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Twisted Hecke L-values and period polynomials ☆

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A B S T R A C T

Let $f_1, \dots, f_d$ be an orthogonal basis for the space of cusp forms of even weight $2k$ on $\Gamma_0(N)$. Let $L(f_i, s)$ and $L(f_i, \chi, s)$ denote the L-function of f_i and its twist by a Dirichlet character χ, respectively. In this note, we obtain a "trace formula" for the values $L(f_i, \chi, m)\overline{L(f_i, n)}$ at integers m and n with $0 < m, n < 2k$ and proper parity. In the case $N = 1$ or $N = 2$, the formula gives us a convenient way to evaluate precisely the value of the ratio $L(f, \chi, m)/L(f, n)$ for a Hecke eigenform f.

1. Introduction and statements of results

Let $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$ be a Hecke eigenform of even weight $2k$ on $\Gamma_0(N)$ and $f_\chi(z) = \sum_{n=1}^{\infty} a_f(n)\chi(n) e^{2\pi i n z}$ be its twist by a Dirichlet character χ. The L-function, defined by $L(f, s) = \sum_{n=1}^{\infty} a_f(n) n^{-s}$ and extended analytically to the whole complex plane, and its twist $L(f, \chi, s) = \sum_{n=1}^{\infty} a_f(n)\chi(n) n^{-s}$ are very important number theoretical objects. For instance, when $f(z)$ is the weight 2 newform associated to a rational elliptic curve E, the Birch and Swinnerton–Dyer conjecture asserts that the rank of the group of rational points $E(\mathbb{Q})$ on E is equal to the order of $L(f, s)$ at $s = 1$.

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In this article, we are concerned with the values of $L(f,s)$ and $L(f,\chi,s)$ at integers inside the critical strip $0<\operatorname{Re} s<2k$. In [14], Manin showed that for a normalized Hecke eigenform $f(z)$ of weight $2k$ on $\mathrm{SL}(2,\mathbb{Z})$, there are two real numbers ω_f^+ and ω_f^-, depending only on f, such that

$$\pi^{-2n}L(f,2n)/\omega_f^+,\qquad \pi^{-(2n-1)}L(f,2n-1)/\omega_f^-$$

are contained in the (totally real) field $\mathbb{Q}(a_f(2),a_f(3),\dots)$ for all integers n with $0<2n,2n-1<2k$. This result was later generalized to newforms on $\Gamma_0(N)$ by Razar [16, Theorem 1]. More generally, it is known that the twisted Hecke L-value $L(f,\chi,n)$ is equal to an algebraic number times either $\pi^n\omega_f^+$ or $\pi^n\omega_f^-$, depending on the parities of n and χ.

The values of $L(f,s)$ and $L(f,\chi,s)$ at the center point $s=k$ are particularly interesting. Assume that the level N is odd and $g(z)=\sum_{n=1}^\infty b_g(n)e^{2\pi inz}$ is the modular form of weight $k+1/2$ lying in Kohnen's plus-space corresponds to $f(z)$ in the sense of Shimura. In [19], Waldspurger proved that $b_g(n)^2$ is essentially proportional to the value of $L(f,\chi_{(-1)^kn},s)$ at $s=k$, where $\chi_D=(\frac{D}{\cdot})$. Later on, Kohnen and Zagier [11] made this result more explicitly by proving

$$\frac{b_g(n)^2}{\langle g,g\rangle}=\frac{n^{k-1/2}\Gamma(k)}{\pi^k}\frac{L(f,\chi_{(-1)^kn},k)}{\langle f,f\rangle}$$

for a normalized Hecke eigenform $f(z)$ on the full modular group $\mathrm{SL}(2,\mathbb{Z})$ and positive integers n such that $(-1)^kn$ is a fundamental discriminant, where $\langle f,f\rangle$ and $\langle g,g\rangle$ denote the Petersson norms of $f(z)$ and $g(z)$, respectively. This result was generalized by [3] and [15] to Hecke eigenforms on $\Gamma_0(N)$.

In this article, we will derive a "trace formula"

$$\sum_{i=1}^s\frac{1}{\langle f_i,f_i\rangle}L(f_i,\chi,m)\overline{L(f_i,n)}$$

for a Dirichlet character χ and integers m and n with proper parity, where $\{f_1,\dots,f_s\}$ is any orthogonal basis for $S_{2k}(\Gamma_0(N))$. In some cases, such as $N=1$ and $N=2$, this formula enables us to compute the exact value of the ratio $L(f,\chi,m)/L(f,m)$ for a Hecke eigenform f.

To achieve our goal, we first express the values of a Hecke L-function $L(f,s)$ as *periods*

$$r_n(f):=\int_0^{i\infty}f(z)z^n\,dz=\frac{n!}{(-2\pi i)^{n+1}}L(f,n+1)$$

of a cusp form f. The periods are studied extensively in [4–6,8,9,12–14,18,21,22]. In particular, for the case of $\mathrm{SL}(2,\mathbb{Z})$, Kohnen and Zagier [12] showed that there is a rational structure associated to the periods that is different from the usual rational structure coming from the Fourier coefficients of cusp forms. The idea is to consider the cusp form characterized by the property

$$r_n(f)=\langle f,R_n\rangle$$

for all $f\in S_{2k}(\Gamma_0(N))$, where $\langle\cdot,\cdot\rangle$ denote the Petersson inner product. Then in [8,12] it is shown that the values of $r_m(R_n)$ can be expressed in terms of the Bernoulli numbers for integers m and n with opposite parity satisfying $0\leqslant m\leqslant 2k-2$ and $1\leqslant n\leqslant 2k-3$. In [7], by considering the natural correspondence of $S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$, its dual, and the space of Dedekind symbols, the first author of the present article found bases for $S_{2k}(\mathrm{SL}(2,\mathbb{Z}))$ in terms of R_n, which in turn give explicit expression for Hecke operators in terms of Bernoulli numbers and sum-of-divisor functions. For the case $\Gamma_0(2)$, this is done in [8] with a different approach.

