國立交通大學

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碩士論文

隨機參數二項式選擇權定價

Binomial Option Pricing with Stochastic Parameters

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摘要

BS 模型已經被發現其隱含波動率是隨著時間變動而變動，也就是說在二項式模型當中的上漲下跌幅度參數也是應該要隨著時間變動而變動。然而在二項式選擇權訂價模型當中，依舊是假設上漲下跌幅度參數是固定常數。因此，在這篇論文中我們將探討在這些參數為隨機參數下，將如何做選擇權定價。

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ABSTRACT

The Black and Scholes (1973) model has been extended in several ways. One of these is to allow the volatility of the underlying asset to change over time. That is, binomial parameters *u* and *d* must also change over time. In this thesis, we consider inference for the binomial option pricing model with stochastic parameters.

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Contents

1. *Introduction*

The continuous-time option-pricing model (OPM) of Black and Scholes (BS) (1973) has been extended in several ways. One of these is to allow the volatility of the underlying asset to change over time (see Hull and White, 1987; Scott, 1987). This is motivated by the recent evidence that the volatility of stock prices changes over time. The closed-form solution to this stochastic volatility option pricing in general is not found and is not useful. However, a quasi-closed-form solution can be derived under certain assumptions (see Hull and White, 1987; Scott, 1987).

On the other hand, the binomial OPM of Cox, Ross, and Rubinstein (CRR) (1979), and Rendleman and Barter (RB) (1979) have been extended in the following ways. Boyle (1988) develops a procedure for the valuation of options when there are two underlying state variables. The procedure allows more than a two-point jump process. Hull and White (1988) and Omberg (1988) apply a different approach to derive the binomial OPM.

Now, we briefly review the binomial OPM of CRR and RB under fixed parameters. The n-period binomial OPM may be written as

$$C = \frac{1}{r^n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}\max[0, u^k d^{n-k}S - E], \quad (1)$$

where

C : the n-period option price,

E : the option exercise price,

S : the current stock price,

n : the number of periods to maturity,

d : one plus the percentage of downward movement in stock price,

u : one plus the percentage of upward movement in stock price,

r : one plus the riskless rate per period, and

$p = \dfrac{r-d}{u-d}$.

It is assumed that $u > r > d$; thus, $0 < p < 1$. Let m be an integer such that

$$u^{m-1} d^{n-(m-1)} S \le E < u^m d^{n-m} S .$$

That is, m is the minimum number of upward stock movements necessary for the option to terminate "in the money" (that is, $u^k d^{n-k} S - E > 0$). Under this notion, (1) may be rewritten as

$$C = S[\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} (\frac{pu}{r})^k (\frac{(1-p)d}{r})^{n-k}] - \frac{E}{r^n}[\sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}]. \qquad (2)$$

Consider the second term in brackets: it is just a complementary binomial distribution with parameters n and p, which is the sum of the binomial tail probabilities. Likewise, via a small algebraic manipulation we can show that the first term in the brackets is also a complementary binomial distribution. This can be done by defining $p^* \equiv (u/r)p$ and $1-p^* \equiv (d/r)(1-p)$. Thus, (2) can be rewritten as

$$C = SB(m;n,p`) - \frac{E}{r^n} B(m;n,p), \qquad (3)$$

where

$$B(m;n,p) = \sum_{k=m}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} .$$

CRR and RB have shown that the binomial OPM converges to the continuous time OPM of Black and Scholes (1973) as n approaches infinity. Notice that in the OPM of (1) or (2), it is assumed that the binomial parameters, u and d, are fixed and known with certainty. This implicitly assumes that the stock volatility is constant over time of the option. Given the recent evidence that stock volatility changes over time, binomial parameters u and d must also change over time.

What is lacking in the discrete time OPM is a generalized model that allows the up and down parameters to be stochastic. In general, it is assumed that these up and down parameters in a binomial distribution are fixed (i.e., known with certainty). This implicitly assumes that the volatility of the underlying stock is constant over time of the option. If the volatility of the underlying stock is assumed to change over time, the up and down parameters must be modeled as random variables.

In this thesis, we consider inference for the binomial option pricing model with stochastic parameters. In section 2, it is assumed that the up and down parameters are independent random variables following beta distributions. Under this independent beta distribution assumption, a closed-form solution can be derived. In section 3, it is assumed that functions of up, down, and riskless rate are trivariate normal-distribution. In section 4, we briefly discuss estimation of parameters and the Markov chain Monte Carlo (MCMC) methods. In section 5, we compare some numerical result. Finally, we conclude this thesis in section 6.

2. *Binomial option pricing model with stochastic parameters u and d*

In this section, we consider the inference for the binomial OPM with stochastic parameters. The binomial option price in (2) may be thought as the option price given u and d, i.e., $C(S, E, n \mid u, d)$. Expressed in this way, the binomial OPM can be easily extended to the stochastic case. That is, if the joint distribution of u and d, $f(u, d)$, are known, the stochastic binomial option price can be easily derived as

$$C(S,E,n)=\iint f(u,d)C(S,E,n\mid u,d)d\underset{\sim}{u}d\underset{\sim}{d}\ .$$

2.1 Assumptions

To derive binomial OPM formally with random parameters, we need to assume that：

(i) The representative investor is risk neutral so that r is the appropriate discount rate given jump parameters, u and d.

(ii) r is fixed and known with certainty, while u and d are stochastic.

(iii) $\frac{d}{r}$ and $\frac{r}{u}$ are independently distributed as

$$x=\frac{d}{r}\sim Beta(.\ ;\ \beta_1,\beta_2),0<x<1,$$

$$y=\frac{r}{u}\sim Beta(.\ ;\ \alpha_1,\alpha_2),0<y<1.$$

With parameter space $\Omega_x=\{(\beta_1,\beta_2);\beta_1>0,\beta_2>0\}$ and

$\Omega_y=\{(\alpha_1,\alpha_2);\alpha_1>0,\alpha_2>0\}$.

Since $u>r>d$, d/r and r/u are random variables defined between 0 and 1, Beta distributions can be suitable candidates for them.

