long-run average variance is positive, and the condition $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$ implies that the unconditional variance of $\{r_t\}$ is finite.

Because the simple GARCH (1, 1) with normal distribution assumption is the most commonly used, we will focus on it with

$$r_t = \varepsilon_t, \quad \varepsilon_t = \sqrt{h_t} \cdot z_t, \quad z_t \sim N(0,1),$$

$$h_t = \alpha_0 + \beta_1 \cdot h_{t-1} + \alpha_1 \cdot \varepsilon_{t-1}^2,$$

$$\alpha_0 > 0, \quad 0 < \alpha_1, \beta_1 < 1, (\alpha_1 + \beta_1) < 1.$$

With the normality assumption, all the parameters can be estimated directly from the data through the maximum likelihood method. It is easy to obtain the parameter estimates by non-linear constrained optimization given the constraints of the boundary and stationarity conditions with packages such as MATLAB or E-views.

Forecast through the GARCH model can then be easily obtained. The conditional variance h_{t+1} depends on the one-step ahead observation and the one-step ahead variance rate:

$$h_{t+1} = \alpha_0 + \beta_1 h_t + \alpha_1 \varepsilon_t^2,$$

where h_t and ε_t are known in time index *t*. Therefore, the 1-step ahead forecast is

$$h_t(1) = \alpha_0 + \beta_1 h_t + \alpha_1 \varepsilon_t^2.$$

For multiple ahead forecasts, we rewrite the volatility equation as

$$h_{t+1} = \alpha_0 + (\alpha_1 + \beta_1) \cdot h_t + \alpha_1 \cdot h_t^2 (\varepsilon_t^2 - 1).$$

Since $E(\varepsilon_{t+1}^2 - 1 | F_t) = 0$, the 2-step ahead volatility forecast at the forecast origin *t* satisfies the equation

$$h_t(2) = \alpha_0 + (\alpha_1 + \beta_1) h_t(1).$$

In general, we have

$$h_t(l) = \alpha_0 + (\alpha_1 + \beta_1) h_t(l-1), l > 1.$$

By repeated substitutions, we obtain the following *l*-step ahead forecast

$$h_t(l) = \frac{\alpha_0\left[1-(\alpha_1+\beta_1)^{l-1}\right]}{1-\alpha_1-\beta_1} + (\alpha_1+\beta_1)^{l-1} h_t(1).$$

Therefore,

$$h_t(l) \to \frac{\alpha_0}{1-\alpha_1-\beta_1}, \quad \text{as} \quad l \to \infty$$

provided that $\alpha_1 + \beta_1 < 1$. Consequently, the multi-step ahead volatility forecasts of a GARCH (1, 1) model converge to the unconditional variance as the forecast horizon increases to infinity provided that the variance exists.

2.2.2 The LRNVR and the GARCH Option Pricing Model

Duan (1995) develops a pricing model for options on an asset whose return follow the GARCH process in a discrete time economy environment. By maximizing the expected utility, he shows the Locally Risk-Neutral Valuation Relationship (LRNVR) holds and derives the following model for log return process.

Duan started by assuming that the asset return dynamic, under the actual probability measure, P, is

$$\ln\left(\frac{X_t}{X_{t-1}}\right) = r + \lambda\sqrt{h_t} - \frac{1}{2}h_t + \varepsilon_t,$$

where ε_t has zero mean and conditional variance h_t under measure P; r is the constant one-period risk-free rate of return (continuously compounded) and λ is the constant unit risk premium. Under conditional log-normality, one plus the conditionally expected rate of return equals $\exp(r + \lambda\sqrt{h_t})$. It thus suggests that λ can be interpreted as the unit risk premium.

Furthermore, ε_t follows a GARCH (p, q) process of Bollerslev (1986) under measure P. That means:

$$\varepsilon_t \middle| F_{t-1} \sim N(0, h_t) \quad \text{under measure P},$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j},$$

where ϕ_t is the information set (σ-field) of all information up to and including time t. For positive variance and stationarity, the following conditions are required: $p \geq 0,\ q \geq 0;\ \alpha_0 > 0,\ \alpha_i \geq 0, i = 1, \ldots, q;\ \beta_j \geq 0, j = 1, \ldots, p;\ \ \sum_{i=1}^{q}\alpha_i + \sum_{j=1}^{p}\beta_j < 1$.

As p = 0 and q = 0, we have

$$r_t = \log\left(\frac{X_t}{X_{t-1}}\right) = r - \frac{1}{2}h_t + a_t, \quad a_t \sim^{iid} N(0,\sigma^2),$$

which implies

$$\log\left(\frac{X_T}{X_0}\right) \sim N\left((r-\frac{1}{2}\sigma^2)\cdot T, \sigma^2 T\right).$$

It is back to the Black-Scholes result. Obviously, the Black-Scholes model is a special case of the GARCH option pricing model.

