國 立 交 通 大 學

物理研究所

碩 士 論 文

玻色-愛因斯坦凝聚態在一維光晶格中的動力學

Dynamics of a Bose-Einstein condensate in one-dimensional optical lattice

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摘要

我們應用Gross-Pitaevskii equation 來描述在一維光晶格中的玻色-愛因斯坦疑聚態，基於光晶格位能的週期性，我們用Bloch function 來展開秩序參數(order parameter)可得到封包函數(envelope function)的一維非線性方程式。接著我們引用K• P 微擾近似的方法(或等效質量理論) 在特定k_0展開能量E(k)將群速度和等效質量帶入我的討論中。在Thomas- Fermi 近似和流體力學(hydrodynamic)的方法可以解得密度變化量的方程式。解此方程式可得激發模組頻率大小，發現與等效質量有關和群速的大小無關，群速度只會影響密度變化的大小。 我們也解了在自由擴散中的隨時間變化的密度函數, 我們也展示在自由擴散時密度和流量密度的變化情形。自由擴散的這些推導是在由流體力學和非線性光學中的方法而得知的。

Dynamics of a Bose-Einstein condensate in one-dimensional optical lattice

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Abstract

We applied the Gross-Pitaevskii equation for BEC condensate in 1D optical lattice. For the character of the periodic potential from optical lattice, the Bloch function is used to expand our order parameter to get a 1D nonlinear equation of the envelop function. We also introduce the K • P perturbation theory (effective mass theory) to expand the energy E to specific point k_0 and bring the group velocity and effective mass into our discussion. To solve the nonlinear differential equation of envelope function we assume the Thomas Fermi approximation and use the hydrodynamic theory to get the equation for the density fluctuation. We solve the equation and get frequency of the excitation mode and find out that the frequency depends on the effective mass and is not influenced by the group velocity. The group velocity will influence only the change of density. The time dependent density of BEC in free expansion is also solved and we show how the density and the current density vary in free expansion. They are derived under the hydrodynamics with another method from nonlinear optics.

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感謝在交大的這些日子中，江進福老師從一進交大就不斷的給我的指導和教誨，也幫助我對原子分子和 BEC 的領域有深入的了解。在此也感謝程思誠老師在研究方面的指導，使我在今年得以畢業。也謝謝資格考和畢業時都擔任我的口試委員的林貴林老師也懷念當年上電動力學的情景。

在物理所的日子中，有許多的同學和好朋友。祝福我研究室的室友宗哲和我們曾經一起渡過的無數夜晚，也謝謝文詢學姐的許多美好意見。最後感謝我的摯愛的妻子的支持以及我的家人們的關心。

于交大　2007 年 1 月

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Chapter 1   Introduction

1.1   Preface

The Bose-Einstein condensation (BEC) was first predicted out in the 1925 when Einstein devoted to the statistical description of the quanta of light. He based on the paper of the Indian physicist S.N. Bose and considered the Bose-Einstein condensation as the condensation of atoms in the state of the lowest energy associated with a phase transition. For the gas of non-interacting massive bosons he concluded that below a certain temperature, a finite fraction of the total number of particles would occupy the lowest-energy single-particle state [1].

In 1938, Fritz London suggested the connection between the superfluidity of liquid ^{4}He and the Bose-Einstein condensate. However, the interaction between helium atoms is strong, and this reduces the number of atoms in the zero-momentum state even at absolute zero. The fact that interactions in liquid helium reduce dramatically the occupancy of the lowest single-particle state led to the search for weakly interacting Bose gases with a higher condensate fraction. The difficulty with most substances is that at low temperatures they do not remain gaseous, but form solids or liquid and the effects of interaction thus become large.

The experimental studies on the dilute atomic gases were much later until the 1970 when the new techniques about magnetic trapping and advanced cooling mechanisms were developed in atomic physics. In a series of experiments hydrogen atoms were coming very close to BEC by being first cooled in a dilute refrigerator and further cooled by evaporation.

In 1980s the cooling and trapping techniques based on laser such as laser cooling and magneto-optical trapping were developed to cool and trap neutral atoms. Alkali atoms are also to be cooled by such method because their optical transition can be excited by available lasers and also have the energy-level structure which can be cooled to very low temperature.

BEC in dilute alkali gas was then first observed by the team of Cornell and Wieman at Boulder and of Ketterle at MIT in 1995. This great achievement also won the Nobel Prize in physics in 2001. Following the successful experimental observations of BECs, more physical properties of BECs had been investigated such as loading BECs in optical lattices [2] generated by interference of laser beams. The first experiment involving the dynamics of BECs in periodic potentials carried out by Anderson and Kasevich was demonstrating a mode-locked atom laser to observe atomic Josephson oscillations [3,4]. In addition, the other properties of coherent macroscopic matter waves in a lattice have been explored [5,6].

An optical lattice is practically perfect periodic potential for atoms, produced by the interference of two or more laser beams. A Bose–Einstein condensate is the ultimate coherent atom source: collective atoms, all in the same state, and with an extremely narrow momentum spread. Combining the BEC and optical lattice gives an opportunity for exploring an analog of electrons in a solid-state crystal but with unprecedented control over both lattice and the particles.

The time-dependent behavior of Bose-Einstein condensed clouds such as collective excitation modes and the expansion of a cloud releasing from a trap is an important source of information about the physical nature of the condensate.

1.2 Motivation

The study of elementary excitations is an important subject in the physics of quantum many-body system. In superfluid helium, Landau, Bogoliubov and Feynman all had pioneering theoretical work on this subject. After the experimental realization of Bose-Einstein condensation in trapped atomic gases, the study of collective excitations has become a popular subject of research in ultracold gases [7, 8]. From the theoretical side, new challenging features emerge for the nonuniform nature of these systems. From the experimental side the high precision of the measurement of collective frequencies provides a unique opportunity for a detailed comparison with theory to point out the role of the interactions and of quantum correlations.

On the other hand, Bose-Einstein condensate is a macroscopic quantum system that is easy to be exactly controlled experimentally. Therefore, many phenomena studied in solid-state system can be re-examined in a more direct and dramatic way. It is now possible to experimentally realize those model systems that had previously been studied theoretically but were impossible to test experimentally. One example of these is the experimental realization of the Hubbard model leading to the demonstration of the superfluid to Mott insulator transition in a Bose condensate of ^{87}Rb atoms [9].

Although the theory and experiment about the excitation and free expansion have been studied for a long time and many results were known, the studies about dynamics of BEC in optical lattice are still not been well discussed. The properties of elementary excitations can be investigated by considering small deviations of the state of the gas from equilibrium and finding periodic solutions of the time-dependent Gross-Pitaevskii equation.

Bose-Einstein condensate in the optical lattice can be considered as the giant matter wave moving in the periodic potential. Therefore it will have group velocity, effective mass and the band structure. We put the BEC into the one dimension optical lattice to see its excitation mode and expansion behavior. Furthermore we want to see how the group velocity υ_g and effective mass m* affect these dynamic behaviors to have better understanding the dynamic of the BEC in optical lattice.

1.3 Organization of the thesis

In the thesis, we first introduce the Gross-Pitaevskii equation and method of effective mass (K• P perturbation) method to apply to the BEC in optical lattice and the basic theory about the excitation and free expansion in hydrodynamic approach in the Chapter 2. After the discussion we get the 1D G-P equation of BEC in optical lattice and then get the differential equation for the density fluctuation and velocity.

Then in Chapter 3, we try to get the analytic solution of excitation modes. The current density and velocity in free expansion of the condensate are also solved in this Chapter. Finally we get some conclusion and further discuss the results.

Chapter 2　Theory and Methodology

At low temperature the condensate evolution is described sufficiently well by the Gross-Pitaevskii (G-P) equation, which is originally three dimensional (3D) but in the case of a cigar-shaped trap potential it is reducible to a 1D nonlinear Schrödinger (NLS). The validity is based on the condition that s-wave scattering length is much smaller than the average distance between atoms and that number of atoms in the condensate is much larger than 1 . Under this condition, the BEC in optical lattice can be simplified as the nonlinear Schrödinger equation with periodic potential. As the numerous experiments of BEC had been observed, many physical properties of BEC might be predicted and investigated to understand the fabulous phenomenon. BEC in optical lattices are affected by the structures of optical lattices. The BEC spectrum has an associated band structure .If the atomic density is high, BEC behaves nonlinearly. The properties of the atoms are characterized by the depth and period of this optically induced potential.