Since

$$p=\frac{r-d}{u-d}=\frac{1-\frac{d}{r}}{\frac{u}{r}-\frac{d}{r}}=\frac{1-x}{\frac{1}{y}-x}=\frac{y(1-x)}{1-xy}\ ,$$

and

$$1-p=1-\frac{y(1-x)}{1-xy}=\frac{1-y}{1-xy},$$

(1) may be rewritten as

$$C=S\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{y^{k}(1-y)^{n-k}(1-x)^{k}}{(1-xy)^{n}}\max[0,\frac{x^{n-k}}{y^{k}}-C^{*}], \tag{4}$$

where $x=d/r, y=r/u, C^{*}=E/r^{n}S$. The option price in (4) is a function of *x* and *y*. By assumption 3 that *x* and *y* are independent beta distributions, we have the joint probability density function of *x* and *y* is

$$f(x,y)=\frac{1}{B(\alpha_1,\alpha_2)B(\beta_1,\beta_2)}x^{\beta_1-1}(1-x)^{\beta_2-1}y^{\alpha_1-1}(1-y)^{\alpha_2-1}, \tag{5}$$

where *B*(.,.) is the beta function.

2.2 The option price under stochastic parameters

Let *C*(*S, E, n* | *x, y*) be the value of (4) on the right-hand side given that random parameters *x* and *y* are known. Then the new option price, denoted by *C*, is equal to

$$C(S,E,n)=\iint f(x,y)C(S,E,n\,|\,x,y)d\underset{\sim}{x}d\underset{\sim}{y}.$$

That is,

$$\begin{aligned}
C&=\int_0^1\int_0^1 S\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{y^{k}(1-y)^{n-k}(1-x)^{k}}{(1-xy)^{n}}\max[0,\frac{x^{n-k}}{y^{k}}-C^{*}]f(x,y)dxdy\\
&=\frac{S}{B(\alpha_1,\alpha_2)B(\beta_1,\beta_2)}\int_0^1\int_0^1\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{y^{\alpha_1+k-1}(1-y)^{n+\alpha_2-k-1}x^{\beta_1-1}(1-x)^{\beta_2 k-1}}{(1-xy)^{n}}\max[0,\frac{x^{n-k}}{y^{k}}-C^{*}]dxdy\\
&=\frac{S}{B(\alpha_1,\alpha_2)B(\beta_1,\beta_2)}\times[\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\iint_{\frac{x^{n-k}}{y^{k}}>C^{*}}\frac{y^{\alpha_1-1}(1-y)^{n+\alpha_2-k-1}x^{n+\beta_1-k-1}(1-x)^{\beta_2 k-1}}{(1-xy)^{n}}dxdy\\
&\quad-C^{*}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\iint_{\frac{x^{n-k}}{y^{k}}>C^{*}}\frac{y^{\alpha_1+k-1}(1-y)^{n+\alpha_2-k-1}x^{\beta_1-1}(1-x)^{\beta_2 k-1}}{(1-xy)^{n}}dxdy].
\end{aligned} \tag{6}$$

Now denote the first integral in (6) as *A*(*k*), and the second as *B*(*k*). As shown in the appendix, they are

$$A(k)=\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{n+l-1}{l}(-1)^{j}\binom{n+\alpha_2-k-1}{j}(C^{*})^{-\frac{\alpha_1+l+j}{k}}\Gamma(\beta_2+k)\Gamma(n+\beta_1+l-k+\frac{(n-k)(\alpha_1+l+j)}{k})}{(\alpha_1+l+j)\Gamma(n+\beta_1+\beta_2+l+\frac{(n-k)(\alpha_1+l+j)}{k})},$$

to be used when $C^{*}>1$, otherwise,

$$A(k)=\sum_{l=0}^{\infty}\frac{\binom{n+l-1}{l}\Gamma(\alpha_1+l)\Gamma(n+\alpha_2-k)\Gamma(\beta_2+k)\Gamma(n+\beta_1+l-k)}{\Gamma(n+\alpha_1+\alpha_2-k+l)\Gamma(n+\beta_1+\beta_2+l)}$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{n+l-1}{l}(-1)^{j}\binom{\beta_2+k-1}{j}(C^{*})^{\frac{n+\beta_1-k+l+j}{n-k}}\Gamma(n+\alpha_2-k)\Gamma(\alpha_1+l+\frac{k(n+\beta_1-k+l+j)}{n-k})}{(n+\beta_1-k+l+j)\Gamma(n+\alpha_1+\alpha_2+l-k+\frac{k(n+\beta_1-k+l+j)}{n-k})};$$

$$B(k)=\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{n+l-1}{l}(-1)^{j}\binom{n+\alpha_2-k-1}{j}(C^{*})^{-\frac{\alpha_1+k+l+j}{k}}\Gamma(\beta_2+k)\Gamma(\beta_1+l-k+\frac{(n-k)(\alpha_1+k+l+j)}{k})}{(\alpha_1+k+l+j)\Gamma(\beta_1+\beta_2+k+l+\frac{(n-k)(\alpha_1+k+l+j)}{k})},$$

to be used when $C^{*}>1$, otherwise,

$$B(k)=\sum_{l=0}^{\infty}\frac{\binom{n+l-1}{l}\Gamma(\alpha_1+l+k)\Gamma(n+\alpha_2-k)\Gamma(\beta_2+k)\Gamma(\beta_1+l)}{\Gamma(n+\alpha_1+\alpha_2+l)\Gamma(\beta_1+\beta_2+l+k)}$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{n+l-1}{l}(-1)^{j}\binom{\beta_2+k-1}{j}(C^{*})^{\frac{\beta_1+l+j}{n-k}}\Gamma(n+\alpha_2-k)\Gamma(\alpha_1+l+k+\frac{k(\beta_1+l+j)}{n-k})}{(\beta_1+l+j)\Gamma(n+\alpha_1+\alpha_2+l+\frac{k(\beta_1+l+j)}{n-k})},$$

where $\Gamma(.)$ is the gamma function.