Now if $\{f_1, \ldots, f_s\}$ is an orthogonal basis for $S_{2k}(\Gamma_0(N))$, then we have

$$R_n = \sum_{i=1}^{s} \frac{\langle R_n, f_i\rangle}{\langle f_i, f_i\rangle} f_i,$$

from which we deduce that

$$r_m(R_n) = C_{m,n} \sum_{i=1}^{s} \frac{1}{\langle f_i, f_i\rangle} L(f_i, m+1)\overline{L(f_i, n+1)}$$

for some complex number $C_{m,n}$ depending only on m and n. In other words, the "trace" of $L(f_i, m+1)\overline{L(f_i, n+1)}/\langle f_i, f_i\rangle$ is essentially $r_m(R_n)$. More generally, if we define the *twisted period* of a cusp form f by

$$r_{m,\chi}(f) := \int_0^{i\infty} f_\chi(z) z^m \, dz,$$

where f_χ denotes the twist of f by χ, then the trace of $L(f_i, \chi, m+1)\overline{L(f_i, n+1)}$ is essentially $r_{m,\chi}(R_n)$.

It turns out that formulas for $r_{m,\chi}(R_n)$ can be more elegantly stated if we write them collectively as *twisted period polynomials*

$$r_\chi(f)(X) := \int_0^{i\infty} f_\chi(z)(X-z)^{2k-2} \, dz$$

of a cusp form f. Before we state our formula for $r_\chi(f)(X)$, let us first fix some notations.

Notation 1.1. Throughout the notes, the letter N will always denote the level of the congruence subgroup $\Gamma_0(N)$, and χ will represent a primitive Dirichlet character modulo D with $D > 1$.

For convenience, we shall write the weight of the space of cusp forms $S_{2k}(\Gamma_0(N))$ under consideration as $2k = w + 2$. For integers m and n, we set

$$\tilde{m} = w - m, \qquad \tilde{n} = w - n.$$

We now recall some definitions related to a Dirichlet character χ.

Definition 1.2. For a non-negative integer k, the kth Bernoulli polynomial $B_k(x)$ is defined by the power series expansion

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k.$$

Naturally, $B_k(x)$ is the zero polynomial if k is a negative integer. For a Dirichlet character χ modulo D, we also define *generalized Bernoulli polynomial* $B_{k,\chi}(x)$ by

$$\sum_{h=0}^{D-1}\chi(h)\frac{te^{t(h+x)}}{e^{Dt}-1}=\sum_{k=0}^{\infty}\frac{B_{k,\chi}(x)}{k!}t^k,$$

that is,

$$B_{k,\chi}(x)=D^{k-1}\sum_{h=0}^{D-1}\chi(h)B_k\big((h+x)/D\big)=\sum_{j=0}^{k}\binom{k}{j}B_{j,\chi}x^{k-j},$$

where $B_{j,\chi}:=B_{j,\chi}(0)$ is the usual generalized Bernoulli number.

For positive integers a,c,k,ℓ satisfying $ka+\ell c=D$ and $(a,c)=1$, we choose integers b and d such that $ad-bc=1$ and set

$$\chi(a,c,k,\ell)=\chi(kb+\ell d).$$

(It is easy to see that the definition does not depend on the choice of b and d.)

Finally, we let

$$\tau(\chi):=\sum_{h=0}^{D-1}\chi(h)e^{2\pi ih/D}$$

denote the Gaussian sum associated to χ.

Now we can describe our first main result.

Theorem 1. *Let R_n, $0<n<w$, be the unique cusp form of weight $w+2$ on $\Gamma_0(N)$ characterized by $r_n(f)=\langle f,R_n\rangle$. Let χ be a primitive Dirichlet character modulo D with $D>1$. Then we have*

$$\begin{aligned}
&r_\chi(R_n)(X)+(-1)^{n-1}\chi(-1)r_\chi(R_n)(-X)\\
&=\frac{(2i)^{w+1}}{\tau(\overline{\chi})}\Bigg(\epsilon_1(-D)^{-\tilde n}\frac{B_{\tilde n+1,\overline{\chi}}(DX)}{\tilde n+1}-D^{-n}\frac{B_{n+1,\overline{\chi}}(DX)}{n+1}\\
&\quad+\epsilon_2(-1)^{n-1}\chi(-N)N^{\tilde n}D^nX^w\frac{B_{\tilde n+1,\chi}(-1/DNX)}{\tilde n+1}+\epsilon_3\chi(-1)D^{\tilde n}X^w\frac{B_{n+1,\chi}(-1/DX)}{n+1}\\
&\quad+G_n(X)+(-1)^{n-1}\chi(-1)G_n(-X)\Bigg),
\end{aligned}$$

where

$$\epsilon_1=\begin{cases}1, & \text{if } N=1,\\ 0, & \text{if } N>1,\end{cases}\qquad \epsilon_2=\begin{cases}1, & \text{if } (N,D)=1,\\ 0, & \text{if } (N,D)>1,\end{cases}\qquad \epsilon_3=\begin{cases}1, & \text{if } N\mid D,\\ 0, & \text{if } N\nmid D,\end{cases}$$

and

$$G_n(X)=\sum_{\substack{a,c,k,\ell>0,(a,c)=1\\ N\mid c,ka+\ell c=D}}\overline{\chi}(a,c,k,\ell)(aX+\ell/D)^n(-cX+k/D)^{\tilde n}.$$

In terms of L-values, Theorem 1 can be rephrased as follows. Here for a cusp form f in $S_{w+2}(\Gamma_0(N))$, we set

$$\Lambda(f,s)=\left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\Gamma(s)L(f,s),\qquad \Lambda(f,\chi,s)=\left(\frac{2\pi}{D\sqrt{N}}\right)^{-s}\Gamma(s)L(f,\chi,s).$$

Note that when $(N,D)=1$, the function $\Lambda(f,\chi,s)$ satisfies a functional equation

$$\Lambda(f,\chi,s)=\epsilon\Lambda(f,\chi,w+2-s)$$

for some root of unity ϵ. When $(N,D)>1$, we need to modify the definition of Λ (replacing $D\sqrt{N}$ in the denominator by other number) in order to get a functional equation of the same symmetry. Here we stick to our definition of Λ to keep the statement of the result simple.