2-1 Bose-Einstein condensates in optical lattices

The dynamic of the Bose Einstein condensate can be described by the Gross-Pitaevskii (G-P) equation

$$i\hbar \frac{\partial \Phi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Phi(r,t) + V(r)\Phi(r,t) + g_0 \left|\Phi(r,t)\right|^2 \Phi(r,t) \tag{2.1}$$

where

$$\mathrm{V(r)} = V_1 + V_2 = \frac{1}{2} m (r_\perp^2 \omega_\perp^2 + x^2 \omega_x) + V_0 \sin^2(\frac{\pi}{L} x) \tag{2.2}$$

The $\Phi(r, t)$ here is the order parameter of the condensate, the x is the direction of the optical lattice and L is the lattice constant. The g_0 is nonlinear coefficient $g_0 = 4\pi a_s \hbar^2 / m$ and a_s is the s-wave scattering length. Here the optical lattice is periodic potential V_2.

When the condensate is under the strongly elongated trap ($\frac{w_x}{w_\perp} \sim 10^{-1}$) the model can be reduced to a one-dimensional G-P equation by the separable solution

$$\Phi(r,t) = \phi(r_\perp)\psi(x,t) \ . \tag{2.3}$$

And the $r_\perp$ direction can be described by the quantum harmonic-oscillation equation:

$$-\frac{\hbar^2}{2m}\nabla_\perp^2 \phi(r_\perp) + \frac{1}{2} m \omega_\perp^2 r_\perp^2 \phi(r_\perp) = E_\perp \phi(r_\perp) \tag{2.4}$$

The first eigen-function and eigen-value are given by

$$\phi(r_\perp)=\sqrt{\frac{m\omega_\perp}{\pi\hbar}}\exp\left[-\frac{m\omega_\perp}{2\hbar}(r_\perp^2)\right]\quad and \quad E_\perp=\hbar\omega_\perp \tag{2.5}$$

By taking the ground-state solution and the normalization condition to integrate the Eq.(2.1), we finally get the one-dimension GP equation:

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V(x)\psi(x,t)+U_0\left|\psi(x,t)\right|^2\psi(x,t) \tag{2.6}$$

Where

$$V(x)=V_1(x)+V_2(x)=\frac{1}{2}mx^2\omega^2{}_x+V_0\sin^2(\frac{\pi}{L}x) \tag{2.7}$$

The potential *V(x)* includes two parts one is the periodic lattice potential and the other is the additional external potential (confining potential) which varying slowly on the scale of the lattice period.

2-2 K • P perturbation method

Here we show how to use the K• P perturbation method (effective mass theory) to introduce the group velocity and effective mass. The G-P equation will be reduced to the equation of the envelope function by expanding the order parameter in Bloch function basis.

We first expand the stationary condensate wave function (order parameter) on the complete set of Bloch function $\phi_{n,k}(x)$. For the periodic potential, it will have the band structure.

$$\psi(x,t)=\psi(x)e^{-i\mu t/\hbar}\quad ;$$

$$\psi(x) = \sum_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A_n(k)\phi_{n,k}(x)\ dk \tag{2.8}$$

The $\phi_{n,k}(x)$ satisfy the equation

$$-\frac{\hbar}{2m}\frac{\partial^2 \phi_{n,k}(x)}{\partial x^2} + V_2\phi_{n,k}(x) = E_{n,k}(x)\phi_{n,k}(x) \tag{2.9a}$$

Where $V_2 = V_0 \sin^2(\frac{\pi}{L}x)$ is the periodic lattice potential and subscripts n and k represent the band index and quasi-momentum, respectively. The Bloch function satisfy the orthonormal condition

$$\int dx\ \phi^*_{m,k}(x)\ \phi_{n,k'}(x) = \delta_{m,n}\delta_{k,k'} \tag{2.9b}$$

The solution of Eq. (2.9a) is called the Bloch function and we use its complete set to expand the order parameter. The expansion in Eq. (2.8) is for the rapid oscillation of the condensate cloud while the slowly varying envelope function can describe the slow part of its motion. Now we start to introduce the k• p perturbation method to get the effective mass equation which is known in semiconductor physics. The "k• p" perturbation approach is a method to relate the energies and wave functions at nearby points in the quasi-momentum space. In the process, we will find out the expressions for the first and second derivatives of $E_{n,k}$ with respect to k. The power expansion around the central wave vector k_0 of the energy-band function $E_{n,k}$ will also be investigated. We proceed by writing the Bloch function $\phi_{n,k}(x)$ as

$$\phi_{n,k}(x) = e^{ikx} u_{n,k}(x)$$

where the $u_{n,k}(x+\lambda) = u_{n,k}(x)$ is periodic function.

For k is very close k_0, the $\phi_{n,k}(x)$ becomes:

$$\phi_{n,k}(x) = e^{ikx}\ u_{n,k}(x) \cong e^{ikx}\ u_{n,k_0}(x)$$
$$= \mathrm{e}^{i(k-k_0)x}\ \phi_{n,k_0}(x)$$

(2.10)

where the $|k-k_0|$ is small, the term $\mathrm{e}^{i(k-k_0)x}$ will vary slowly.

The $\psi(x)$ in Eq. (2.8) can be modified following one band approximation

$$\psi(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A(k)\ e^{i(k-k_0)x}\ \phi_{k_0}(x)\ dk = \phi_{k_0}(x) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A(k)\ e^{i(k-k_0)x}\ dk$$
$$= \left[\phi_{k_0}(x)\right] f(x)$$

(2.11a)

where

$$f(x) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A(k)\ e^{i(k-k_0)x}\ dk$$

The f(x) is the envelope function.

We now go back to the Eq. (2.6) by using

$$\psi(x,t) = e^{iEt/\hbar} \psi(x) = e^{i\mu t/\hbar} \phi_{k_0}(x) f(x) \qquad (2.11b)$$

The equation becomes:

$$E\phi_{k_0}(x)f(x)=\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V_2(x)\right]\phi_{k_0}(x)f(x)$$
$$+\left[\quad V_1(x)+U_0\left|\phi_{k_0}(x)f(x)\right|^2\right]\phi_{k_0}(x)f(x)$$

We use the power expansion around k_0 up to second order for the energy and rewrite the equation.

$$k=k_0+\delta k,$$

$$E(k)=E(k_0)+\delta k\,(\frac{\partial E(k)}{\partial k})\bigg|_{k_0}+\frac{1}{2}(\delta k)^2\,(\frac{\partial^2 E(k)}{\partial k^2})\bigg|_{k_0}$$

$$E(k)\;\phi_{k_0}(x)f(x)$$
$$=\left[E(k_0)+\upsilon_g(k_0)\hat{p}+\frac{\hat{p}^2}{2m^*}\right]\phi_{k_0}(x)f(x)$$

Then we integral the equation over coordinate x with multiplying $\phi_k{}^*(x)$ and use the normalization condition Eq. (2.9b) to get the effective mass equation

$$i\hbar(\frac{\partial f(x,t)}{\partial t}+\tilde{\upsilon}_g\frac{\partial f(x,t)}{\partial x})=-\frac{\hbar^2}{2\tilde{m}^*}\frac{\partial^2 f(x,t)}{\partial x^2}+\tilde{V}(x)f(x,t)+\tilde{U}_0\left|f(x,t))\right|^2 f(x,t) \qquad (2.12)$$

Where $\tilde{V}(x)=E(k_0)+\frac{1}{2}mx^2\omega_x^2$, $\tilde{U}_0(x)=U_0\int\left|\phi_k(x)\right|^4 dx$

Where the potential only have the confined potential term and a constant $E(k_0)$ and the mass become the effective mass $\tilde{m}*$ and here the group velocity υ_g was shown into the equation.