Following the approach in Whittaker and Watson (1962), it can be shown that $A(k)$ and $B(k)$ are absolutely convergent. Thus, the new option price can be represented as

$$C=\frac{S}{B(\alpha_1,\alpha_2)B(\beta_1,\beta_2)}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}[A(k)-C^{*}B(k)],\qquad(7)$$

where $A(k)$ and $B(k)$ are given in the Appendix. Equation (7) is a generalized binomial option price model under stochastic up and down parameters. Although complicated, $A(k)$ and $B(k)$ are double summations of absolute convergent series. Hence the computation of the option price can be carried out. The computation of $A(k)$ and $B(k)$ has been successful executed in Section 5 and verified with the results by the MCMC methods.

The results obtained so far are under the assumption of independence between

X=d/r, and *Y=r/u,* which could be unrealistic. We will next derive a new OPM when X and Y are not assumed independent. Instead, we can assume that

$$X^* = \ln(\frac{X}{1-X}), -\infty < X^* < \infty,$$
$$Y^* = \ln(\frac{Y}{1-Y}), -\infty < Y^* < \infty,$$

where

$$\begin{pmatrix} X^* \\ Y^* \end{pmatrix} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_{X^*} \\ \mu_{Y^*} \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{X^*}{}^2 & \sigma_{X^*Y^*} \\ \sigma_{X^*Y^*} & \sigma_{Y^*}{}^2 \end{bmatrix}.$$

Since X and Y are random variables defined between 0 and 1. X^* and Y^* are random variables between -∞ and ∞. Bivariate normal-distribution can be suitable candidates for X^* and Y^*.

Since

$$p = \frac{r-d}{u-d} = \frac{1-\frac{d}{r}}{\frac{u}{r}-\frac{d}{r}} = \frac{e^{Y^*}}{1+e^{X^*}+e^{Y^*}},$$

and

$$1-p = 1-\frac{e^{Y^*}}{1+e^{X^*}+e^{Y^*}} = \frac{1+e^{X^*}}{1+e^{X^*}+e^{Y^*}},$$

(1) may be rewritten as

$$C = S\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(1+e^{X^*}+e^{Y^*})^{-n} e^{kY^*}(1+e^{X^*})^{n-k} \max[0, \frac{(1+e^{-Y^*})^k}{(1+e^{-X^*})^{n-k}} - C^*],$$

where $X^* = \ln(\frac{d}{r-d})$, $Y^* = \ln(\frac{r}{u-r})$, $C^* = E/r^n S$. The option price in (8) is a function of X^* and Y^*. By assumption that X and Y are bivariate normally distributed, we have the following joint probability density function of X and Y,

$$f(X,Y) = (2\pi)^{-1} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}, \quad (9)$$

where

$$\mathbf{x} = \begin{pmatrix} X^* \\ Y^* \end{pmatrix}, \underset{\sim}{\boldsymbol{\mu}} = \begin{pmatrix} \mu_{X^*} \\ \mu_{Y^*} \end{pmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{X^*}{}^2 & \sigma_{X^*Y^*} \\ \sigma_{X^*Y^*} & \sigma_{Y^*}{}^2 \end{bmatrix}.$$

$$C = \iint S \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(1+e^{X^*}+e^{Y^*})^{-n} e^{kY^*}(1+e^{X^*})^{n-k} \max[0, \frac{(1+e^{-Y^*})^k}{(1+e^{-X^*})^{n-k}} - C^*] f(X,Y) d\underset{\sim}{X}^* d\underset{\sim}{Y}^* \cdot$$

We will show the result and the comparison between the dependent and the independent assumption in section 5.

2.3 Special cases

Now we consider two special cases if *k=0* or *k=n*, in which we have

$$A(k=0) = \sum_{l=0}^{\infty} \frac{\binom{n+l-1}{l}\Gamma(\alpha_1+l)\Gamma(n+\alpha_2)\Gamma(n+\beta_1+l)\Gamma(\beta_2)}{\Gamma(n+\alpha_1+\alpha_2+l)\Gamma(n+\beta_1+\beta_2+l)}$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{n+l-1}{l}\Gamma(\alpha_1+l)\Gamma(n+\alpha_2)(-1)^j \binom{\beta_2-1}{j}(C^*)^{\frac{n+l+\beta_1+j}{n}}}{\Gamma(n+\alpha_1+\alpha_2+l)(n+l+\beta_1+j)}, \; C^* \leq 1;$$

$$B(k=0) \quad = \sum_{l=0}^{\infty} \frac{\binom{n+l-1}{l}\Gamma(\alpha_1+l)\Gamma(n+\alpha_2)\Gamma(\beta_2)\Gamma(\beta_1+l)}{\Gamma(n+\alpha_1+\alpha_2+l)\Gamma(\beta_1+\beta_2+l)}$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{n+l-1}{l}(-1)^j \binom{\beta_2-1}{j}(C^*)^{\frac{\beta_1+l+j}{n}}\Gamma(n+\alpha_2)\Gamma(\alpha_1+l)}{(\beta_1+l+j)\Gamma(n+\alpha_1+\alpha_2+l)};$$

$$A(k=n) = \sum_{l=0}^{\infty} \frac{\binom{n+l-1}{l}\Gamma(\alpha_1+l)\Gamma(\alpha_2)\Gamma(\beta_1+l)\Gamma(\beta_2+n)}{\Gamma(\alpha_1+\alpha_2+l)\Gamma(n+\beta_1+\beta_2+l)}, C^* \leq 1;$$

$$B(k=n) \quad = \sum_{l=0}^{\infty} \frac{\binom{n+l-1}{l}\Gamma(n+\alpha_1+l)\Gamma(\alpha_2)\Gamma(n+\beta_2)\Gamma(\beta_1+l)}{\Gamma(n+\alpha_1+\alpha_2+l)\Gamma(n+\beta_1+\beta_2+l)}.$$

Next section will derive the new binomial OPM with stochastic parameters, *u*, *d*, and *r*.