Theorem 2. *Let $\{f_1,\dots,f_s\}$ be an orthogonal basis for $S_{w+2}(\Gamma_0(N))$, and let χ be a primitive Dirichlet character modulo D with $D>1$. Let m and n be integers satisfying $0\leqslant m\leqslant w$, $0<n<w$, and $(-1)^{m+n+1}\chi(-1)=1$. Then we have*

$$\begin{aligned}
&\sum_{i=1}^{s}\frac{1}{\langle f_i,f_i\rangle}\Lambda(f_i,\chi,m+1)\overline{\Lambda(f_i,n+1)}\\
&\quad=(-D)^{m+1}(i\sqrt{N})^{m+n+2}r_{m,\chi}(R_n)\\
&\quad=\frac{(2i)^{w+1}i^{m+n+2}DN^{(n+1)/2}}{2\binom{w}{m}\tau(\overline{\chi})}\times\Bigg(\epsilon_1(-1)^{n+1}\binom{\tilde n}{\tilde m}D^n\frac{B_{\tilde n-\tilde m+1,\overline{\chi}}}{\tilde n-\tilde m+1}\\
&\qquad+\binom{n}{\tilde m}D^{\tilde n}\frac{B_{n-\tilde m+1,\overline{\chi}}}{n-\tilde m+1}+\epsilon_2(-1)^{n+m}\binom{\tilde n}{m}\chi(-N)N^{\tilde n-m}D^n\frac{B_{\tilde n-m+1,\chi}}{\tilde n-m+1}\\
&\qquad+\epsilon_3(-1)^{m+1}\binom{n}{m}\chi(-1)D^{\tilde n}\frac{B_{n-m+1,\chi}}{n-m+1}+2(-1)^{m+1}\sum_{\substack{a,c,k,\ell>0,(a,c)=1\\ N|c,D=ka+\ell c}}\overline{\chi}(a,c,k,\ell)\\
&\qquad\times\sum_{r=0}^{\tilde m}(-1)^r\binom{n}{r}\binom{\tilde n}{\tilde m-r}a^rc^{\tilde m-r}\ell^{n-r}k^{\tilde n-\tilde m+r}\Bigg),
\end{aligned}$$

where ϵ_1, ϵ_2, and ϵ_3 are given as in Theorem 1.

2. Examples

2.1. Example 1

Let $N=1$, $D=3$, $w+2=12$, and $\chi=(\frac{-3}{\cdot})$. Since $S_{12}(\mathrm{SL}(2,\mathbb{Z}))$ is one-dimensional, we expect that for odd n with $1\leqslant n\leqslant 9$, the polynomials $g_n(X)=r_\chi(R_n)(X)+(-1)^n r_\chi(R_n)(-X)$ should be scalar multiples of each other. Indeed, we have $B_{k,\chi}=0$ for even k, and

$$B_{1,\chi}=-\frac{1}{3},\qquad B_{3,\chi}=\frac{2}{3},\qquad B_{5,\chi}=-\frac{10}{3},\qquad B_{7,\chi}=\frac{98}{3},\qquad B_{9,\chi}=-\frac{1618}{3},$$

which give

$$B_{2,\chi}(x) = -\frac{2}{3}x, \qquad B_{4,\chi}(x) = -\frac{4}{3}x^3 + \frac{8}{3}x, \qquad B_{6,\chi}(x) = -2x^5 + \frac{40}{3}x^3 - 20x,$$
$$B_{8,\chi}(x) = -\frac{8}{3}x^7 + \frac{112}{3}x^5 - \frac{560}{3}x^3 + \frac{784}{3}x,$$
$$B_{10,\chi}(x) = -\frac{10}{3}x^9 + 80x^7 - 840x^5 + 3920x^3 - \frac{1618}{3}x.$$

The tuples (a, c, k, ℓ) contributing to $G_n(x)$ are

$$(1,1,1,2), \quad (1,1,2,1), \quad (1,2,1,1), \quad (2,1,1,1)$$

with $\chi(a, c, k, \ell)$ being $-1, 1, 1, -1$, respectively. We find that

$$g_1 = -\frac{2048}{\sqrt{3}}\left(-1536X^9 + 128X^7 - \frac{128}{81}X^3 + \frac{512}{2187}X\right),$$
$$g_3 = g_7 = -\frac{25}{48}g_1, \qquad g_5 = \frac{5}{12}g_1, \qquad g_9 = g_1.$$

This gives us

$$\frac{\Lambda(\Delta,\chi,2)\Lambda(\Delta,2)}{\|\Delta\|^2} = -\frac{2^{18}3^2}{5}\sqrt{3} = -\frac{\Lambda(\Delta,\chi,10)\Lambda(\Delta,2)}{\|\Delta\|^2},$$
$$\frac{\Lambda(\Delta,\chi,4)\Lambda(\Delta,2)}{\|\Delta\|^2} = -\frac{2^{14}3^2}{5}\sqrt{3} = -\frac{\Lambda(\Delta,\chi,8)\Lambda(\Delta,2)}{\|\Delta\|^2}$$

and $\Lambda(\Delta, \chi, 6) = 0$. (Note that the sign for the functional equation of $\Lambda(\Delta, \chi, s)$ is -1, so that it vanishes at $s = 6$.)