It is convenient to use the dimensionless quantities by

$$\tilde{t} = t/T_0 , \qquad \tilde{x} = \frac{x}{(L/2)} , \qquad \tilde{f} = \frac{f}{L_1^{1/2}}$$

$$T_0 = \frac{mL^2}{4\hbar} , \quad L_1 = \frac{\omega_\perp |a_s|}{2\hbar} mL^2 , \quad E'_0 = \frac{E(k_0)}{E_r}$$

$$E_r = 4\hbar^2 / mL^2$$

After these transformation equation (2.12) become

$$i(\frac{\partial\tilde{f}(\tilde{x},\tilde{t})}{\partial\tilde{t}} + \upsilon_g \frac{\partial\tilde{f}(\tilde{x},\tilde{t})}{\partial\tilde{x}}) = -\frac{1}{2m*}\frac{\partial^2\tilde{f}(\tilde{x},\tilde{t})}{\partial\tilde{x}^2} + V'(\tilde{x})\tilde{f}(\tilde{x},\tilde{t}) + u_0\left|\tilde{f}(\tilde{x},\tilde{t}))\right|^2 \tilde{f}(\tilde{x},\tilde{t}) \qquad (2.13)$$

Where

$V'(\tilde{x}) = E'(k_0) + \frac{1}{2}\Omega^2\tilde{x}^2;\ \ \Omega^2 = \frac{m^2\omega_x^2 L^4}{16\hbar^2}$ *and* $\upsilon_g, m*, V'$ *and* u_0 are dimensionless.

$$\upsilon_g = \tilde{\upsilon}_g \frac{mL}{2\hbar}, m* = \frac{\tilde{m}*}{m}, u_0 = U_o \frac{mL_1L^4}{4\hbar^2}$$

For convenience, we will use x, t, f for $\tilde{x}, \tilde{t}, \tilde{f}$ and all the variables in the discussion below will be dimensionless.

We begin with the equation

$$i\left(\frac{\partial f}{\partial t} + \upsilon_g \frac{\partial f}{\partial x}\right) = -\frac{1}{2m^*}\frac{\partial^2 f}{\partial x^2} + V'(x) f + u_0 |f|^2 f \qquad (2.14)$$

Where $V'(x) = E_n'(k_0) + \frac{1}{2}\Omega^2 x^2$ and $\psi(x,t) = f(x,t)\phi_{k_0}(x)\exp(iE'(k_0)t)$

Eq. (2.14) is the G-P equation under the effective mass theory.

2-3 Excitation of BEC

The Eq. (2.14) is the equation of envelope function *f*(x) which involved the group velocity and effective mass and the confined potential of BEC.

Multiply the Eq. (2.14) by f^* and use the complex conjugate of the equation, we get the equation

$$\frac{\partial n}{\partial t} + \frac{\partial (n\ v)}{\partial x} + \left(v_g \cdot \frac{\partial n}{\partial x}\right) = 0 \qquad (2.15)$$

Where $n = |f|^2, v = \frac{1}{2m^* i}\frac{f^* \frac{\partial}{\partial x} f - f \frac{\partial}{\partial x} f^*}{|f|^2} = \frac{1}{m^*}\frac{\partial \phi}{\partial x}$

We define the new velocity v' :

$$v' = v + v_g \qquad (2.16)$$

The equation then becomes

$$\frac{\partial}{\partial t}n+\frac{\partial}{\partial x}(n\ v')=0 \tag{2.17}$$

We can use the hydrodynamic theory to express the order parameter in terms of its modulus and phase.

$$f=\rho e^{i\phi} \tag{2.18}$$

Insert the Eq. (2.18) to Eq. (2.15), we have two equations from imaginary and real parts.

$$\frac{\partial\rho^2}{\partial t}=-\frac{1}{m^*}\frac{\partial}{\partial x}(\rho^2\frac{\partial}{\partial x}\phi)+(v_g\frac{\partial}{\partial x}\rho^2) \tag{2.19}$$

$$-(\frac{\partial\phi}{\partial t}+v_g\frac{\partial}{\partial x}\phi)=-\frac{1}{2m^*\rho}\frac{\partial^2}{\partial x^2}\rho+\frac{1}{2}m^*v^2+V'(r)+u_0\rho^2 \tag{2.20}$$

The Eq. (2.19) is the same equation to Eq. (2.17) and now we can rewrite the Eq. (2.20):

$$m^* \frac{\partial v'}{\partial t} = -\frac{\partial}{\partial x}(\mu' + \frac{1}{2} m^* v'^2) \tag{2.21}$$

Where $$\mu' = -\frac{1}{2m^* \sqrt{n}} \frac{\partial^2}{\partial x^2} \sqrt{n} + V'(r) + n \cdot (u_0) \tag{2.22}$$

For the excitation of BEC, we give a small perturbation quantity for the density to see how it evolves.

Here the quantities n, v', μ' can be considered as the variables (n_0, v_g and μ_0) add a small perturbation.

$$n = n_0 + \delta n, \quad v' = v_g + v, \quad \mu' = \mu_0 + \delta\mu \tag{2.23}$$

Insert back to equation and take the zero order:

$$\frac{\partial n_0}{\partial t} + \frac{\partial}{\partial x}(n_0 v_g) = 0 \tag{2.24}$$

$$\frac{\partial}{\partial x}\mu_0 = 0 \tag{2.25}$$

Then the first order:

$$\frac{d\delta n}{d\tau}+\frac{\partial}{\partial x}(n_0 v)=0 \qquad (2.26)$$

$$m^*\frac{dv}{d\tau}= -\frac{\partial}{\partial x}(\delta\mu) \qquad (2.27)$$

Where we define $\frac{d}{d\tau}=\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial x}$ (2.28)

When the number of the atoms in the trap is sufficiently large, the density becomes smooth and the kinetic energy pressure term can be neglected. That is so-called Thomas-Fermi expression:

$$n_0=\frac{1}{u_0}(\mu-V')=\frac{1}{u_0}(\mu-E_n'(k_0)+\frac{1}{2}\Omega^2x^2), \quad \delta\mu=u_0\delta n \qquad (2.29)$$

The TF approximation fails at the tails of density distribution where the density decays exponentially as $|x|\to\infty$ instead of vanishing at finite distance $|x|\to\sqrt{2(\mu-E'(k_0))}/\Omega$ (called TF radius) according to Eq. (2.29).Since a small part of the condensate's mass is concentrated in these tails, in the limit $\mu\gg\Omega$ distribution in Eq. (2.29) is assumed to be a good approximation of the initial density distribution of BEC before switching off the external potential.

Under Thomas-Fermi, we deal with the Eq. (16) and Eq. (17)

$$\frac{d^2\delta n}{d\tau^2} = \frac{u_0}{m^*}\frac{\partial}{\partial x}(n_0\frac{\partial \delta n}{\partial x}) \tag{2.30}$$

We can rewrite Eq. (2.28) by using Eq. (2.27)

$$(\frac{\partial^2}{\partial t^2} + 2v_g\frac{\partial}{\partial t}\frac{\partial}{\partial x} + v_g^2\frac{\partial^2}{\partial x^2})\delta n = \frac{(\mu - V)}{m^*}\frac{\partial^2 \delta n}{\partial x^2} - \frac{1}{m^*}\frac{\partial V}{\partial x}\frac{\partial \delta n}{\partial x} \tag{2.31}$$

Here we have find the equation governing the density change δn and we see the effective mass and group velocity are involved.

2-4 Free expansion of BEC

Now we focus to another topic of "free expansion" in the dynamics of the BEC. Here we start from the hydrodynamic description of BEC in 1D optical lattice. By the hydrodynamic representation of the G-P Equation where the order parameter was treated in terms of its modulus and phase, we have the Eq. (2.18) and Eq. (2.10). When the number of the atoms in the trap is sufficiently large (which will let the density n_0 becomes smooth), the kinetic energy pressure term can be neglected.