For pricing purpose, we introduced the LRNVR of Duan (1995):

Definition: The risk-neutral probability measure Q is said to satisfy the *local risk-neutral valuation relationship* (LRNVR) if the following conditions are satisfied:

(i) Measure Q is mutually continuous with respect to measure P;

(ii) Log return is normally distributed under Q,

with $E^Q\left(\frac{X_t}{X_{t-1}}|F_{t-1}\right) = e^r$,

and $Var^Q\left(\ln\left(X_t/X_{t-1}\right)|F_{t-1}\right) = Var^P\left(\ln\left(X_t/X_{t-1}\right)|F_{t-1}\right)$ almost surely with respect to measure P.

In the above definition, the conditional variances under the two measures are required to be the same. It is desirable because one can observe and hence estimate the conditional variance under measure P. This property and the fact that the conditional mean can be replaced by the risk-free rate obtained a well-specified model that does not depend on preference. However, the local risk neutralization is not sufficient to eliminate the preference parameters. Under the model setting, it is nevertheless strong enough to reduce all preference consideration to the risk premium. In the case of a homoscedastic lognormal process, i.e., p =

0 and q = 0, the conditional variances become the same constant and the LRNVR reduces to the conventional risk-neutral valuation relationship.

Duan (1995) also proved in the view of equilibrium that, under some combinations of the distributions and the preferences, the LRNVR holds its validity.

Theorem: If the representative agents is an expected utility maximizer and the utility function is time separable and additive, then the LRNVR holds under any of the following conditions:

(i) The utility function is of constant relative risk aversion and changes in the logarithm aggregate consumption are distributed normally with constant mean and variance under measure P.

(ii) The utility function is of constant absolute risk aversion and changes in the aggregate consumption are distributed normally with constant mean and variance under measure P.

(iii) The utility function is linear.

The first and second conditions ensure that the implied interest rate is constant. This is the same with the constant interest rate assumption earlier. It is possible to develop the model with stochastic interest rates, but the resulting model will become much more complicated.

Under the local risk-neutralized probability measure Q, the asset return dynamic becomes

$$\ln\left(\frac{X_t}{X_{t-1}}\right) = r - \frac{1}{2}h_t + \xi_t,$$

where

$\xi_t | F_{t-1} \sim N(0, h_t)$ under measure Q,

$$h_t = \alpha_0 + \sum_{i=1}^{q}\alpha_i(\xi_{t-i}^2 - \lambda\sqrt{h_{t-1}})^2 + \sum_{j=1}^{p}\beta_j h_{t-j}.$$

The result implies that the form of the GARCH (p, q) process remains largely intact with respect to local risk neutralization. The variance innovation is governed by q non-central chi-square random variables with one degree of freedom and non-central parameter λ, whereas the GARCH process under P can be seen as the process governed by q central chi-square innovations. It suggests that the unit risk premium, λ, influences the conditional variance process globally although the risk has been locally neutralized under measure Q.

Under GARCH model, a European call option with exercise price K maturing at time T has the following time-t value,

$$C_t = e^{-r(T-t)} E^Q \left[\max\left(X_T - K, 0 \right) \middle| F_t \right].$$

3. Description of the Data

To examine the empirical performance of the GARCH option pricing model, we applied the model to daily closing prices of the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) and its corresponding TAIEX options. We use the index and its corresponding options based on the following consideration. The first reason is that the index and the option data are freely available. The data were obtained from the official website of Taiwan Stock Exchange Corporation (TSEC):

http://www.tse.com.tw

and the official website of Taiwan Futures Exchange (TAIFEX):

http://www.taifex.com.tw .

Second, the TAIEX index option is the most actively traded European-style option in Taiwan. Thus, the TAIEX option market is chosen to test the empirical performance of the Black-Scholes model and the GARCH option pricing model.

We will use the TAIEX index with the sample period from January 4, 2000 to December 10, 2003 to establish the GARCH volatility dynamic. There are 617 observations. Figure 2 describes the evolution of the daily TAIEX index levels, which fluctuated dramatically during this period. One can easily see that the daily TAIEX index levels in Figure 2 demonstrate a decreasing trend. The mean index value is about 5,800, while the maximum index value is more than 10,000, and the lowest value is less than 3,500. Starting year 2000, Taiwanese economy has been affected by many political and non-political issues. The first shift of Taiwanese political party, the suspension and re-establishment of nuclear power station, and the unhealthy financial system had all affected the stock market. Figure 3 describes the daily return for the sample period. It shows that the mean of the return series appears to be constant whereas the variance clearly changes over time. Volatility clustering is also observed in the plot, a large value tends to follow by another large value. This is known as the conditional heteroscedasticity.

The TAIEX option is a European style option, and it expires at the opening of the Thursday following the third Wednesday of each contract month. The expiration month is the next two calendar months followed by two additional months from the March quarterly cycle (March, June, September, and December). Their strike price intervals are 100 points. On each trading day, we report the first call contract happened in the time interval of 1:00-1:25 which is near the closing time for each strike price and the time-to-maturity. The time-to-maturity is measured as the number of calendar days from the trading date to the Wednesday immediately preceding the Thursday when the option expires date, because TAIEX index options expire at the opening of trading. The reported index level is the closing price of the TAIEX index. After establishing the GARCH dynamic, we will compute the index option prices from September 1, 2003 to December 10, 2003.