3. *Binomial option pricing model with stochastic parameters u , d and r*

The binomial option price in (2) may also be thought as the option price given *u*, *d* and *r*, i.e., *C*(*S, E, n* | *u, d, r*). Expressed in this way, the binomial OPM can be easily extended to the stochastic case. That is, if the joint distribution of *u*, *d* and *r*, *f* (*u, d, r*), are known, the stochastic binomial option price can be easily derived as

$$C(S,E,n)=\iiint f(u,d,r)C(S,E,n\,|\,u,d,r)d\underset{\sim}{u}d\underset{\sim}{d}d\underset{\sim}{r}\ .$$

3.1 Assumptions

We will derive the new binomial OPM with stochastic parameters, *u*, *d*, and *r* by assuming：

$$X=\ln(\frac{X^*}{1-X^*}),-\infty<X<\infty, X^*=\frac{d}{r},$$

$$Y=\ln(\frac{Y^*}{1-Y^*}),-\infty<Y<\infty, Y^*=\frac{r}{u},$$

$$Z=\ln(\frac{r^*}{1-r^*}),-\infty<Z<\infty,\ r^*=r-1,$$

where

$$\begin{pmatrix}X\\Y\\Z\end{pmatrix}\sim N_3(\boldsymbol{\mu},\boldsymbol{\Sigma}),\quad \boldsymbol{\mu}=\begin{pmatrix}\mu_X\\\mu_Y\\\mu_Z\end{pmatrix}\text{ and }\boldsymbol{\Sigma}=\begin{bmatrix}\sigma_X{}^2 & \sigma_{XY} & \sigma_{XZ}\\ \sigma_{XY} & \sigma_Y{}^2 & \sigma_{YZ}\\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z{}^2\end{bmatrix}.$$

Since $u>r>d$, X^*,Y^*, and r^* are random variables defined between zero and one. Then *X*, *Y*, and *Z* are random variables defined between -∞ and ∞. Trivariate normal distribution would be a suitable candidate for (*X*, *Y*, *Z*). Notice that estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ can be functions of X^*,Y^* *and* r^*.

Since

$$p=\frac{r-d}{u-d}=\frac{1-\frac{d}{r}}{\frac{u}{r}-\frac{d}{r}}=\frac{1-\frac{1}{1+e^{-X}}}{1+e^{-Y}-\frac{1}{1+e^{-X}}}=\frac{e^{-X}}{e^{-X}+e^{-Y}+e^{-(X+Y)}}=\frac{e^{Y}}{1+e^{X}+e^{Y}},$$

and

$$1-p=1-\frac{e^{Y}}{1+e^{X}+e^{Y}}=\frac{1+e^{X}}{1+e^{X}+e^{Y}},$$

(1) may be rewritten as

$$C=S\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(1+e^{X}+e^{Y})^{-n}e^{kY}(1+e^{x})^{n-k}\max[0,\frac{(1+e^{-Y})^{k}}{(1+e^{-X})^{n-k}}-C^{*}], \quad (10)$$

where $X=\ln(\frac{d}{r-d})$, $Y=\ln(\frac{r}{u-r})$, $Z=\ln(\frac{r-1}{2-r})$, $C^{*}=\frac{E}{Sr^{n}}$. The option price in (10) is a function of *X*, *Y* and *Z*. By assumption that *X*, *Y* and *Z* are trivariate normally distributed, we have the following joint probability density function of *X*, *Y* and *Z*,

$$f(X,Y,Z)=(2\pi)^{-\frac{3}{2}}|\mathbf{\Sigma}|^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{'}\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\}, \quad (11)$$

where

$$\mathbf{x}=\begin{pmatrix}X\\Y\\Z\end{pmatrix},\ \boldsymbol{\mu}=\begin{pmatrix}\mu_X\\\mu_Y\\\mu_Z\end{pmatrix},\ \mathbf{\Sigma}=\begin{bmatrix}\sigma_X{}^2 & \sigma_{XY} & \sigma_{XZ}\\ \sigma_{XY} & \sigma_Y{}^2 & \sigma_{YZ}\\ \sigma_{XZ} & \sigma_{YZ} & \sigma_Z{}^2\end{bmatrix}.$$

Thus,

$$C=\iiint S\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(1+e^{X}+e^{Y})^{-n}e^{kY}(1+e^{X})^{n-k}\max[0,\frac{(1+e^{-Y})^{k}}{(1+e^{-X})^{n-k}}-C^{*}]f(X,Y,Z)d\underset{\sim}{X}d\underset{\sim}{Y}d\underset{\sim}{Z}\cdot$$

3.2 The option price under stochastic parameters

Let *C*(*S, E, n* | *X, Y, Z*) be the value of the (10) on the right-hand side given that random parameters *X*, *Y* and *Z* are known. Then the new option price, denoted by *C*, is equal to

$$C(S,E,n)=\iiint f(X,Y,Z)C(S,E,n\,|\,X,Y,Z)d\underset{\sim}{X}d\underset{\sim}{Y}d\underset{\sim}{Z}.$$

That is,

$$C=\iiint S\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(1+e^{X}+e^{Y})^{-n}e^{kY}(1+e^{X})^{n-k}\max[0,\frac{(1+e^{-Y})^{k}}{(1+e^{-X})^{n-k}}-C^{*}]f(X,Y,X)dXdYdZ\cdot \quad (12)$$

Equation (12) is a binomial option price model under stochastic up, down and riskless rate parameters. Since equation (12) is hard to expand, we will only use equation (12) to get some numerical integral result. We will show some numerical results by using MCMC in section 5.

4. *Empirical Analysis*

This section starts with an introduction of the MCMC methods.

4.1 MCMC methods

The Markov chain Monte Carlo (MCMC) methods are used in the generation of random variables and have proved extremely useful for doing complicated calculations, involving integrations and maximizations.

As the name suggests, these methods are based on Markov chain, a probabilistic structure that we have not explored. The sequence of random variables X_1, X_2, … is a *Markov chain* if

$$P(X_{k+1} \in A \mid X_1,...,X_k) = P(X_{k+1} \in A \mid X_k) \quad ,$$

that is, the distribution of the present random variable depends, at most, on the immediate past random variable. Note that this is a generalization of independence. The *Ergodic Theorem*, which is generalization of the Law of Large Numbers, says that if the Markov chain X_1, X_2,… satisfied some regularity conditions (which are often satisfied in statistical problems), then

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) \to Eh(X_i), n \to \infty \,,$$

provided the expectation exists.