The result can be verified numerically as follows. The L-values can be approximated by the standard method. We have

$$\Lambda(\Delta, 2) \approx 0.003707710464948,$$
$$\Lambda(\Delta, \chi, 2) \approx -228.22304046813742,$$
$$\Lambda(\Delta, \chi, 4) \approx -14.263940029258589.$$

To get an approximation for the inner product $\langle \Delta, \Delta \rangle$, we consider the Poincaré series

$$P_k(z) = \frac{1}{2} \sum_{c,d\in\mathbb{Z},(c,d)=1} \frac{e^{2\pi ik(az+b)/(cz+d)}}{(cz+d)^{12}},$$

where in the summand a and b are any integers satisfying $ad - bc = 1$. The Poincaré series is characterized by the property that if $f(z) = \sum_{k=1}^{\infty} a_k(f)e^{2\pi ikz}$, then

$$\langle f, P_k \rangle = \frac{\Gamma(11)}{(4\pi k)^{11}} a_f(k).$$

From this we can easily deduce that

$$P_k(z) = 2\pi\,\Gamma(11)\frac{\tau(k)}{(2k)^{11}}\frac{\Delta(z)}{\|\Delta\|^2}$$

and

$$\langle P_k, P_m\rangle = 4\pi^2\Gamma(11)^2\frac{\tau(k)\tau(m)}{(4km)^{11}\|\Delta\|^2},$$

where $\tau(k)$ is the kth Fourier coefficient of $\Delta(z)$. Now there is a well-known formula for the inner product $\langle P_k, P_m\rangle$ in terms of the Kloosterman sums and the Bessel functions. (See [10, Corollary 3.4].) For instance, evaluating $\langle P_1, P_1\rangle$, we get

$$\frac{1}{\|\Delta\|^2} \approx 965845.709168185. \tag{1}$$

Then

$$\frac{\Lambda(\Delta,\chi,2)\Lambda(\Delta,2)}{\|\Delta\|^2} \approx -817284.10841880 \approx -\frac{2^{18}3^2}{5}\sqrt{3}.$$

Note that the approximation (1) can also be obtained using the formula

$$\sum_{m=1}^{\infty}\frac{\tau(m)^2}{m^{20}} = \frac{2}{245}\frac{4^{20}\pi^{29}}{20!}\frac{\zeta(9)}{\zeta(18)}\|\Delta\|^2$$

given in [20, p. 2].

2.2. Example 2

Let $N=1$ and $w+2=24$. The normalized Hecke eigenforms are

$$E_4(z)^3\Delta(z) + (-156 \pm 12\sqrt{144169})\Delta(z)^2.$$

Let f_1 and f_2 denote these two functions. In this example we will work out the ratio $\Lambda(f_i,\chi,m)/\Lambda(f_i,12)$ for $\chi = (\frac{5}{\cdot})$ and even m. We first express f_i in terms of R_n.

Let $r_m(f)$ denote $\int_0^{i\infty} f(z)z^m\,dz$. By computing the determinant of the matrix

$$\begin{pmatrix} r_1(R_2) & r_3(R_2) \\ r_1(R_4) & r_3(R_4) \end{pmatrix}$$

using the formula in Theorem 1 of [12], we easily see that R_2 and R_4 form a basis for $S_{24}(\mathrm{SL}(2,\mathbb{Z}))$. Let T_2 denote the second Hecke operator on $S_{24}(\mathrm{SL}(2,\mathbb{Z}))$. We can determine the numbers a,b,c,d such that

$$\begin{pmatrix} T_2R_2 \\ T_2R_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} R_2 \\ R_4 \end{pmatrix}$$

by considering the relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r_1(T_2R_2) & r_3(T_2R_2) \\ r_1(T_2R_4) & r_3(T_2R_4) \end{pmatrix} \begin{pmatrix} r_1(R_2) & r_3(R_2) \\ r_1(R_4) & r_3(R_4) \end{pmatrix}^{-1},$$

which, using the formulas in Theorem 2.8 of [7], is shown to be

$$\begin{pmatrix} -716424 & -6894720 \\ 1416492/19 & 717504 \end{pmatrix}.$$

From this we deduce that

$$118041R_2 + (1135193 \pm 19\sqrt{144169})R_4$$

are (unnormalized) Hecke eigenforms, i.e., scalar multiples of f_1 and f_2, respectively. Now we have

$$\Lambda(f, \chi, m+1) = (-5i)^{m+1} r_{m,\chi}(f), \qquad \Lambda(f, m+1) = (-i)^{m+1} r_m(f)$$

for any cusp form f of weight 24. Thus, using the formulas from Theorem 1 of [12] and our Theorem 1, we find that

$$\frac{\Lambda(f_i, \chi, 2)}{\Lambda(f_i, 12)} = \frac{454494815973561283200 \mp 495053625411273600\sqrt{144169}}{11\sqrt{5}},$$

$$\frac{\Lambda(f_i, \chi, 4)}{\Lambda(f_i, 12)} = \frac{1710371411434851840 \mp 1874940923128320\sqrt{144169}}{11\sqrt{5}},$$

$$\frac{\Lambda(f_i, \chi, 6)}{\Lambda(f_i, 12)} = \frac{7923984224047200 \mp 8900924205600\sqrt{144169}}{11\sqrt{5}},$$

$$\frac{\Lambda(f_i, \chi, 8)}{\Lambda(f_i, 12)} = \frac{46543863219840 \mp 56895592320\sqrt{144169}}{11\sqrt{5}},$$

$$\frac{\Lambda(f_i, \chi, 10)}{\Lambda(f_i, 12)} = \frac{359949679200 \mp 545421600\sqrt{144169}}{11\sqrt{5}},$$

$$\frac{\Lambda(f_i, \chi, 12)}{\Lambda(f_i, 12)} = \frac{469261440 \mp 789120\sqrt{144169}}{\sqrt{5}}.$$

Now recall that Theorem 1 of [11] implies that the ratio $\Lambda(f_i, \chi, 12)/\sqrt{5}\Lambda(f_i, 12)$ is a square in the ring of integers of $\mathbb{Q}(\sqrt{144169})$. Indeed, we find that

$$\frac{\Lambda(f_i, \chi, 12)}{\sqrt{5}\Lambda(f_i, 12)} = (3288 \mp 24\sqrt{144169})^2.$$

2.3. *Hecke eigenforms on* $\mathrm{SL}(2, \mathbb{Z})$

For the convenience of the reader, here we tabulate the (unnormalized) Hecke eigenforms in terms of R_n for the case $\dim S_k(\mathrm{SL}(2, \mathbb{Z})) = 2$. For each weight k, we give two bases, one with even n and the other with odd n.

weight	bases
24	$133705R_1 + (1421844 \pm 12\sqrt{144169})R_3$ or $118041R_2 + (1135193 \pm 19\sqrt{144169})R_4$
28	$357271915R_1 + (5430899304 \pm 26568\sqrt{18209})R_3$ or $166985R_2 + (2335719 \pm 23\sqrt{18209})R_4$
30	$339215569R_1 + (6031600980 \pm 6360\sqrt{51349})R_3$ or $39282705R_2 + (646717136 \pm 1352\sqrt{51349})R_4$
32	$18559684975R_1 + (381717886692 \pm 12876\sqrt{18295489})R_3$ or $20837993R_2 + (398996469 \pm 27\sqrt{18295489})R_4$
34	$17696951272R_1 + (416907865575 \pm 20925\sqrt{2356201})R_3$ or $8056833785R_2 + (177566376094 \pm 17806\sqrt{2356201})R_4$
38	$67449635297R_1 + (2033146500360 \pm \sqrt{63737521})R_3$ or $1231612816525R_2 + (35003462442636 \pm 146676\sqrt{63737521})R_4$

2.4. Example 3

Let $N = 2$. To obtain exact values of ratios between twisted L-values of newforms, we can follow the following procedure.