The following criteria are employed to filter the option data:

First, general arbitrage violations must be eliminated from the data; otherwise there might be a negative implied volatility. A transaction has to satisfy the following no-arbitrage relationships:

$$C_t \geq \max(0, S - K \cdot e^{-r(T-t)}).$$

Second, very short-term options and very long-term options are excluded. Options with less than 7 days to expiration are excluded because they are very sensitive to liquidity-related biases and their prices are generally very volatile. Options with time to maturity longer than 40 days are also excluded because they are not actively traded and thus excluded from the sample.

Third, very deep out-of-the-money and very deep in-the-money options are excluded.

This criterion is based on the same considerations as discussed in the above paragraph. Options with deep out-of-the-money and very deep in-the-money options may contain little information about the volatility process. Moreover, these options are not traded actively. An option is defined as very deep in- or out-of-the-money if its moneyness is greater than 1.2 or lower than 0.8. The option moneyness is defined as the ratio between the TAIEX level and the strike price:

$$Moneyness = S/K.$$

Our exclusionary criteria yield a final daily sample of 584 observations for 70 days. On average we have about 9 option prices available on each day. Since option prices are not very sensitive to the interest rate and the change of interest rate is small on daily basis, we shall just assume the risk free rate as 2% per year.

It is a common practice in the literature to divide options into different moneyness categories to study their price behavior because option prices are very sensitive to their exercise prices. We divide the option data into 5 categories according to the moneyness.

We define a call option is said to be at-the-money if the moneyness is between (0.98, 1.02), in-the-money if the moneyness is between (1.02, 1.05), out-of-the-money if the moneyness is between (0.95, 0.98) and deep in-the-money if the moneyness is greater than 1.05 and deep out-of-the-money if the moneyness is less than 0.95.

Table 1 provides the average and standard deviation of call option prices reported for each moneyness category, and also shows the numbers of observations in these categories for the period from September 1, 2003 to December 10, 2003. About 25% of the samples are at-the-money, 19% of the samples are out-of-the-money and 19% of the samples are in-the-money. Only 8% of the samples are deep out-of-the-money. The most unusual thing is the high proportion of the deep in-the-money (26%). The demand for in-the-money options is higher than the demand for out-of-the-money options, which indicates that investors are cynical about future market increases, since in-the-money call options will be valuable only if the market decreases significantly in the future. The overall average call price in our sample period is NT$210.51 with a standard deviation of NT$197.61.

4. Empirical Analysis

The GARCH Model

As mentioned in Chapter 2, a European call option with exercise price K maturing at time T has the following time-t value under measure Q,

$$C_t = e^{-r(T-t)} E^Q\left[\max\left(X_T - K, 0\right)\middle| F_t\right].$$

Thus the Monte Carlo simulation is used in the computation of the GARCH option prices. Use of the Monte Carlo method to compute option prices can be traced back to Boyle (1977). It is a convenient method for the GARCH option pricing model because the distribution for the temporally aggregated asset return can not be derived easily. To simulate the risk-neutralized GARCH (1, 1) asset returns for the option pricings at time t, we recognize that the asset price S_t and the conditional volatility h_{t+1} can together serve as sufficient statistics.

We tried two different ways to employ the GARCH option pricing model: one is to assume that the implied parameters are constant in the following period, i.e. the structure will not change in the following period. Then, we input the period's implied parameters to the GARCH option model and price all the option contracts. And another is to assume that the implied parameters change daily, we input the previous day's implied parameters to the GARCH option pricing model and compute current day's model-determined option prices, it predicts one day ahead.

In the first case, the GARCH-M model specified as Duan (1995) with $p = 1$ and $q = 1$ is fitted to the TAIEX daily index series from January 4, 2000 to September 30, 2003. The estimated parameters are $\hat{\alpha}_0 = 2.072 \times 10^{-5}$, $\hat{\alpha}_1 = 0.075$, $\hat{\beta}_1 = 0.867$, and $\hat{\lambda} = 0.007$, respectively. The GARCH parameter values together imply that the annualized standard deviation is 29.09%. The risk-free rate is fixed at 2% for simplicity. Each contract was obtained by carrying out fifty thousand Monte Carlo simulation runs.

In the second case, the GARCH model will be re-estimated daily. There are two ways to predict volatility via GARCH model period by period: one is to "rolling GARCH model", that is, we fix the number of observations, every time we add a new observation, we eliminate the oldest observation; the other one is to "updating GARCH model", every period, add a new observation into the original observations. We adopt the rolling GARCH model method which uses the same data length. The estimated parameter values were varying period by period. In Figure 4, the parameter patterns shows there were few changes in the model. All the fitted models were stationary GARCH (1, 1) model, i.e. $\alpha_1 + \beta_1 < 1$. The estimates of the risk-premium parameter, λ, are all very close to zero.