The two equations therefore become:

$$n_t + (n\upsilon')_x = 0 \tag{2.32}$$

$$\upsilon'_t + n_x \frac{u_0}{m^*} + \upsilon' \upsilon'_x = 0 \qquad (2.33)$$

Where $\upsilon' = \upsilon + \upsilon_g$ and initial condition $n(x,0) = n_0$, $\upsilon(x,0) = 0$

Compare the two equations above with that getting from the situation without optical lattice trap, we find that the difference is that the m become m* and the velocity *v* become *v*'. We can consider that the condensate is moving with the velocity *v* respect to a moving frame with velocity v_g.

For the situation of free expansion, the potential is switched off. The condensate evolves according to Eq. (2.14) with $\Omega = 0$. If evolution does not lead to wave breaking of the pulse, then we still can neglect the dispersion effects and use the hydrodynamics approximation to describe the evolution.

Here the equation (2.32) and (2.33) are the similar to the problem in the nonlinear optics [13, 14] for the opposite sign of the n_x in Eq(2.32) and (2.33) which corresponds to evolution of an optical beam in a focusing Kerr medium. We will apply the same method [12] to the problem of BEC expansion.

From Eq. (2.29):

$$n_0 = \frac{\mu_0 - V'(r)}{u_0} = \frac{\mu_0 - E'(k_0) - \frac{1}{2}\Omega^2 x^2}{u_0} = \frac{\mu_0 - E'_0}{u_0}[1 - \frac{\Omega^2 x^2}{2(\mu_0 - E'_0)}]$$

(2.34)

We then look for the solution of the form:

$$n(x,t) = \frac{(\mu - E_0)}{u_0 f(t)}\left(1 - \frac{\Omega^2 (x - \upsilon_g t)^2}{2(\mu - E_0) f(t)^2}\right), \tag{2.34}$$

$$\upsilon'(x,t) = (x - \upsilon_g t)\ \phi(t) + \upsilon_g \tag{2.35}$$

Where the function $f(t)$ and $\phi(t)$ satisfy the initial condition

$$f(0) = 1,\ \ \phi(0) = 0 \tag{2.36}$$

Here it is reasonable only when this function $f(t)$ will increase by the time which having the effect to make the density spread and broaden out to agree with the situation of the free expansion of BEC.

Substitution *n* and v' into the Eq. (2.32) and Eq. (2.33) and it will give the relations between $f(t)$ and $\phi(t)$:

$$\phi(t) = \frac{1}{f}\frac{\partial f}{\partial t} \tag{2.37}$$

$$f''f^2 = \frac{\Omega^2}{m^*} \tag{2.38}$$

Where $$\Omega = \frac{m\omega_x L^2}{4\hbar}$$

Now we try to solve the two equations and first we integral the Eq. (2.38).

$$(f')^2 = -\frac{2\Omega^2}{m^* f} + C \tag{2.39}$$

Using the initial condition $f(0)=1, f'(0)=2f(0)\phi(0)=0$, we then get the differential equation of the f:

$$f' = \sqrt{\frac{2\Omega}{m^*}}\sqrt{1-\frac{1}{f}} \tag{2.40}$$

Integrate the equation above:

$$\int \frac{f\,df}{\sqrt{f(f-1)}} = \int \sqrt{\frac{2\Omega}{m^*}}\,dt + c$$

Using the initial condition $f(0)=1, f'(0)=2f(0)\phi(0)=0$ the Eq. (2.37) and (2.38) can be resolved and give the Eq. (2.41) and (2.42):

$$2\sqrt{f(f-1)} + \ln\left|2\sqrt{f(f-1)} + (2f-1)\right| = 2\sqrt{\frac{2\Omega}{m^*}}t \tag{2.41}$$

Using Eq.(2.37) and Eq. (2.39), We get the ϕ as a function of f:

$$\phi = \sqrt{\frac{2\Omega}{m^*}}\,(\frac{1}{f}\sqrt{1-\frac{1}{f}}) \qquad (2.42)$$

We finally get the f and ϕ which will influence the evolution of BEC in the expansion. Some further result will be discussed in the next Chapter.

Chapter 3 Discussion and result

3-1 Excitation mode

We discuss "excitation" of the Bose Einstein condensate. The "excitation" here can be considered as "the coherent fluctuations in the density distribution". The density change is what we considered about and we have already found the equation which influenced the density distribution.

From the Eq. (2.31), we seek to find the excitation mode and see what are the factors affecting those modes. We show the Eq. (2.31) again:

$$(\frac{\partial^2}{\partial t^2} + 2v_g\frac{\partial}{\partial t}\frac{\partial}{\partial x} + v_g^2\frac{\partial^2}{\partial x^2})\delta n = \frac{(\mu - V)}{m^*}\frac{\partial^2 \delta n}{\partial x^2} - \frac{1}{m^*}\frac{\partial V}{\partial x}\frac{\partial \delta n}{\partial x}$$

We use $\delta n = A(x)\cos wt + B(x)\sin wt$ to find the collective mode of the BEC in the lattice trap.

Since the cos (wt) and sin (wt) are orthogonal, we have two equations:

$$(-w^2 A + 2v_g w\frac{\partial B}{\partial x} + v_g^2\frac{\partial^2 A}{\partial x^2})\cos wt = (\frac{(\mu - V)}{m^*}\frac{\partial^2 A}{\partial x^2} - \frac{1}{m^*}\frac{\partial V}{\partial x}\frac{\partial A}{\partial x})\cos wt$$

(2.43)

$$(-w^2 B - 2v_g w\frac{\partial A}{\partial x} + v_g^2\frac{\partial^2 B}{\partial x^2})\sin wt = (\frac{(\mu - V)}{m^*}\frac{\partial^2 B}{\partial x^2} - \frac{1}{m^*}\frac{\partial V}{\partial x}\frac{\partial B}{\partial x})\sin wt$$

(2.44)

Now we seek the series solution for A(x) and B(x) of the Eq. (2.43) and Eq. (2.44). We use the following series to describe A(x) and B(x).

$$A(x) = \sum_{i=0}^{\eta} A_{\eta-i}x^{\eta-i} = A_\eta x^\eta + A_{\eta-1}x^{\eta-1}\ldots + A_0 x^0 ,\qquad (2.45)$$

$$B(x) = \sum_{i=0}^{\eta} B_{\eta-i}x^{\eta-i} = B_\eta x^\eta + B_{\eta-1}x^{\eta-1}\ldots + B_0 x^0$$

(2.46)

At first, we put the A(x) and B(x) into the Eq. (2.43) and it then becomes

$$-w^2\sum_{i=0}^{\eta} A_{\eta-i}x^{\eta-i} + 2v_g w\sum_{i=0}^{\eta-1}(\eta-i)B_{\eta-i}x^{\eta-i-1} + v_g^2\sum_{i=0}^{\eta-2}(\eta-i)(\eta-i-1)A_{\eta-i}x^{\eta-i-2}$$
$$=\frac{\Omega^2}{2m*}[\lambda^2\sum_{i=0}^{\eta-2}(\eta-i+2)(\eta-i+1)A_{\eta-i}x^{\eta-i-2} - \sum_{i=1}^{\eta}(\eta-i)(\eta-i-1)A_{\eta-i}x^{\eta-i}]$$
$$-\frac{\Omega^2}{m*}\sum_{i=1}^{\eta-1}(\eta-i)A_{\eta-i}x^{\eta-i} \tag{2.47}$$

where

$$V = E(k_0) + \frac{1}{2}\Omega^2 x^2,\ \ \Omega = \frac{m\omega_x L^2}{4\hbar},\ \ \ \lambda^2 = (X^2 - \frac{2E(k_0)}{\Omega^2})$$