Theorem 1.4 of [8] asserts that if we let d_w denote the dimension of $S_{w+2}(\Gamma_0(2))$, then each of the sets

$$\{R_{2i}\colon i = 1, \ldots, d_w\}, \qquad \{R_{2i-1}\colon i = 1, \ldots, d_w\}$$

is a basis for $S_{w+2}(\Gamma_0(2))$. Using Theorems 1.1 and 1.3 of [8], we can find the matrices for the Hecke operators with respect to the above bases. Diagonalizing the matrices, we obtain expressions of newforms in terms of R_n. Then an application of Theorem 1 gives us the values of ratios between twisted L-values of newforms.

Let us consider the case $w + 2 = 16$. The space $S_{16}(\Gamma_0(2))$ has dimension 3 and is spanned by

$$f_1 = \Delta(z)E_4(z), \qquad f_2 = \Delta(2z)E_4(2z), \qquad f_3 = \eta(z)^{16}\eta(2z)^{16}.$$

By a direct computation, we find that the unique normalized newform is

$$f = f_1 + 256f_2 - 600f_3 = q - 128q^2 + 6252q^3 + \cdots$$

whose eigenvalue for the Atkin–Lehner involution w_2 is $+1$. Let $D > 1$ be a fundamental discriminant. We now compute $\Lambda(f, \chi_D, 8)/\Lambda(f, 8)$ with $\chi_D = (\frac{D}{\cdot})$ for the first few D. Note that if $(D, 2) = 1$, then the functional equation for $L(f, \chi_D, s)$ has sign $\chi_D(-2)$. Thus, if $D \equiv 5 \bmod 8$ and $D > 0$, we know that $L(f, \chi_D, 8) = 0$.

Proceeding as in Example 2 and using Theorems 1.1 and 1.3 of [8], we find that

$$T_3\begin{pmatrix} R_2 \\ R_4 \\ R_6 \end{pmatrix} = \left[r_{2j-1}(T_3R_{2i})\right]\left[r_{2j-1}(R_{2i})\right]^{-1}\begin{pmatrix} R_2 \\ R_4 \\ R_6 \end{pmatrix}$$
$$= \begin{pmatrix} 154348 & 2478080 & 3784704 \\ -11648 & -186388 & -279552 \\ 1456 & 22880 & 31596 \end{pmatrix}\begin{pmatrix} R_2 \\ R_4 \\ R_6 \end{pmatrix},$$

where T_3 denote the third Hecke operator. The characteristic polynomial of the above matrix is $(x + 3348)^2(x - 6252)$. The eigenfunction $7R_2 + 110R_4 + 168R_6$ associated to the eigenvalue 6252 must be a newform. Applying Theorem 1.1 of [8] and Theorem 2 we obtain

D	$D^{-1/2}\Lambda(f,\chi_D,8)/\Lambda(f,8)$
8	$2(2^7\cdot 3^2)^2$
12	$2(2^8\cdot 3\cdot 7)^2$
17	$(2^6\cdot 3^2\cdot 7^2)^2$
24	$2(2^8\cdot 3\cdot 7\cdot 29)^2$
28	$2(2^{10}\cdot 3\cdot 7\cdot 19)^2$
33	$(2^6\cdot 3\cdot 7\cdot 11\cdot 23)^2$
40	$2(2^8\cdot 3\cdot 5\cdot 7\cdot 61)^2$
41	$(2^9\cdot 3^2\cdot 7\cdot 23)^2$
44	$2(2^8\cdot 3^2\cdot 11\cdot 41)^2$
56	$2(2^9\cdot 3^2\cdot 5\cdot 7^2)^2$
57	$(2^6\cdot 3\cdot 15671)^2$
60	$2(2^{10}\cdot 3\cdot 5\cdot 43)^2$
65	$(2^8\cdot 3^2\cdot 5\cdot 13\cdot 23)^2$

Remark 2.1. Apparently, when D is odd, the ratio $D^{-1/2}\Lambda(f,\chi_D,8)/\Lambda(f,8)$ is a perfect square, and when D is even, the ratio is 2 times a perfect square. As pointed out by the referee, there is a representation-theoretical explanation for this extra factor 2. See [3], in particular, formula (1.5) in Theorem 1.2 of [3].

To check the correctness, we note that a half-integral weight cusp form on $\Gamma_0(8)$ corresponding to the normalized newform of weight 16 on $\Gamma_0(2)$ is

$$\begin{aligned}&-\frac{11}{252}\big[E_6(4\tau),\theta(\tau)\big]_1+\frac{32}{252}\big[E_6(8\tau),\theta\big]_1-88\eta(4\tau)^8\eta(8\tau)^8\theta(\tau)\\&\quad=q-128q^4-2^7\cdot 3^3q^8+4065q^9-2^8\cdot 3\cdot 7q^{12}+2^{14}q^{16}-2^6\cdot 3^2\cdot 7^2q^{17}+\cdots,\end{aligned}$$

where $[g,h]_r$ denotes the Rankin–Cohen bracket, $E_6(\tau)$ is the usual Eisenstein series of weight 6 on $\mathrm{SL}(2,\mathbb{Z})$, and $\theta(\tau)=\sum_{n\in\mathbb{Z}}e^{2\pi i n^2\tau}$ is the Jacobi theta function.