Duan also mentioned that it is an opportunity to examine the impacts of the initial conditional variance on the price of the option. To compare the impacts of conditional volatility, three levels of initial conditional standard deviations are studied. They are the stationary level, 20% below the stationary level, and 20% above, respectively. Similar to his parameter setting, our result shows that although the different initial values effect the option pricing result, but the difference is limited, as indicated in Tables 3A and 3B. Therefore, the following analysis will focus on the initial value being the standard conditional volatility, i.e. $\sqrt{h_0} = \sigma$.

<u>The Black-Scholes Model</u>

The only unobservable variable in the Black-Scholes formula is the volatility; therefore, the value of implied volatility will determine the pricing result. The original model assumption limits the implied volatility to be a constant. However, it has been widely recognized that financial asset return processes possess heavy-tailed marginal distributions and volatility clustering. Various improvements had been employed. Similar to the GARCH model, we tried the non-update and update volatility for the Black-Scholes model.

We impose the original Black-Scholes' assumption in the non-update case: the constant volatility. Estimation of implied volatility is 20.35% from the option contracts in August, 2003. And in the case we update the volatility, for each contract we computed the implied volatility of the previous day by numerical method, and used their average as the estimate of the volatility to value the current day's options.

Comparison

Table 2 lists the mean of the market option prices, the update Black-Scholes, non-update Black-Scholes, update GARCH, and non-update GARCH prices by the moneyness categories. As mentioned above, the GARCH option pricing model is computed via 50,000 Monte Carlo simulation runs. The difference between the non-update and update GARCH model is in the parameter estimates being obtained. The implied volatilities of the update Black-Scholes model were obtained from the previous day, while for the non-update model it was obtained from our option sample available in the previous day.

When we are comparing the performance of the option pricing model, there is something to note about. As we can see from Table 1, the amplitudes of option prices of the different moneyness categories are significantly different. The relative pricing error is a decreasing function of moneyness(*S/K*). It is reasonable because the option price is relative high when they are (deep) in-the-money, thus the pricing errors are relatively small to the option price. Therefore, when we compare the pricing performance, it is not proper to just judge them in either pricing error or percentage pricing error. The pricing error may be huge when the price of option is large, while the percentage pricing error may be huge when the price of option is small. The pricing error is defined as the difference between the market option price and the model-determined price while the percentage pricing error is the pricing error divided by the market price, within each moneyness category.

For a closer look, we compare the pricing performance of different moneyness categories. In Figures 5A and 5B, we visualize the comparison of Tables 4A and 4B with plots. In Figure 5A, it seems that the update GARCH model underestimates the option prices in all moneyness categories. And the Update-GARCH model performs better than the Black-Scholes model in the deep-out-of-the-money, Out-of-money, and At-the-money categories. While the update Black-Scholes model always has the best performance.

The update and non-update Black-Scholes models overestimate all the options besides deep-in-the-money. One probable explanation is the volatility smiles. When we take the "average" implied volatility as the estimate, there might be an underestimation to the implied volatility for the out-of-the-money category.

Also, the total pricing error could be eliminated once some of them were positive while some were negative. Thus, it is natural to consider absolute pricing error and the absolute percentage pricing error. The absolute pricing error, defined as the absolute value of the difference between the market option price and the model-determined price while the absolute percentage pricing error is the absolute pricing error divided by the market price, within each moneyness category. In Tables 5A and 5B, the absolute and absolute percentage pricing errors of the alternative models are shown with the corresponding plots given in Figures 6A and 6B.

In Table 5A, the update Black-Scholes model has an overall average absolute error of 15.59 NT dollars, while the non-update Black-Scholes model has 17.56 NT dollars, the non-update GARCH model has NT 20.76 dollars and update GARCH has NT 23.27 dollars.

In Table 5B, the overall absolute percentage pricing error of the non-update Black-Scholes model is 26%, while the error of the update Black-Scholes model is 23%, the non-update GARCH model is 24%, and update GARCH model is 27%. The performance of the GARCH model is similar or worse than the update Black-Scholes model.

5. Conclusion

A central hypothesis of the Black-Scholes model is that the return on the underlying asset distributed log-normally with constant volatility. However, it has been widely recognized that financial asset return processes possess heavy-tailed marginal distributions and volatility clustering. These features are interpreted as the evidence of the stochastic volatility of financial assets, and estimating the term structure of volatility has become an important issue in finance engineering.

We introduced the GARCH option pricing model of Duan (1995), using the LRNVR change measure to price options by Monte Carlo simulation. We then evaluate the empirical performance of different option pricing models on TAIEX options. We had considered the improved and constant volatility (non-update) Black-Scholes models and the update, non-update GARCH option pricing models. We then compare their pricing performance according to the absolute and percentage pricing errors.

Under Duan's model setting, we compute the option prices according to the information set of the underlying asset, say, stock index; while for the Black-Scholes the information set of index options. There are a lot of authors utilizing information from the option data, for examples, Heston and Nandi (2000), Brigo and Mercurio (2001). They estimated their model by non-linear least square method, and the performance could depend on the parameter dimension.