The equation can change as

$$-w^2\sum_{i=0}^{\eta} A_{\eta-i}\, x^{\eta-i} + 2v_g w\sum_{i=1}^{\eta}(\eta-i+1)\, B_{\eta-i+1}\, x^{\eta-i} + v_g^2\sum_{i=2}^{\eta}(\eta-i+2)(\eta-i+1)\, A_{\eta-i+2}\, x^{\eta-i}$$
$$=\frac{\Omega^2}{2m*}[\lambda^2\sum_{i=2}^{\eta}(\eta-i+2)(\eta-i+1)\, A_{\eta-i+2}\, x^{\eta-i} - \sum_{i=0}^{\eta-2}(\eta-i)(\eta-i-1)\, A_{\eta-i}\, x^{\eta-i}]$$
$$-\frac{\Omega^2}{m*}\sum_{i=0}^{\eta-1}(\eta-i)\, A_{\eta-i}\, x^{\eta-i} \tag{2.48}$$

The equation for all value of x can exist only when each of the coefficients equals to zero. We begin to use i = 0 and set the coefficient of the x^η to be zero.

$$-w^2 A_\eta - \frac{\Omega^2}{2m*}\left[-\eta(\eta-1)A_\eta\right] + \frac{\Omega^2}{m*}\eta A_\eta = 0$$

$$\left\{-w^2 + \frac{\Omega^2}{2m*}[\eta(\eta+1)]\right\}A_\eta = 0$$

When $A_\eta \neq 0$,

$$w^2 = \frac{\Omega^2}{2m^*}[\eta(\eta+1)],\ \eta = 1,2.. \tag{2.49}$$

We also can change the frequency to original form and can compare to the dimensionless form above.

$$w = \sqrt{\frac{1}{2}\eta(\eta+1)}\ \sqrt{\frac{m}{m^*}}\ w_x\ ,\ \eta = 1,2... \tag{2.50}$$

Here we have the low lying excitation modes by putting $\eta = 1,2$ and the frequencies are $\sqrt{\frac{m}{m^*}}\ w_x$ and $\sqrt{\frac{3m}{m^*}}\ w_x$.

Such result can compare the result for the papers about the excitation of Bose-Einstein condensate. In Ref. [10] for the quasi-1D system and $w_\perp / w_x \gg 1$, the result shows that the collective modes are $\Omega_l = \sqrt{\frac{1}{2}l(l+1)}\ w_x$, l=1, 2....

The influence of adding the optical lattice only has the difference by multiplying the factor $\sqrt{\frac{m}{m^*}}$.

To this result of our research it also shows that the group velocity υ_g does not affect the collective modes. In Fig. 1 we show the relation between the collective mode and effective mass where the frequency of the mode is proportional to the inverse of square root of effective mass.

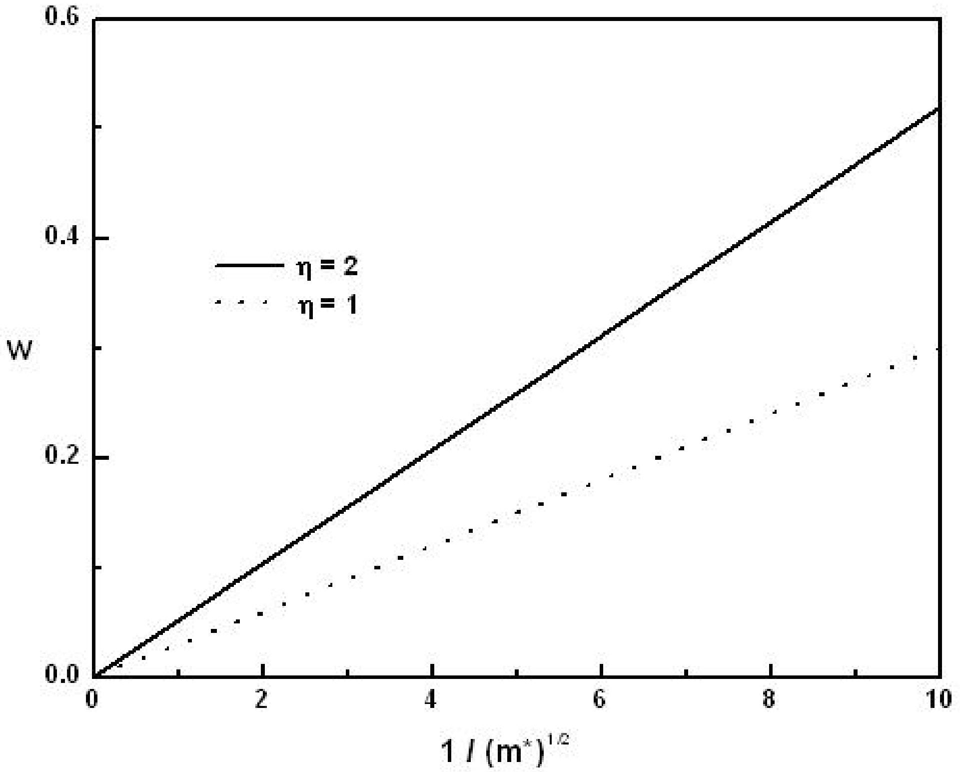

Fig. 1 The excitations of the lowest two modes are shown. It takes η =1,2 to the Eq. (2.35) and frequency is proportional to inverse of the square root of effective mass.

Furthermore, we let the coefficient of the $x^{\eta-1}$ to be zero by set i =1 to collect the terms of $x^{\eta-1}$.

$$-w^2 A_{\eta-1} + (2v_g w)\eta B_\eta + \frac{\Omega^2}{2m^*}[-(\eta-1)(\eta-2)A_{\eta-1}] + \frac{\Omega^2}{m^*}(\eta-1)A_{\eta-1} = 0 \tag{2.51}$$

$$\eta\left(-\frac{\Omega^2}{2m^*}A_{\eta-1} + 2v_g w B_\eta\right) = 0 \tag{2.52}$$

We then find the relation of B_η and $A_{\eta-1}$.

$$B_\eta = \frac{\Omega^2}{2m^* v_g w} A_{\eta-1} \tag{2.53}$$

Noticed the η here is a parameter and B_η and A_η is the highest term of each series.

Finally, we can find the formula by use $i = n$ and $\eta \geq n \geq 2$ we get another formula

$$\left[\frac{\Omega^2}{2m^*}(n)(n-2\eta-1)\right] A_{\eta-n} + (\eta-n+2)(\eta-n+1)\left[\upsilon_g - \frac{\Omega}{2m^*}\right] A_{\eta-n+2} + 2\upsilon_g w(\eta-n+1) B_{\eta-n+1} = 0 \tag{2.54}$$

Repeat the process to another formula of Eq. (2.43) and get the two another formula of the relation of series. Collect these four formulas

$$A_\eta = -\frac{\Omega^2}{2m^* v_g w} B_{\eta-1} \tag{2.55}$$

$$B_\eta = \frac{\Omega^2}{2m^* v_g w} A_{\eta-1} \tag{2.56}$$

$$\left[\frac{\Omega^2}{2m^*}(n)(n-2\eta-1)\right] A_{\eta-n} + (\eta-n+2)(\eta-n+1)\left[\upsilon_g - \frac{\Omega}{2m^*}\right] A_{\eta-n+2} + 2\upsilon_g w(\eta-n+1) B_{\eta-n+1} = 0 \tag{2.57}$$

$$\left[\frac{\Omega^2}{2m^*}(n)(n-2\eta-1)\right] B_{\eta-n} + (\eta-n+2)(\eta-n+1)\left[\upsilon_g - \frac{\Omega}{2m^*}\right] B_{\eta-n+2} - 2\upsilon_g w(\eta-n+1) A_{\eta-n+1} = 0$$

(2.58)

We then use η =2 for the mode $w = \sqrt{\frac{3m}{m^*}}\, w_x$ to find the series of A(x) and B(x). The result is

$$\delta n = A(x)\cos wt + B(x)\sin wt$$
$$= \left[(\frac{5}{3}\chi + \frac{\lambda^2}{18\chi})e + bx + (-\frac{1}{6\chi})e\, x^2 \right]\cos wt + \left[(-\frac{5}{3}\chi - \frac{\lambda^2}{18\chi})b + ex + (\frac{1}{6\chi})b\, x^2 \right]\sin wt$$

(2.59)

where

$$A(x) = (\frac{5}{3}\chi + \frac{\lambda^2}{18\chi})e + bx + (-\frac{1}{6\chi})e\, x^2$$
$$B(x) = (-\frac{5}{3}\chi - \frac{\lambda^2}{18\chi})b + ex + (\frac{1}{6\chi})b\, x^2$$

$\chi = \frac{\upsilon_g}{w}$ and b, e are constants.