To fully understand MCMC methods it is necessary to understand more about Markov chain, which we will not do here. There already a vast literature on MCMC methods, encompassing both theory and applications. Tanner (1996) provides a good introduction to computational methods in statistics, as does Robert (1994), who provides a more theoretical treatment with a Bayesian flavor. An easier introduction to this topic via the Gibbs sampler (a particular MCMC method) is given by Casella and George (1992). The Gibbs sampler is, perhaps, the MCMC method that is still the most widely used and is responsible for the popularity of this method.

Now we use the MCMC method to solve (6) in section 2 and (10) in section 3.

4.2 Description of data

First, we see the market description.

4.2.1 Market description

The Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) is the most widely quoted of all TSEC (Taiwan Stock Exchange Corporation) indices. The base year value as of 1966 was set at 100. TAIEX is adjusted in the event of new listing, de-listing and new shares offering to offset the influence on TAIEX owing to non-trading activities. TAIEX covers all of the listed stocks excluding preferred stocks, full-delivery stocks and newly listed stocks, which are listed for less than one calendar month. The TAIEX is computed by Taiwan Stock Exchange Co., Ltd. and takes as its component sample all common stocks listed for trading.

The TAIEX contract is a European option with trading during Spot month, the next two calendar months followed by two additional months from the March quarterly cycle (March, June, September, and December). The last trading day is the third Wednesday of the delivery month. Trading occurs from 8:45 AM to 1:45 PM Taiwan time Monday through Friday of the regular Taiwan Stock Exchange business days. The exercise prices are given in 100index point intervals. It is important to point out that liquidity is concentrated in the nearest expiration contract.

4.2.2 The data

For this thesis, our database is comprised of call options on the TAIEX traded daily on TSEC during the period from July 1, 2003 through October 31, 2003. Given the concentration on liquidity, our daily set of observations includes only calls with the nearest expiration day. Moreover, we eliminate all transactions taking place during the last week before expiration (to avoid the expiration-related priced price effects). We select the calls from 13:00 to 13:25, and only the last transaction for each contract. The criteria yield a final sample of 663 daily observations.

Table 1：Sample characteristics.

	Moneyness (*E*/*S*)	Average price	Number of observations
Deep OTM calls	1.03~1.08	42.6	140
OTM calls	1.01~1.03	84.4	92
ATM calls	0.99~1.01	129.32	99
ITM calls	0.97~0.99	199.33	93
Deep ITM calls	0.90~0.97	372.77	239
All calls		202.36	663

Average prices and the number of available calls are reported for each moneyness category. All calls are selected from 13:00 to 13:25, and only the last data for each strike price during the period from July 1, 2003 through October 31, 2003. *E* is the exercise price and *S* denotes the stock price. Moneyness is defined as the ratio of the exercise price to stock price (*E*/*S*).

Table 1 describes the sample properties of the call option prices used in this work. Average prices and the number of available calls are reported for each moneyness category. A call option is said to be deep out-of-the-money if the ratio $1.03 \le \frac{E}{S} < 1.08$; out-of-the-money (OTM) if the ratio $1.01 \le \frac{E}{S} < 1.03$; at the money (ATM) when $0.99 \le \frac{E}{S} < 1.01$; in-the-money (ITM) if $0.97 \le \frac{E}{S} < 0.99$; deep in-the-money if $0.90 \le \frac{E}{S} < 0.97$. As indicated, there are 663 call option observations, with OTM, ATM and ITM options, respectively, accounting for 35%, 15% and 50%. The average call prices range from 42.6 for deep OTM options to 372.77 for deep ITM options.

4.3 Estimation

To estimate the parameters α_1 、α_2 、β_1 、β_2 in equation (6) and $\boldsymbol{\mu}$ 、$\boldsymbol{\Sigma}$ in (11), the implied u (one plus the percentage of upward movement in stock price) for each of our 663 options is estimated first. In order to estimate the implied u of each option in our sample, we use call options to compute implied u for each day from July 1, 2003 through October 31, 2003 that minimized the squared error between the theoretical value according to (1) and the market price of the call options. These implied u will be used to estimate the parameters α_1 、α_2 、β_1 、β_2 in (6) and $\boldsymbol{\mu}$ 、$\boldsymbol{\Sigma}$ in (11).

Since $X = \frac{d}{r}, \mathrm{Y} = \frac{r}{u}$. We use method of moments to estimate the parameters α_1 、α_2 、β_1 、β_2. That is,

$$X \sim beta(\alpha, \beta), \hat{\alpha}_{MME} = \bar{x}(\frac{(1-\bar{x})}{S^2} - 1), \hat{\beta}_{MME} = (1-\bar{x})(\frac{(1-\bar{x})}{S^2} - 1),$$

where $\bar{x}$ is the sample mean and S^2 is the sample variance.

Using this method we can estimate the parameters as shown in Table 2.

Table 2：The binomial parameters estimated by our database.

	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$
ITM	856	1.2	880	1.3
ATM	900	1.3	928	1.4
OTM	876	1.5	892	1.6

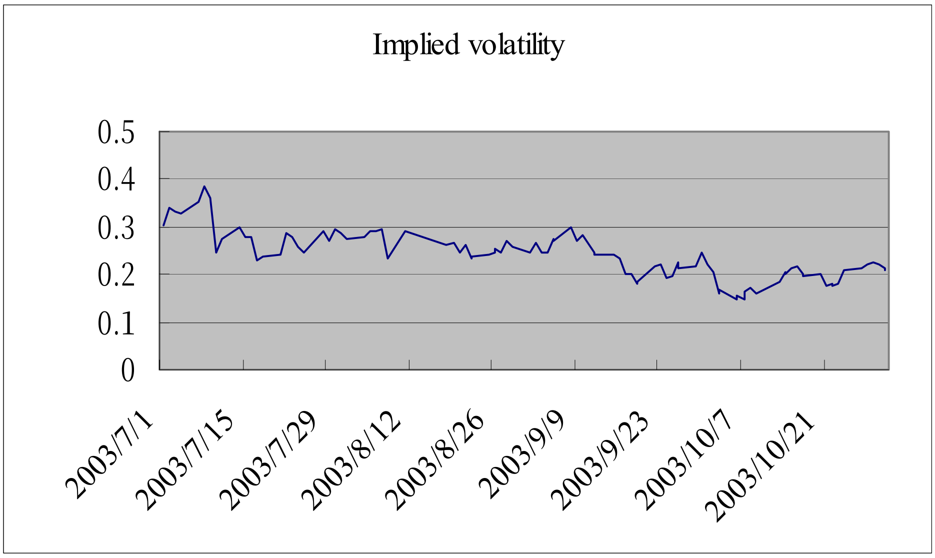
Now, we estimate the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ by $\hat{\boldsymbol{\mu}}$ = the sample mean and $\hat{\boldsymbol{\Sigma}}$ = the sample variance and covariance matrix as shown in Table 3.