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Appendix A. Tables

Table 1: Summary Statistics for TAIEX Call Options (2003/9/1-2003/12/10)*

Table 1.		Moneyness (S/K)					
	Stat.	DOTM	OTM	ATM	ITM	DITM	Over all
Market Price	Average	18.35	44.16	118.15	252.45	501.58	210.52
	Std. Dev.	10.90	23.93	43.71	45.35	162.85	197.61
	Number	52	121	160	113	138	584

*The summary statistics of TAIEX call option near closing prices are reported for each moneyness category. Moneyness is defined as *S/K*, where *S* denotes the closing value of the TAIEX and *K* denotes the exercise price of the option. The sample period is from September 1, 2003 to December 10, 2003 with a total of 584 call options.

Table 2: The market option prices and the estimated prices of the alternative models+

Table 2	Moneyness (*S/K*)					
Model	DOTM	OTM	ATM	ITM	DITM	Over all
Market Price	18.35	44.16	118.15	252.45	501.58	210.52
Non-update BS	29.02	62.57	136.16	260.63	489.41	218.93
Update BS	29.34	57.76	130.36	255.40	488.15	215.06
Non-update GARCH	29.75	60.62	130.43	259.54	490.28	217.02
Update GARCH	12.72	38.28	102.10	240.08	485.80	198.28

+The GARCH option pricing model is computed via 50,000 Monte Carlo simulation runs. The only difference between the non-update and update GARCH model is in the parameters being obtained. The implied volatilities of the update Black-Scholes model were obtained from the previous day, while the non-update model was obtained from the available sample.

Table 3A: The option prices with various initial conditional variance ratios in the GARCH option pricing model (Non-Update case)

Table 3A	Moneyness (*S/K*)					
$\sqrt{h_0}/\sigma$	DOTM	OTM	ATM	ITM	DITM	Over all
0.8	29.91	60.59	130.14	258.61	511.08	235.11
1	29.75	60.62	130.43	258.65	511.20	235.22
1.2	29.93	60.89	131.02	258.48	511.20	235.41

Table 3B: The option prices with various initial conditional variance ratios in the GARCH option pricing model (Update case)

Table 3B	Moneyness (*S/K*)					
$\sqrt{h_0}/\sigma$	DOTM	OTM	ATM	ITM	DITM	Over all
0.8	12.68	37.94	101.87	239.90	507.62	217.38
1	12.72	38.28	102.10	239.76	507.51	217.45
1.2	12.62	38.23	102.25	240.50	507.44	217.59

Table 4A: The Pricing Error of Alternative Option Pricing Models

Table 4A.	Moneyness (*S/K*)					
Moneyness	DOTM	OTM	ATM	ITM	DITM	Overall
S/K	<0.95	(0.95, 0.98)	(0.98, 1.02)	(1.02, 1.05)	>1.05	
BS	10.68	18.42	18.02	8.18	-12.17	8.41
update BS	10.99	13.60	12.22	2.95	-13.43	4.54
Non-update GARCH	11.40	16.46	12.28	7.09	-11.30	6.49
Update GARCH	-5.63	-5.88	-16.05	-12.37	-15.78	-12.24

Table 4B: The Percentage Pricing Error of Alternative Option Pricing Models

Table 4B.	Moneyness (*S/K*)					
Moneyness	DOTM	OTM	ATM	ITM	DITM	Overall
S/K	<0.95	(0.95, 0.98)	(0.98, 1.02)	(1.02, 1.05)	>1.05	
BS	0.68	0.55	0.20	0.04	-0.02	0.23
update BS	0.74	0.39	0.12	0.02	-0.02	0.18
Non-update GARCH	0.57	0.30	0.10	0.03	-0.02	0.14
Update GARCH	-0.43	-0.26	-0.18	-0.05	-0.03	-0.16

Table 5A: The Absolute Pricing Error of the Alternative Option Pricing Models

Table 5A.	Moneyness (*S/K*)					
Moneyness	DOTM	OTM	ATM	ITM	DITM	Overall
S/K	<0.95	(0.95, 0.98)	(0.98, 1.02)	(1.02, 1.05)	>1.05	
BS	10.74	18.80	20.00	15.76	17.69	17.56
update BS	11.97	16.19	16.55	13.87	16.71	15.59
Non-update GARCH	12.30	20.50	22.22	21.81	21.63	20.76
Update GARCH	9.55	19.54	29.14	25.47	23.11	23.27

Table 5B: The Percentage Absolute Pricing Error of Alternative Option Pricing Models

Table 5B.	Moneyness (*S/K*)					
Moneyness	DOTM	OTM	ATM	ITM	DITM	Overall
S/K	<0.95	(0.95, 0.98)	(0.98, 1.02)	(1.02, 1.05)	>1.05	
BS	0.69	0.56	0.22	0.07	0.04	0.26
update BS	0.79	0.47	0.16	0.06	0.03	0.23
Non-update GARCH	0.66	0.47	0.20	0.09	0.05	0.24
Update GARCH	0.58	0.52	0.29	0.10	0.05	0.27

Appendix B. Figures

Figure 1: The volatility smile of the TAIEX options in September, 2, 2003.