2.5. Newforms on $\Gamma_0(2)$

For the convenience of the reader, here we tabulate newforms on $\Gamma_0(2)$ in terms of R_n for the first few w. Note that for $w+2=8,10$, we have $\dim S_{w+2}(\Gamma_0(2))=1$ and each R_n is a newform. Also, for $w+2=12$, the space of newforms has dimension 0.

weight	bases
14	$21R_1+220R_3$, R_1+12R_3 or R_2+8R_4, $11R_2+120R_4$
16	$49R_1+936R_3+1872R_5$ or $7R_2+110R_4+168R_6$
18	$11R_1+300R_3+1056R_5$ or $15R_2+364R_4+1232R_6$
20	$11R_1+416R_3+2576R_5+2816R_7$, $3861R_1+123488R_3+321776R_5-622336R_7$ or $51R_2+1722R_4+9464R_6+8448R_8$, $R_2+30R_4+104R_6$
22	$143R_1+6612R_3+48640R_5+63232R_7$, $1105R_1+52524R_3+425472R_5+708864R_7$ or $113R_2+4624R_4+28896R_6+29952R_8$, $19R_2+816R_4+6048R_6+9984R_8$
24	$10R_3+459R_5+3264R_7+3536R_9$ or $693R_2+34010R_4+228480R_6+16320R_8-311168R_{10}$

Remark 2.2. For $N = 2$, it is shown in [8] that the first few R_{2i} or the first few R_{2i-1} is a basis for $S_{w+2}(\Gamma_0(N))$. Our computation suggests that the same is true for $N = 3, 4, 5$. In these cases, we can compute the ratios of twisted L-values of newforms using the above approach. For $N \geqslant 6$, the method no longer works, as the dimension of $S_{w+2}(\Gamma_0(N))$ already exceeds the number of R_n.

3. Proof of theorems

3.1. Preliminary

Let $f \in S_{w+2}(\Gamma_0(N))$ and χ be a Dirichlet character modulo D with $D > 1$. Recall that if χ is primitive, then we have

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{h=0}^{D-1} \overline{\chi}(h) e^{2\pi i h n/D}.$$

It follows that

$$f_\chi(z) = \frac{1}{\tau(\overline{\chi})} \sum_{h=0}^{D-1} \overline{\chi}(h) f(z + h/D)$$

and

$$r_{m,\chi}(f) = \frac{1}{\tau(\overline{\chi})} \sum_{h=0}^{D-1} \overline{\chi}(h) r_{m,h/D}(f),$$

where

$$r_{m,h/D}(f) = \int_0^{i\infty} f(z + h/D) z^m \, dz.$$

Before we proceed to evaluate $r_{m,h/D}(R_n)$, let us recall the following properties of the Bernoulli polynomials. (See [1, pp. 804–805].) Here in the lemma, the notation $\{x\}$ represents the fractional part of a real number x.

Lemma 3.1. *For two real numbers a and x and an integer k, we have*

$$B_k(a + x) = \sum_{j=0}^{k} \binom{k}{j} B_j(a) x^{k-j}.$$

Moreover, the Fourier expansion for the Bernoulli function $B_k(\{x\})$, $k \geqslant 2$, is given by

$$B_k\bigl(\{x\}\bigr) = -\frac{k!}{(2\pi i)^k} \sum_{r \in \mathbb{Z}, r \neq 0} \frac{e^{2\pi i r x}}{r^k},$$

and

$$-\frac{1}{2\pi i}\lim_{T\to\infty}\sum_{0<|r|<T}\frac{e^{2\pi irx}}{r}=\begin{cases}B_1(\{x\})=\{x\}-1/2, & \textit{if } x\notin\mathbb{Z},\\ 0, & \textit{if } x\in\mathbb{Z}.\end{cases}$$

To evaluate $r_{m,h/D}(R_n)$, we shall utilize the following expression for R_n.

Lemma 3.2. *Let $R_n(z)$ be the cusp form of weight $w+2$ on $\Gamma_0(N)$ characterized by the property $r_n(f)=\langle f, R_n\rangle$ for all $f\in S_{w+2}(\Gamma_0(N))$. We have*

$$R_n(z)=c_n^{-1}\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)}\frac{1}{(az+b)^{\tilde{n}+1}(cz+d)^{n+1}},\quad c_n=(-1)^n4\pi i(2i)^{-w-1}\binom{w}{n}.$$

Proof. See [8, Lemma 2.1]. (See also [2, Proposition 1].) □

From the above lemma, we have

$$c_nR_n(z+h/D)=\sum_{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)}\frac{1}{(az+ah/D+b)^{\tilde{n}+1}(cz+ch/D+d)^{n+1}}.$$

We shall consider the cases

(1) $a=0$ (with $N=1$),
(2) $c=0$,
(3) $(a,b)=\pm(D,-h)$ (with $(N,D)=1$),
(4) $(c,d)=\pm(D,-h)$ (with $N|D$),
(5) $ac(ah/D+b)(ch/D+d)<0$,
(6) $ac(ah/D+b)(ch/D+d)>0$,

separately. For $j=1,\ldots,6$ and $h\in\mathbb{Z}$ with $(h,D)=1$, we let

$$S_{j,h}(z)=\sum_{\substack{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)\\ \text{satisfying the } j\text{th condition}}}\frac{1}{(az+ah/D+b)^{\tilde{n}+1}(cz+ch/D+d)^{n+1}}.$$

Note that we have

$$S_{j,h+D}(z)=S_{j,h}(z). \tag{2}$$

This is because a matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ contributes to the sum $S_{j,h+D}(z)$ if and only if the matrix $\left(\begin{smallmatrix}a&a+b\\c&c+d\end{smallmatrix}\right)$ contributes to the sum $S_{j,h}(z)$. In the following sections, we will obtain formulas for

$$I_{j,h,m}:=\int_0^{i\infty}\big(S_{j,h}(z)+(-1)^{m+n+1}S_{j,-h}(z)\big)z^m\,dz$$

and

$$F_{j,h}(X) := \int_0^{i\infty} S_{j,h}(z)(X-z)^w\,dz + (-1)^{n-1}\int_0^{i\infty} S_{j,-h}(z)(X+z)^w\,dz$$
$$= \sum_{m=0}^{w}(-1)^m\binom{w}{m} I_{j,h,m} X^{\tilde{m}}.$$