Table 3：The normal parameters estimated by our database.

	$\hat{\mu}$	$\hat{\Sigma}$
ITM	$\begin{pmatrix} 8.5 \\ 7.2 \\ -9.71 \end{pmatrix}$	$\begin{bmatrix} 1.59 & 0.0167 & -1.6\times10^{-30} \\ 0.0167 & 1.4 & -1.4\times10^{-30} \\ -1.6\times10^{-30} & -1.4\times10^{-30} & 0.003 \end{bmatrix}$
ATM	$\begin{pmatrix} 8.0 \\ 7.2 \\ -9.71 \end{pmatrix}$	$\begin{bmatrix} 1.59 & 0.0167 & -1.6\times10^{-30} \\ 0.0167 & 1.4 & -1.4\times10^{-30} \\ -1.6\times10^{-30} & -1.4\times10^{-30} & 0.003 \end{bmatrix}$
OTM	$\begin{pmatrix} 7.5 \\ 7.1 \\ -9.71 \end{pmatrix}$	$\begin{bmatrix} 1.59 & 0.017 & -1.6\times10^{-30} \\ 0.017 & 1.3 & -1.3\times10^{-30} \\ -1.6\times10^{-30} & -1.3\times10^{-30} & 0.003 \end{bmatrix}$

We also need to estimate the implied volatility for the BS model. We use call options to compute implied volatility for each day from July 1, 2003 through October 31, 2003 that minimized the mean squared error between the theoretical value according to Black and Scholes (1973) option pricing formula and the market price of the call options. Figure 1 shows the implied volatility for each day from July 1, 2003 through October 31, 2003.

Figure1：The implied volatility for the BS model.



Next section, we will show the option pricing results using these parameter estimates as the parameter values.

5. *Numerical results and comparison*

This section describes the out-of-sample (the calls from 13:00 to 13:25 during the period from November 1, 2003 through November 28, 2003) comparisons of the binomial option pricing having stochastic parameters model with the BS model and the binomial option pricing model. Most of our numerical results are obtained by the MCMC methods with the exception of the binomial with stochastic parameters (u, d) in which the result given in (7) has been used to verify our MCMC results.

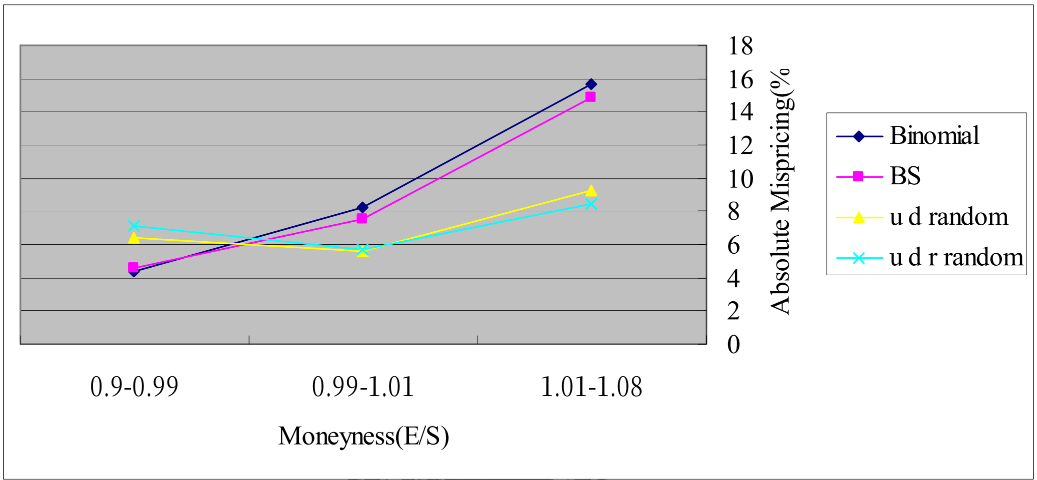
We will first check numerically the adequacy of the independence assumption between $X=d/r$ and $Y=r/u$. It turns out that absolutely percentage errors between theoretical and actual option prices are 0.0724 and 0.0719 for independent and dependent models, respectively. Hence, the two models perform almost equally for our data.

Next, Table 4 reports the out-of-sample absolute errors and absolute percentage errors for various models. The absolute errors are 9.69, 9.85, 12.17, and 13.30, respectively for the BS, binomial, binomial with stochastic parameters (u, d), and binomial with stochastic parameters (u, d, r). For the out-of-sample performance, the BS is best performer out-of-sample, and binomial with parameters (u, d, r) is worst. Looking at Figure 2, we can find that the pricing errors for the new model are bigger than BS model and Binomial model. This is why the new models have bigger absolute errors. The absolute percentage errors are 0.085, 0.089, 0.072, and 0.071, respectively for the BS model, binomial model, binomial with stochastic parameters (u, d) model, and binomial with stochastic parameters (u, d, r) model. Binomial model is worst, and binomial with parameters (u, d, r) model is best. The new models in this thesis perform better.

Table 4：Out-of-sample pricing error for option pricing models: absolute error and absolute percentage error.