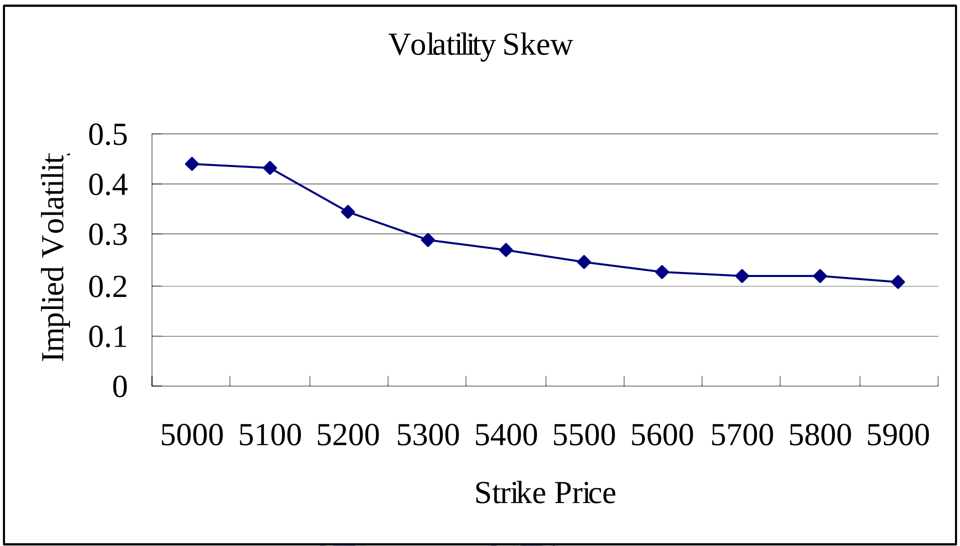


Figure 2: The Daily Closing price of TAIEX (2000/1/1 to 2003/12/10)

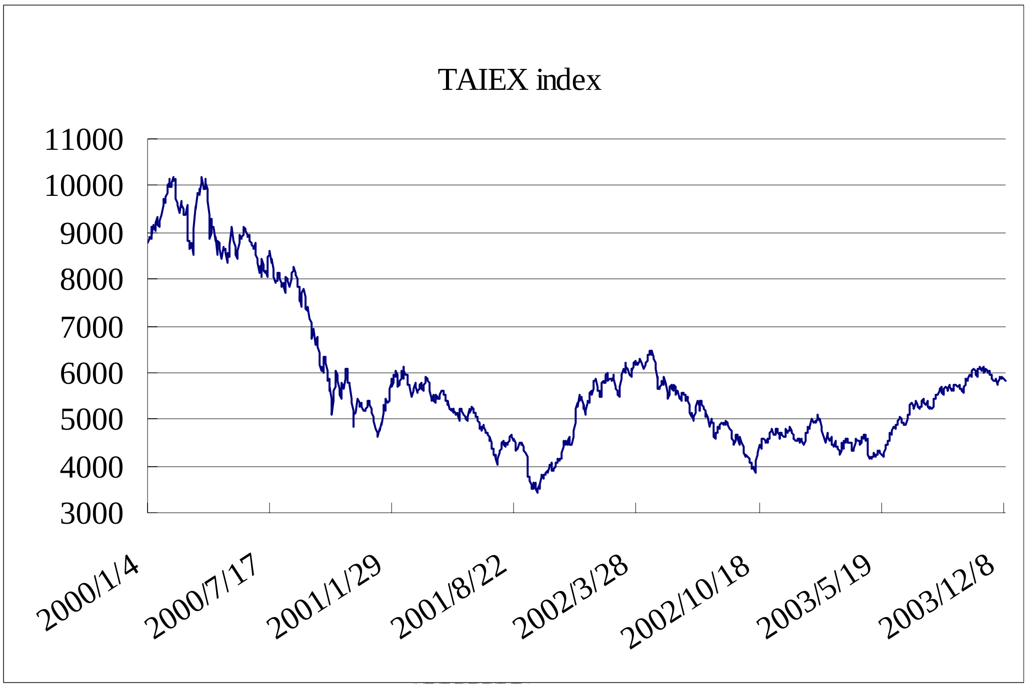


Figure 3: The Daily Log Return series of the TAIEX (2000/1/1 to 2003/12/10)