We can see where the group velocity will have the influence and the group velocity can let the condensate looking like moving with velocity υ_g respect to the rest frame. In the later discussion we will see how the group velocity affects dynamics of the condensate.

Normalization condition can help to decide the parameters b and e.

$$\psi(x) = \; = \left[\phi_{k_0}(x) \right] f(x) \qquad (2.60)$$

where $\phi_{k_0}(x)$ is the Bloch function with k=k$_0$ and the function satisfy the equation

$$\left[-\frac{1}{2m}\frac{\partial^2}{\partial x^2}+V_0\sin^2(\frac{\pi}{2}x)\right]\phi_{k_0}(x)=E(x)\phi_{k_0}(x) \tag{2.61}$$

$$\phi_{k_0}(x)=e^{ik_0x}(\alpha+\beta e^{-i\pi x}) \tag{2.62}$$

Where $\alpha=\frac{1}{\sqrt{2\pi}}\frac{1}{[1+\frac{4}{V_0}(k_0^2+V_0-2E)]^{1/2}}$

$\beta=\frac{2\alpha}{V_0}(k_0^2+V_0-2E)$

We use the normalization condition the total number to be N.

$$\int|\psi(x,t)|^2dx=N\ ,\quad \int n_0\cdot|\varphi_{k_0}(x)|^2dx=N \tag{2.63}$$

The formula can change to include the small perturbation:

$$\int|\psi(x,t)|^2dx=\int(n_0+\delta n)\cdot|\varphi_{k_0}(x)|^2dx=N \tag{2.64}$$

We finally get the equation below:

$$\int\delta n\cdot|\varphi_{k_0}(x)|^2dx=0 \tag{2.65}$$

When t =0 $\delta n(x,0)=A(x)$ and satisfy the normalization condition

$$\int A\ \left|\varphi_{k_0}(x)\right|^2 dx = \int \{(\frac{5}{3}\chi\frac{\lambda^2}{18\chi})e + bx - (\frac{1}{6\chi})ex^2\}[(\alpha+\beta e^{-i\pi x})(\alpha+\beta e^{i\pi x})]^2 dx = 0$$

(2.66)

The condition for the integral to be zero is only when $e = 0$.

The equation of Eq. (2.59) then becomes:

$$\delta n = b\left\{x\cdot\cos wt + [(-\frac{5}{3}\chi - \frac{\lambda^2}{18\chi}) + \frac{1}{6\chi}x^2]\sin wt\right\} \tag{2.67}$$

where $\chi = \frac{\upsilon_g}{w} = \frac{\upsilon_g m^*}{\sqrt{3}\ \Omega}$.

We try to plot the δn respect to x in Eq. (2.67) with the $wt = \frac{\pi}{4}$. The result is shown in Fig. 2.

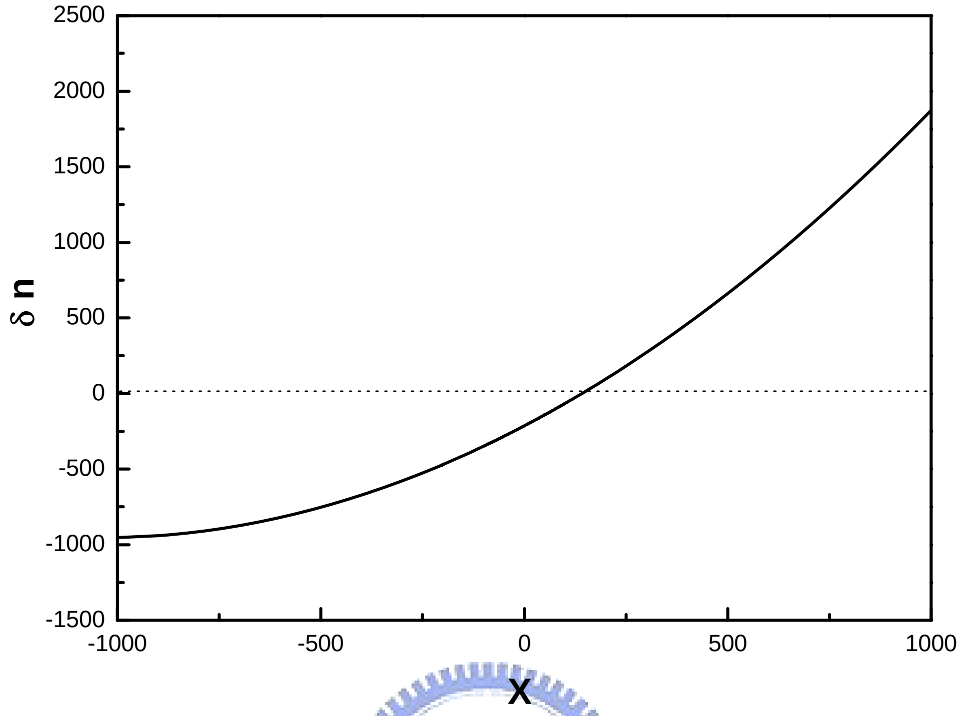

Fig.2 The density fluctuation in x direction is not symmetric with the zero point. The possible reason is that the group velocity makes the condensate move in the x direction.

3-2 Current density in free expansion

Now we start to get some result in free expansion. We found that substitute $f(t)$ and $\phi(t)$ of Eq. (2.40) and (2.41) into Eq.(2.34) and Eq.(2.35) will give us the $n(x,t)$ and $\upsilon'(x,t)$.

Here are those equations:

$$n(x,t) = \frac{(\mu - E_0)}{u_0 f(t)}(1 - \frac{\Omega^2 (x - \upsilon_g t)^2}{2(\mu - E_0) f(t)^2}),$$

$$\upsilon'(x,t) = (x - \upsilon_g t)\ \phi(t) + \upsilon_g$$

where

$$2\sqrt{f(f-1)} + \ln\left|2\sqrt{f(f-1)} + (2f-1)\right| = 2\sqrt{\frac{2\Omega}{m^*}}\ t$$

$$\phi = \sqrt{\frac{2\Omega}{m^*}}\ (\frac{1}{f}\sqrt{1-\frac{1}{f}})$$

We plot the f(t) and t and find the f(t) is almost linear to t in Fig 3. When time is larger it can consider as a line with slope about 1.38.

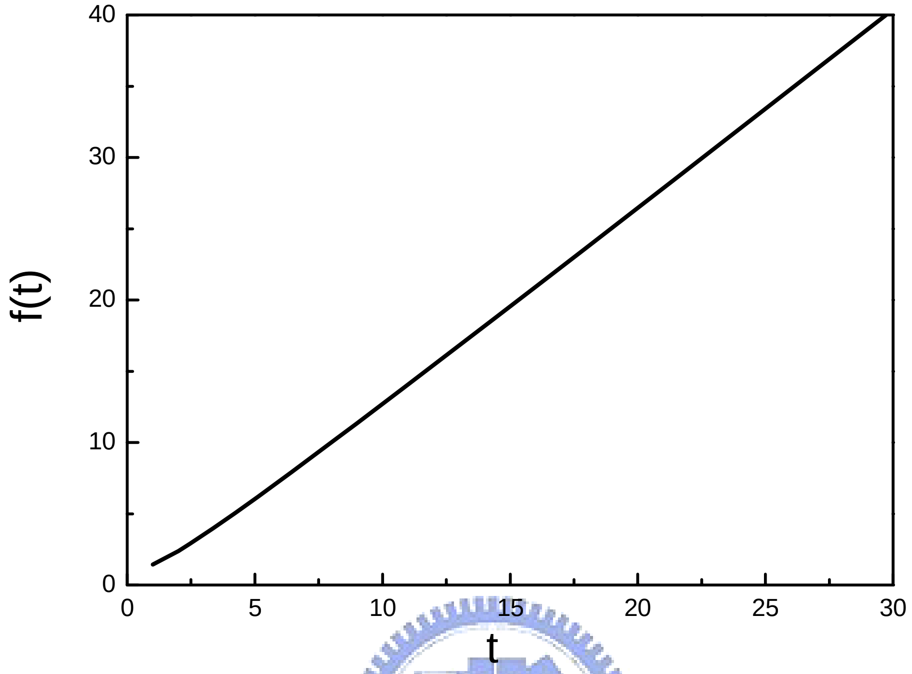

Fig. 3 f(t) as a function of time. It is almost a straight line beside the point near origin point.