3.2. Case $a=0$ (and $N=1$)

In this section, we shall evaluate the integral

$$\int_0^{i\infty} S_{1,h}(z) z^m\,dz.$$

3.2.1. Case $m+1>n$

If $a=0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}, \quad d\in\mathbb{Z}.$$

We have

$$S_{1,h}(z) = \sum_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & d \end{smallmatrix}\right)\in \mathrm{SL}(2,\mathbb{Z})} \frac{(-1)^{\tilde{n}+1}}{(z+h/D+d)^{n+1}} + \sum_{\left(\begin{smallmatrix} 0 & 1 \\ -1 & d \end{smallmatrix}\right)\in \mathrm{SL}(2,\mathbb{Z})} \frac{(-1)^{n+1}}{(z+h/D-d)^{n+1}}$$
$$= \frac{2(-1)^{n+1}(-2\pi i)^{n+1}}{\Gamma(n+1)}\sum_{r=1}^{\infty} r^n e^{2\pi i r(z+h/D)}.$$

(Here we have used the formula [17, p. 51] for $\sum_{d\in\mathbb{Z}}(\tau+d)^{-n}$.) It follows that

$$\int_0^{i\infty} S_{1,h}(z) z^m\,dz = 2(-1)^{n+1}(-2\pi i)^{n-m}\frac{m!}{n!}\sum_{r=1}^{\infty}\frac{e^{2\pi i r h/D}}{r^{m-n+1}}.$$

From this and Lemma 3.1, we obtain

$$I_{1,h,m} = \frac{(-1)^m(4\pi i)m!}{n!(m-n+1)!} B_{m-n+1}\big(\{h/D\}\big)$$

for positive integers m and n with $m+1>n$. (Note that the integral-sum is no longer absolutely convergent in the case $m=n$. However, the conclusion remains valid in view of the bounded convergence theorem.)

3.2.2. Case $m+1<n$

It is easy to check that

$$S_{1,h}(z) \ll \begin{cases} 1, & \text{if } |z| \ll 1, \\ |z|^{-n}, & \text{if } |z| \gg 1. \end{cases}$$

Therefore, if $m+1<n$, we may integrate term by term. Now we have

$$S_{j,-h}(z) = (-1)^{n+1} S_{j,h}(-z) \tag{3}$$

for $j = 1, \ldots, 6$. Hence,

$$\int_0^{i\infty} \big(S_{1,h}(z) + (-1)^{m+n+1} S_{1,-h}(z)\big) z^m \, dz = \int_{-i\infty}^{i\infty} S_{1,h}(z) z^m \, dz.$$

We then integrate term by term. By shifting the path of integration to $\operatorname{Re} z = \infty$ or $\operatorname{Re} z = -\infty$ depending on whether $h/D+d$ is positive or negative, we find that the integral for each term is zero. This shows that the contribution of $S_{1,h}$ in the case $m+1<n$ is 0.

3.2.3. Case $m=n-1$

From (3), we obtain

$$I_{1,h,n-1} = \int_{-i\infty}^{i\infty} S_{1,h}(z) z^{n-1} \, dz = 2(-1)^{n+1} \lim_{U\to\infty} \sum_{d\in\mathbb{Z}} \int_{-iU}^{iU} \frac{z^{n-1}\, dz}{(z+h/D+d)^{n+1}}.$$

If $n=1$, then we have

$$\begin{aligned} I_{1,h,n-1} &= -2 \lim_{U\to\infty} \sum_{d\in\mathbb{Z}} \left(\frac{1}{iU+h/D+d} - \frac{1}{-iU+h/D+d} \right) \\ &= 4i \lim_{U\to\infty} \sum_{d\in\mathbb{Z}} \frac{U}{(h/D+d)^2+U^2} = 4i \lim_{U\to\infty} \frac{1}{U} \sum_{d\in\mathbb{Z}} \frac{1}{((h/D+d)/U)^2+1}. \end{aligned}$$

Interpreting the last sum as a Riemann sum, we arrive at

$$I_{1,h,n-1} = 4i \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 4\pi i.$$

If $n \geqslant 2$, we apply integration by parts once and get

$$\begin{aligned} I_{1,h,n-1} = 2(-1)^{n+1} \lim_{U\to\infty} \sum_{d\in\mathbb{Z}} \bigg(&-\frac{(iU)^{n-1}}{n(iU+h/D+d)^n} + \frac{(-iU)^{n-1}}{n(-iU+h/D+d)^n} \\ &+ \frac{n-1}{n} \int_{-iU}^{iU} \frac{z^{n-2}\, dz}{(z+h/D+d)^n} \bigg). \end{aligned}$$

Again, the sum

$$\sum_{d\in\mathbb{Z}}\left(-\frac{(iU)^{n-1}}{(iU+h/D+d)^n}+\frac{(-iU)^{n-1}}{(-iU+h/D+d)^n}\right)$$

can be interpreted as a Riemann sum for some integral whose value turns out to be 0. Thus, we have

$$I_{1,h,n-1}=2(-1)^{n+1}\frac{n-1}{n}\lim_{U\to\infty}\sum_{d\in\mathbb{Z}}\int_{-iU}^{iU}\frac{z^{n-2}\,dz}{(z+h/D+d)^n}.$$

Integrating by parts repeatedly, we eventually obtain

$$I_{1,h,n-1}=(-1)^{n+1}\frac{4\pi i}{n}.$$

3.2.4. Summary for the case $a=0$

We now combine the computations in Sections 3.2.1–3.2.3. We have

$$\begin{aligned}F_{1,h}(X)&:=\int_0^{i\infty}S_{1,h}(z)(X-z)^w\,dz+(-1)^{n-1}\int_0^{i\infty}S_{1,-h}(z)(X+z)^w\,dz\\&=\sum_{m=0}^{w}(-1)^m\binom{w}{m}I_{1,h,m}X^{w-m}.\end{aligned}$$