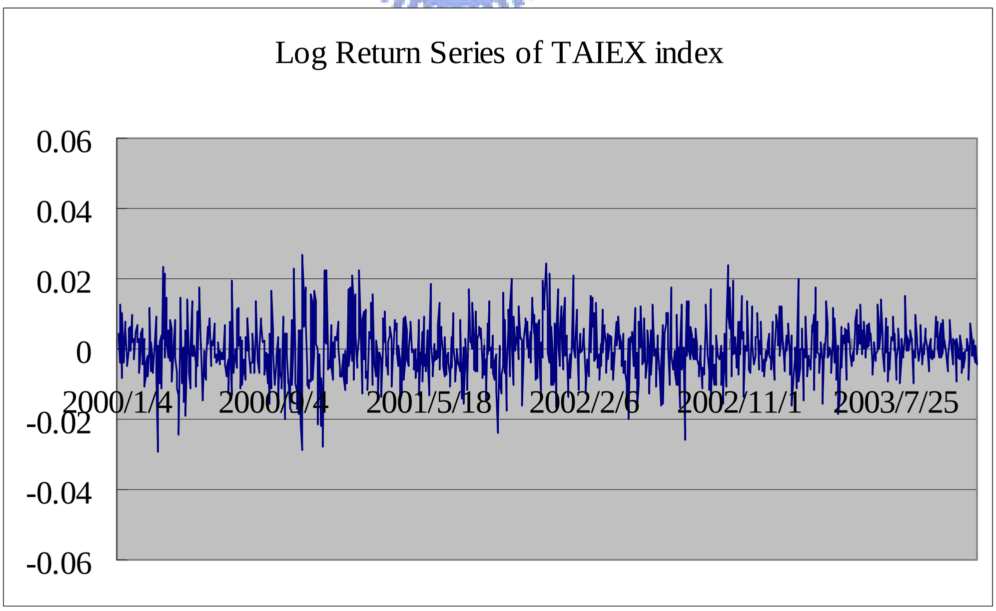
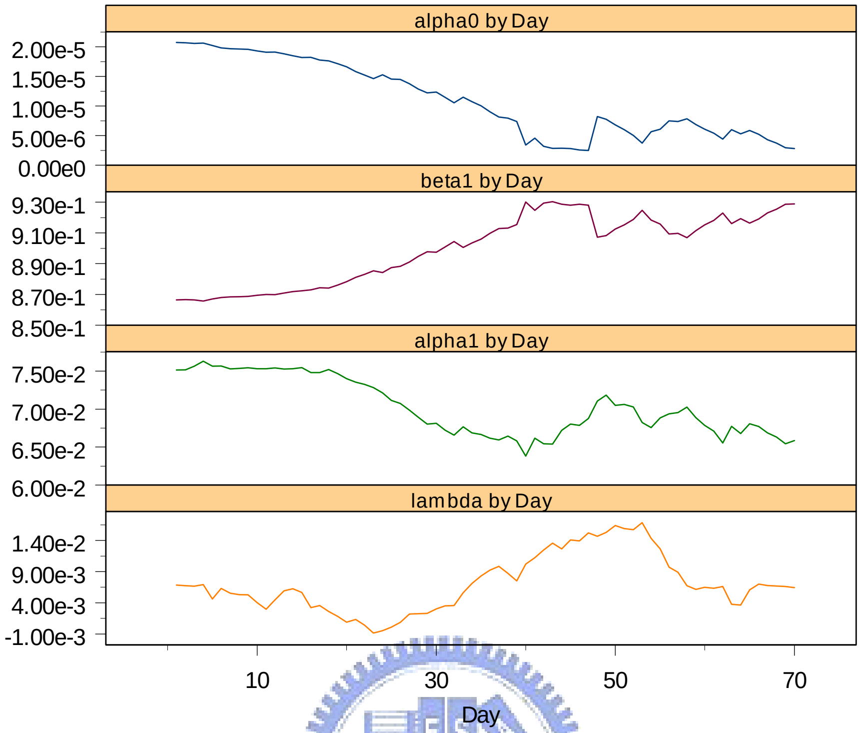


Figure 4: Maximum Likelihood Estimations of update GARCH (1, 1) process*



*The parameters of the GARCH model were obtained by constrained optimization. Since we use the "rolling the GARCH model" method, the parameter estimations changed daily. All the estimation sets satisfy the stationary GARCH conditions, i.e. $\alpha_1 + \beta_1 < 1$. The estimates of the risk-premium parameter, λ, are all very close to zero.