Then we show in Fig. 4 the density *n* and position *x* at the t =1 (which is dimensionless unit = 0.3ms) k=0.7854, E_0=0.7834, v_g=2.2623, f(1)=1.4376

The density of condensate for different time is shown in Fig. 5. Therefore, we can know how the density changes in the condition of free expansion.

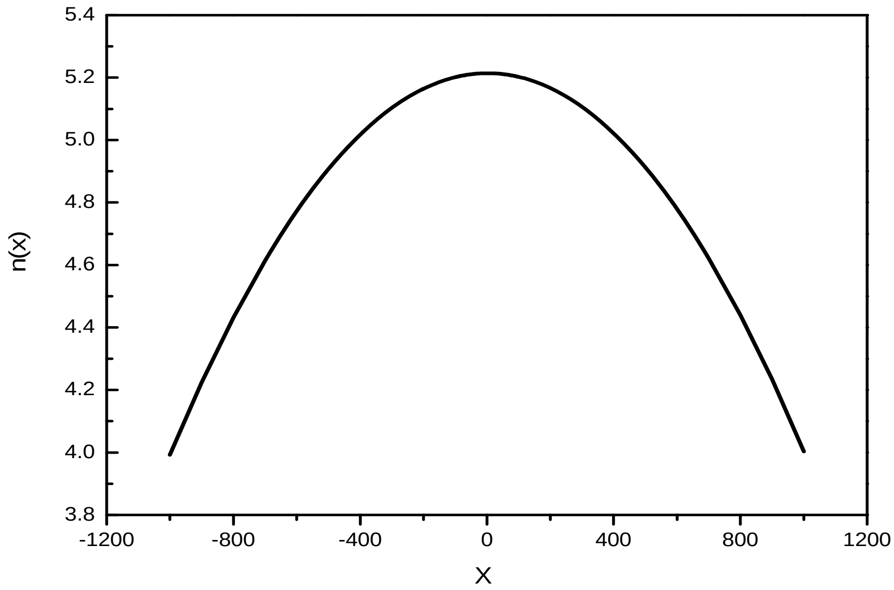

Fig. 4 The function of density for different x when t =0.3 *ms*.

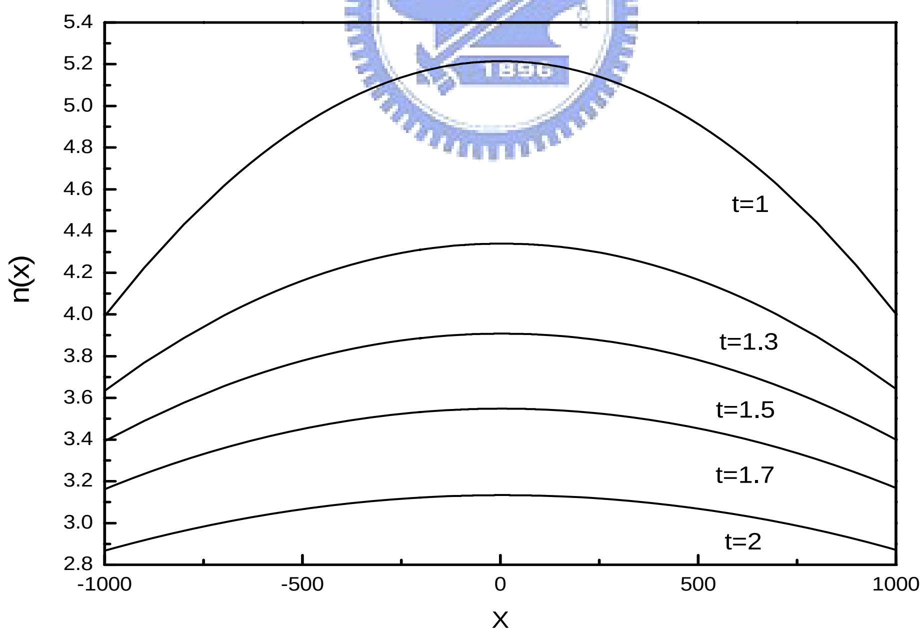

Fig.5 The density change with respect to time. Where time scale is for the dimensionless unit and one unit is equal to 0.3 *ms*. . The density becomes flat and starts the free expansion for large time.

If the optical lattice was removed, the result will be

$$n(x,t) = \frac{\mu}{2f(t)}(1 - \frac{\Omega^2 x^2}{2\mu f(t)^2}) \quad (2.42)$$

$$\upsilon'(x,t) = x\,\phi(t) \quad (2.43)$$

Now we can get the condensate front velocity by using the Eq. (2.35), Eq. (2.36), Eq. (2.37) and Eq. (2.38).

$$\upsilon_f(t) = 2\sqrt{\frac{(\mu - E_0)}{m^*}(1 - \frac{1}{f})} \quad (2.44)$$

We can also calculate the current density $J(x,t)$:

$$J(x,t) = \frac{(\mu - E_0)}{U_0 f(t)}[(1 - \frac{\Omega^2 (x - \upsilon_g t)^2}{2(\mu - E_0) f(t)^2})((x - \upsilon_g t)\phi + \upsilon_g)] \quad (2.45)$$

We plot the current density for different k (wave number) in Fig.6 and it show how the increase of k will make the current density shift from the origin. The group velocity for different k is also shown in Fig.7. It give us the conclusion the higher group velocity will give the current density more shift from origin. Therefore it also means the amount of atoms move toward the direction +x and –x is not equal for group velocity is not zero.

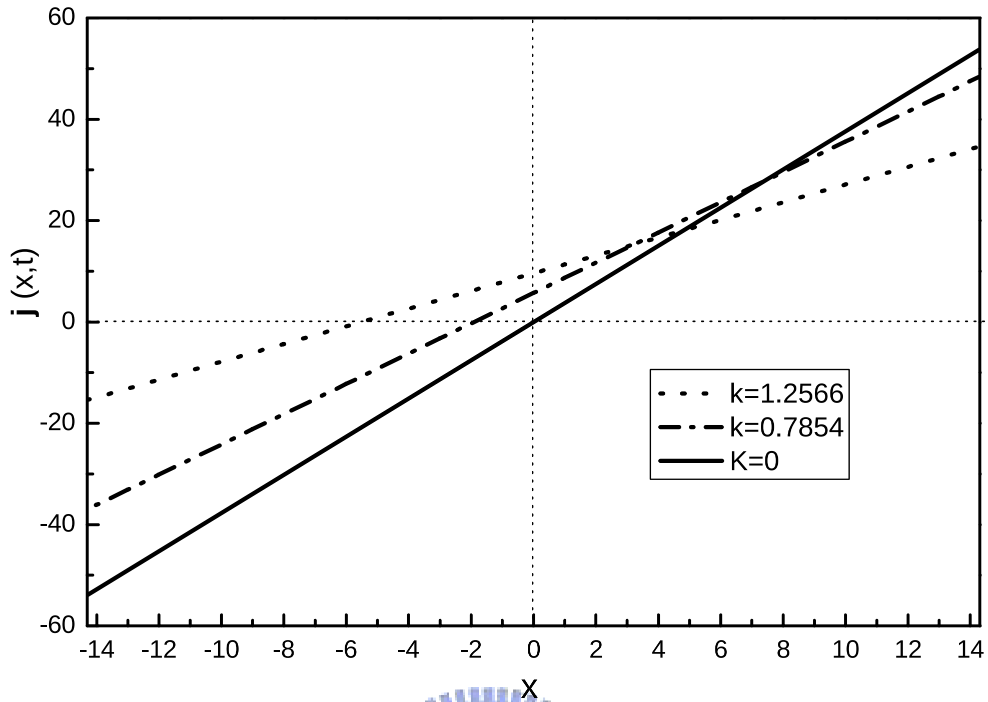

Fig. 6　Current density for different k (wave vector) which related to different group velocity and effective mass. For k=0 it pass the point of origin and for high k also the velocity is high the current density that positive part is higher than negative part and where t =1 (0.3ms) and f =1.4376.