	BS	Binomial	*u* and *d* are random	*u*, *d*, and *r* are random
Absolute error	9.69	9.85	12.17	13.30
Absolute percentage error	8.49%	8.86%	7.24%	7.10%

Figure 2：This figure shows the percentage out-of-sample pricing errors.



Looking at the absolute valuation errors by moneyness (see Table 5), we find that the binomial with stochastic parameters models are able to value out-of-the money options ($1.01 \le \frac{E}{S}$) better than the BS and binomial model. For example, the percentage errors for the two new models are 0.0925 and 0.0844 respectively, and 0.1489 for the BS model. For at the money options ($0.99 \le \frac{E}{S} < 1.01$), the results for the four models are all having small errors. Note that the binomial option pricing with stochastic parameters (u, d) model is the best performer among the model compared.

Table 5：This table shows the absolute percentage errors by moneyness.

	BS	Binomial	u and d are random	u, d, and r are random
OTM	14.89	15.66	9.25	8.44
ATM	7.52	8.27	5.58	5.70
ITM	4.54	4.41	6.42	7.08

6. Conclusions

This paper presents a closed-form solution for binomial option pricing model where the up and down parameters follow independent beta distribution. Moreover, we also derive the formula for binomial option pricing model where the riskless rate, and the up and down parameters are stochastic.

On average, in terms of absolute percentage error, the binomial models with stochastic parameters perform better than the BS and binomial models. Significant improvements are found in at-the-money and out-of-the-money categories for the proposed stochastic parameter models, although they do not perform well in in-the-money category.

Appendix

$$A(k) = \iint\limits_{\frac{x^{n-k}}{y^k} > C^*} \frac{y^{\alpha_1 - 1}(1-y)^{n+\alpha_2-k-1} x^{n+\beta_1-k-1}(1-x)^{\beta_2+k-1}}{(1-xy)} dxdy$$

$$= \iint\limits_{\frac{x^{n-k}}{y^k} > C^*} y^{\alpha_1 - 1}(1-y)^{n+\alpha_2-k-1} x^{n+\beta_1-k-1}(1-x)^{\beta_2+k-1} \sum_{l=0}^{\infty} \frac{(n+l-1)!}{l!(n-1)!}(xy)^l \, dxdy$$

$$= \sum_{l=0}^{\infty} \frac{(n+l-1)!}{l!(n-1)!} \int_0^1 x^{n+\beta_1+l-k-1}(1-x)^{\beta_2+k-1} \int_0^{(C^*)^{\frac{-1}{k}} x^{\frac{n-k}{k}}} y^{\alpha_1+l-1}(1-y)^{n+\alpha_2-k-1} \, dydx$$

$$= \sum_{l=0}^{\infty} \frac{(n+l-1)!}{l!(n-1)!} \int_0^1 x^{n+\beta_1+l-k-1}(1-x)^{\beta_2+k-1} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{n+\alpha_2-k-1}{j} (C^*)^{\frac{-(\alpha_1+l+j)}{k}}}{\alpha_1+l+j} x^{\frac{(n-k)(\alpha_1+l+j)}{k}} dx$$

$$= \sum_{l=0}^{\infty} \frac{(n+l-1)!}{l!(n-1)!} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{n+\alpha_2-k-1}{j} (C^*)^{\frac{-(\alpha_1+l+j)}{k}}}{\alpha_1+l+j} \int_0^1 x^{n+\beta_1+l-k-1+\frac{(n-k)(\alpha_1+l+j)}{k}} (1-x)^{\beta_2+k-1} dx$$

$$= \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{n+\alpha_2-k-1}{j} (n+l-1)!(C^*)^{\frac{-(\alpha_1+l+j)}{k}} \Gamma(n+\beta_1+l-k+\frac{(n-k)(\alpha_1+l+j)}{k})\Gamma(\beta_2+k)}{l!(n-1)!(\alpha_1+l+j)\Gamma(n+\beta_1+l+\frac{(n-k)(\alpha_1+l+j)}{k}+\beta_2)} . (A.1)$$

In the derivation we note that :

1. $(1-r)^{-n} = \sum_{l=0}^{\infty} \frac{(n+l-1)!}{l!(n-1)!} r^l$,

   $\int_0^p x^{a-1}(1-x)dx = \int_0^1 (py)^{a-1}(1-py)^{b-1} dy.$

2. $(1-py)^{b-1} = \sum_{j=0}^{\infty} \binom{b-1}{j} (-py)^j$.

Alternatively,

$$A(k)=\iint_{\frac{x^{n-k}}{y^{k}}>C^{*}}\frac{y^{\alpha_1-1}(1-y)^{n+\alpha_2-k-1}x^{n+\beta_1-k-1}(1-x)^{\beta_2+k-1}}{(1-xy)}dxdy$$

$$=\iint_{\frac{x^{n-k}}{y^{k}}>C^{*}}y^{\alpha_1-1}(1-y)^{n+\alpha_2-k-1}x^{n+\beta_1-k-1}(1-x)^{\beta_2+k-1}\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}(xy)^{l}\,dxdy$$

$$=\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}\int_0^1 y^{\alpha_1+l-1}(1-y)^{n+\alpha_2-k-1}\int_{(C^{*})^{\frac{1}{n-k}}y^{\frac{k}{n-k}}}^{1}x^{n+\beta_1+l-k-1}(1-x)^{\beta_2+k-1}dydx$$

$$=\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}\int_0^1 y^{\alpha_1+l-1}(1-y)^{n+\alpha_2-k-1}[B^{*}-\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{\beta_2+k-1}{j}(C^{*})^{\frac{n+\beta_1+l-k+j}{n-k}}}{n+\beta_1+l-k+j}y^{\frac{k(n+\beta_1+l-k+j)}{n-k}}]dy$$

$$=\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}B^{*}\int_0^1 y^{\alpha_1+l-1}(1-y)^{n+\alpha_2-k-1}dy$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{n+l-1}{l}\binom{\beta_2+k-1}{j}(C^{*})^{\frac{n+\beta_1+l-k+j}{n-k}}}{n+\beta_1+l-k+j}\int_0^1 y^{\alpha_1+l-1+\frac{k(n+\beta_1+l-k+j)}{n-k}}(1-y)^{n+\alpha_2-k-1}dy$$