Figure 5A: The Pricing Error of Alternative Option Pricing Models

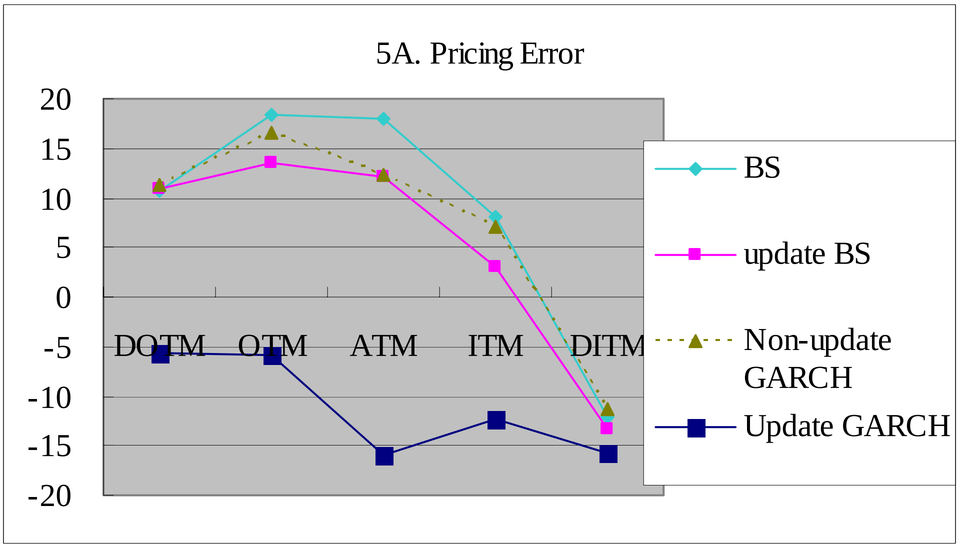


Figure 5B: The Percentage Pricing Error of Alternative Option Pricing Models

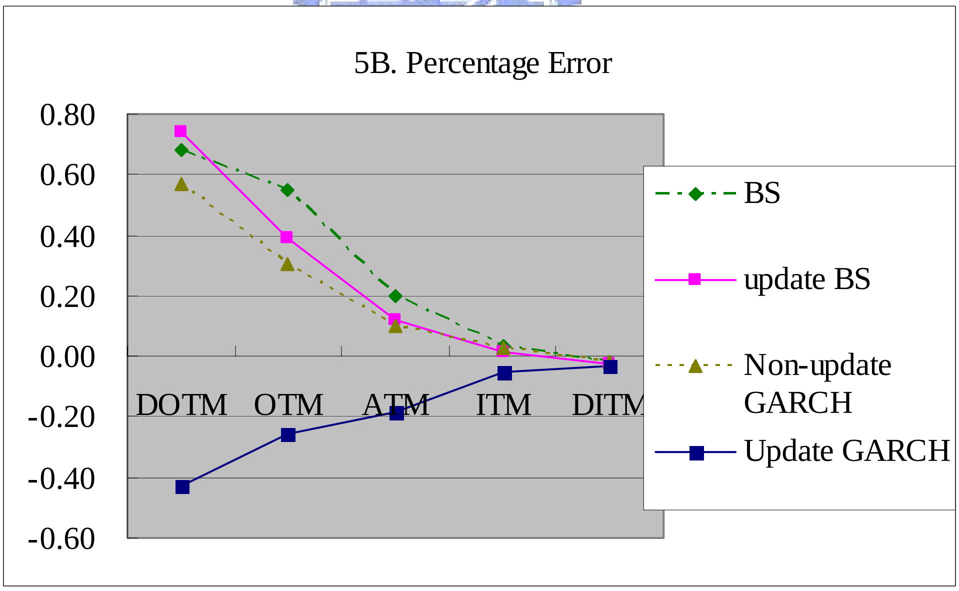


Figure 6A: The Absolute Pricing Error of Alternative Option Pricing Models.

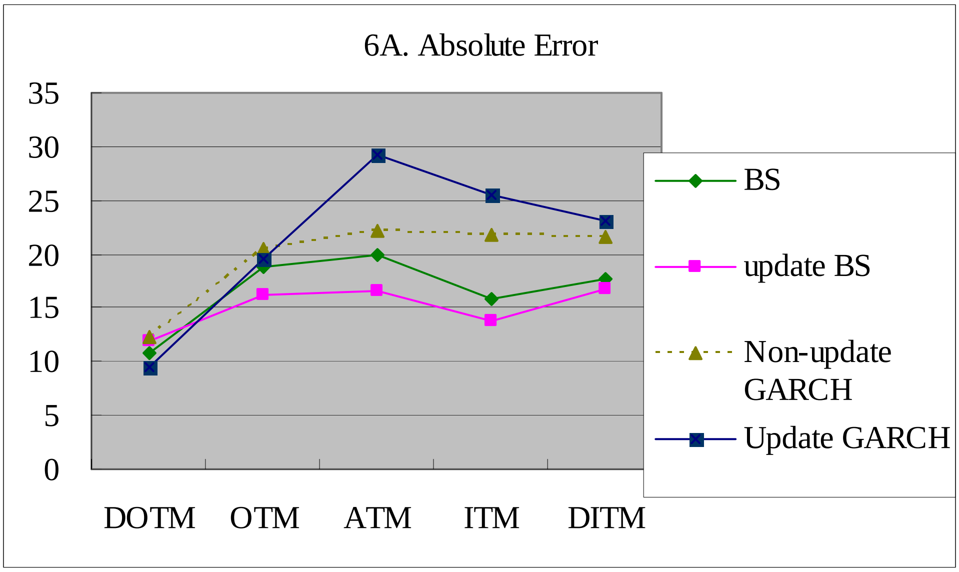


Figure 6B: The Absolute Percentage Pricing Error of Alternative Option Pricing Models.