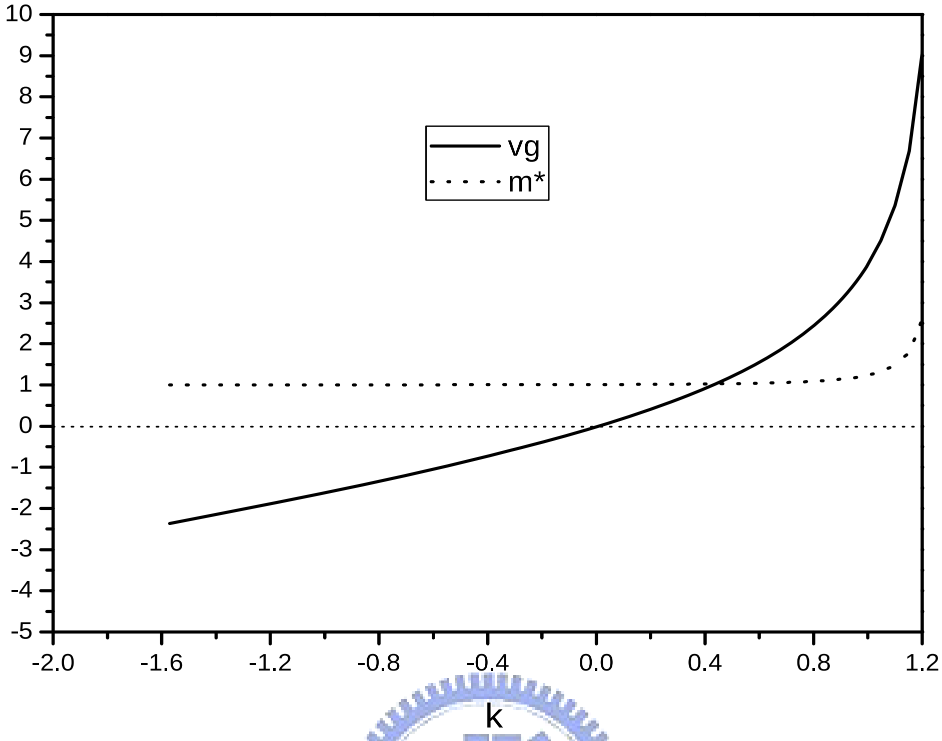

Fig. 7 The group velocity and effective mass for different k are shown. T he group velocity increase proportional to k and the effective mass is almost constant and rise up dramatically only when it above 1.0.

From Fig.7, we find the group velocity is negative for negative k. The current density may therefore have the different change for the negative k. We then show the current density for –k and the result is shown in Fig.8. It shows that for negative group velocity there will have another shift of the current density and it means the amount of atoms moving toward the direction -x is more than the direction of +x while the group velocity is negative.

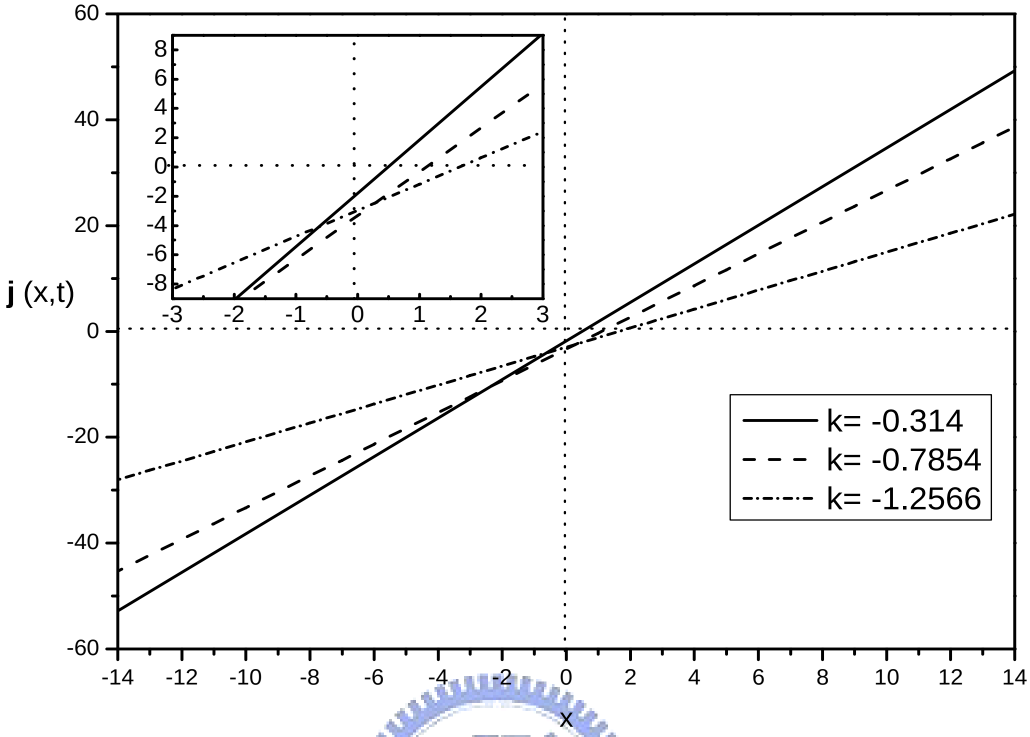

Fig. 8 The current density for negative k is shown. We can see the negative k make the current density have the shift from the origin in different way comparing with positive k.

In asymptotic limit of large time (when $f(t) \gg f(0) = 1$), the Eq. (2.40) and Eq. (2.40) can be reduced:

$$f = \sqrt{\frac{2\Omega}{m^*}}\, t \,, \qquad \phi = {}^{1}\!/_{t} \tag{2.46}$$

In this limit the density and the velocity can be expressed by

$$n(x,t) = \frac{(\mu - E_0)}{U_0 t}\sqrt{\frac{m^*}{2\Omega}}\,(1 - \frac{(x-\upsilon_g t)^2 m^*}{4(\mu - E_0)\, t^2}) \tag{2.47}$$

$$\upsilon'(x,t) = \frac{x}{t} \tag{2.48}$$

The current density will be

$$J(x,t) = \frac{(\mu - E_0)}{U_0 t}\sqrt{\frac{m^*}{2\Omega}}\,[(1 - \frac{(x-\upsilon_g t)^2 m^*}{4(\mu - E_0)\, t^2})((\frac{x}{t} - \upsilon_g) + \upsilon_g)]$$

Chapter 4 Conclusion and Perspective

4-1 Conclusion

In the thesis, we use the G-P equation and K•P perturbation method to study the dynamic of BEC in one dimensional optical lattice. Our approach was under the method of the hydrodynamics. We get analytic solution of the excitation mode and realize the frequency is related to both the frequency of optical lattice and effective mass, while the group velocity affects the density fluctuation. We then introduce two time dependent function f(t) and ϕ(t) to seek the time dependent function of density and velocity for free expansion and finally get the result of the analytic from. We then know how the density and velocity change in free expansion.

The current density can also be derived and we show the influence on current density from group velocity.

4-2 Perspective

We have applied the K•P perturbation method (effective mass theory) to dynamics of BEC condensate in 1D optical lattice. We get some results for excitation mode and also know more about free expansion. In future work, we can try to solve the same question for 3D system instead of the strong confinement in transverse direction to compare the result. Our method (effective mass theory) is only be used in the situation of optical potential not too high and we can also search for another method for the situation for highly trap such as tight binding method to compare the result. Also the researches related to BEC in optical lattice are large and many related topics can be also discussed.

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