$$=\sum_{l=0}^{\infty}\binom{n+l-1}{l}\frac{\Gamma(\alpha_1+l)\Gamma(n+\alpha_2-k)}{\Gamma(n+\alpha_1+\alpha_2-k+l)}B^{*}$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{n+l-1}{l}\binom{\beta_2+k-1}{j}(C^{*})^{\frac{n+\beta_1+l-k+j}{n-k}}\Gamma(\alpha_1+l+\frac{k(n+\beta_1+l-k+j)}{n-k})\Gamma(n+\alpha_2-k)}{(n+\beta_1+l-k+j)\Gamma(n+\alpha_2-k+\alpha_1+l+\frac{k(n+\beta_1+l-k+j)}{n-k})},(A.2)$$

where $B^{*}=\frac{\Gamma(\beta_2+k)\Gamma(n+\beta_1-k+l)}{\Gamma(n+\beta_1+\beta_2+l)}$.

It is noted that if $C^{*}>1$, we use $(A.1)$ to compute $A(k)$ and otherwise we use $(A.2)$ to compute $A(k)$.

$$
\begin{aligned}
B(k) &= \iint\limits_{\frac{x^{n-k}}{y^k} > C^*} \frac{y^{\alpha_1+k-1}(1-y)^{n+\alpha_2-k-1}x^{\beta_1-1}(1-x)^{\beta_2+k-1}}{(1-xy)} dxdy \\
&= \iint\limits_{\frac{x^{n-k}}{y^k} > C^*} y^{\alpha_1+k-1}(1-y)^{n+\alpha_2-k-1}x^{\beta_1-1}(1-x)^{\beta_2+k-1}\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}(xy)^l dxdy \\
&= \sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}\int_0^1 x^{\beta_1+l-1}(1-x)^{\beta_2+k-1}\int_0^{(C^*)^{\frac{-1}{k}}x^{\frac{n-k}{k}}} y^{\alpha_1+k+l-1}(1-y)^{n+\alpha_2-k-1} dydx \\
&= \sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}\int_0^1 x^{\beta_1+l-1}(1-x)^{\beta_2+k-1}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{n+\alpha_2-k-1}{j}(C^*)^{\frac{-(\alpha_1+l+k+j)}{k}}}{\alpha_1+l+k+j} x^{\frac{(n-k)(\alpha_1+k+l+j)}{k}} dx \\
&= \sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{n+\alpha_2-k-1}{j}(C^*)^{\frac{-(\alpha_1+l+k+j)}{k}}}{\alpha_1+l+k+j}\int_0^1 x^{\beta_1+l-1+\frac{(n-k)(\alpha_1+l+k+j)}{k}}(1-x)^{\beta_2+k-1} dx \\
&= \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{n+l-1}{l}\binom{n+\alpha_2-k-1}{j}(C^*)^{\frac{-(\alpha_1+l+k+j)}{k}}\Gamma(\beta_1+l+\frac{(n-k)(\alpha_1+l+k+j)}{k})\Gamma(\beta_2+k)}{(\alpha_1+l+k+j)\Gamma(\beta_1+l+\frac{(n-k)(\alpha_1+l+k+j)}{k}+\beta_2+k)} . (A.3)
\end{aligned}
$$

Alternatively,

$$B(k)=\iint_{\frac{x^{n-k}}{y^k}>C^*}\frac{y^{\alpha_1+k-1}(1-y)^{n+\alpha_2-k-1}x^{\beta_1-1}(1-x)^{\beta_2+k-1}}{(1-xy)}dxdy$$

$$=\iint_{\frac{x^{n-k}}{y^k}>C^*}y^{\alpha_1+k-1}(1-y)^{n+\alpha_2-k-1}x^{\beta_1-1}(1-x)^{\beta_2+k-1}\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}(xy)^l\,dxdy$$

$$=\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}\int_0^1 y^{\alpha_1+k+l-1}(1-y)^{n+\alpha_2-k-1}\int_{(C^*)^{\frac{1}{n-k}}y^{\frac{k}{n-k}}}^1 x^{\beta_1+l-1}(1-x)^{\beta_2+k-1}dydx$$

$$=\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}\int_0^1 y^{\alpha_1+k+l-1}(1-y)^{n+\alpha_2-k-1}[B^*-\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\beta_2+k-1}{j}(C^*)^{\frac{\beta_1+l+j}{n-k}}}{\beta_1+l+j}y^{\frac{k(\beta_1+l+j)}{n-k}}]dy$$

$$=\sum_{l=0}^{\infty}\frac{(n+l-1)!}{l!(n-1)!}B^*\int_0^1 y^{\alpha_1+k+l-1}(1-y)^{n+\alpha_2-k-1}dy$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{n+l-1}{l}\binom{\beta_2+k-1}{j}(C^*)^{\frac{\beta_1+l+j}{n-k}}}{\beta_1+l+j}\int_0^1 y^{\alpha_1+l-1+\frac{k(\beta_1+l+j)}{n-k}}(1-y)^{n+\alpha_2-k-1}dy$$

$$=\sum_{l=0}^{\infty}\binom{n+l-1}{l}\frac{\Gamma(\alpha_1+k+l)\Gamma(n+\alpha_2-k)}{\Gamma(n+\alpha_1+\alpha_2+l)}B^*$$

$$-\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{n+l-1}{l}\binom{\beta_2-k-1}{j}(C^*)^{\frac{\beta_1+l+j}{n-k}}\Gamma(\alpha_1+k+l+\frac{k(\beta_1+l+j)}{n-k})\Gamma(n+\alpha_2-k)}{(\beta_1+l+j)\Gamma(n+\alpha_2+\alpha_1+l+\frac{k(\beta_1+l+j)}{n-k})},(A.4)$$

where $B^*=\frac{\Gamma(\beta_1+l)\Gamma(\beta_2+k)}{\Gamma(\beta_1+\beta_2+k+l)}$.

Similar to $A(k)$, if $C^*>1$, we use ($A.3$) compute $B(k)$ and otherwise we use ($A.4$) to compute $B(k)$.

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