The result in Section 3.2.1 shows that the contribution from the terms with $m\geqslant n$ is

$$\begin{aligned}&\frac{4\pi i}{n!}\sum_{m=n}^{w}\binom{w}{m}\frac{m!}{(m-n+1)!}X^{w-m}B_{m-n+1}\big(\{h/D\}\big)\\&\quad=\frac{4\pi i}{\tilde{n}+1}\binom{w}{n}\sum_{m=n}^{w}\binom{w-n+1}{m-n+1}X^{w-m}B_{m-n+1}\big(\{h/D\}\big).\end{aligned}$$

In view of Lemma 3.1, this is equal to

$$\frac{4\pi i}{\tilde{n}+1}\binom{w}{n}\big(B_{\tilde{n}+1}\big(\{h/D\}+X\big)-X^{w-n+1}\big).$$

From Section 3.2.2, we know that the contribution from the terms with $m+1<n$ to (4) is 0, while Section 3.2.3 shows that the term $m=n-1$ yields

$$(-1)^{n-1}\binom{w}{n-1}X^{w-n+1}(-1)^{n+1}\frac{4\pi i}{n}=\frac{4\pi i}{\tilde{n}+1}\binom{w}{n}X^{w-n+1}.$$

Combining everything, we get the following formula for $S_{1,h}(z)$.

Lemma 3.3. *Let c_n be defined as in Lemma 3.2. We have*

$$c_n^{-1} I_{1,h,m} = (-1)^{n+m}(2i)^{w+1}\binom{w}{m}^{-1}\binom{\tilde{n}}{\tilde{m}}\frac{B_{\tilde{n}-\tilde{m}+1}(\{h/D\})}{\tilde{n}-\tilde{m}+1},$$

or equivalently,

$$c_n^{-1} F_{1,h}(X) = (-1)^{n}(2i)^{w+1}\frac{B_{\tilde{n}+1}(\{h/D\}+X)}{\tilde{n}+1}.$$

3.3. Case $c=0$

The evaluation for the case $c=0$ is very similar to the case $a=0$, so the proof will be very sketchy. We have

$$S_{2,h}(z) = 2\sum_{b\in\mathbb{Z}}\frac{1}{(z+h/D+b)^{\tilde{n}+1}} = \frac{2(-2\pi i)^{\tilde{n}+1}}{\tilde{n}!}\sum_{r=1}^{\infty} r^{\tilde{n}} e^{2\pi i r(z+h/D)}.$$

Thus, when $m+1>\tilde{n}$,

$$\int_0^{i\infty} S_{2,h}(z) z^m\,dz = 2(-2\pi i)^{\tilde{n}-m}\frac{m!}{\tilde{n}!}\sum_{r=1}^{\infty}\frac{e^{2\pi i r h/D}}{r^{m-\tilde{n}+1}},$$

and

$$I_{2,h,m} = \frac{(-1)^{m+n+1}(4\pi i)m!}{\tilde{n}!(m-\tilde{n}+1)!}B_{m-\tilde{n}+1}\big(\{h/D\}\big).$$

When $m+1<\tilde{n}$, we find that

$$I_{2,h,m}=0.$$

The case $m=\tilde{n}-1$ yields

$$I_{2,h,\tilde{n}-1} = \frac{4\pi i}{\tilde{n}}.$$

In summary, the contribution of the case $c=0$ to the period polynomial is the following.

Lemma 3.4. *We have*

$$c_n^{-1} I_{2,h,m} = (-1)^{m-1}(2i)^{w+1}\binom{w}{m}^{-1}\binom{n}{\tilde{m}}\frac{B_{n-\tilde{m}+1}(\{h/D\})}{n-\tilde{m}+1},$$

and

$$c_n^{-1} F_{2,h}(X) = -(2i)^{w+1}\frac{B_{n+1}(\{h/D\}+X)}{n+1}.$$

3.4. Case $(a, b) = \pm(D, -h)$ (with $(N, D) = 1$)

Let c and d be any two integers satisfying $dD + cNh = 1$. Then the matrices in $\Gamma_0(N)$ with an upper row $\pm(D, -h)$ are

$$\pm\begin{pmatrix} D & -h \\ N(c+kD) & d-khN \end{pmatrix}$$

for integers $k \in \mathbb{Z}$. Then

$$S_{3,h}(z) = 2\sum_{k\in\mathbb{Z}} \frac{1}{(Dz)^{\tilde{n}+1}((c+kD)Nz+1/D)^{n+1}}.$$

Now we make a change of variable $z \mapsto -1/ND^2z$ in the integral

$$\begin{aligned}\int_0^{i\infty} S_{3,h}(z)z^m\,dz &= 2\int_0^{i\infty}\left(\sum_{k\in\mathbb{Z}} \frac{z^m}{(Dz)^{\tilde{n}+1}((c+kD)Nz+1/D)^{n+1}}\right)dz \\ &= 2(-1)^{m+n}N^{\tilde{n}-m}D^{w-2m}\int_0^{i\infty}\left(\sum_{k\in\mathbb{Z}} \frac{z^{\tilde{m}}}{(z-k-c/D)^{n+1}}\right)dz.\end{aligned}$$

At this point, we are basically back to the previous cases. For $\tilde{m}+1 > n$, we have

$$\int_0^{i\infty} S_{3,h}(z)z^m\,dz = 2(-1)^{m+n}N^{\tilde{n}-m}D^{w-2m}(-2\pi i)^{n-\tilde{m}}\frac{\tilde{m}!}{n!}\sum_{r=1}^{\infty}\frac{e^{-2\pi irc/D}}{r^{\tilde{m}-n+1}}.$$

Then Lemma 3.1 yields

$$I_{3,h,m} = -\frac{4\pi iN^{\tilde{n}-m}D^{w-2m}\tilde{m}!}{n!(\tilde{m}-n+1)!}B_{\tilde{m}-n+1}\bigl(\{-c/D\}\bigr).$$

For m with $\tilde{m}+1 < n$, arguing as in Section 3.2.2, we find that

$$I_{3,h,m} = 0.$$

When $\tilde{m}+1 = n$ (i.e., $m = \tilde{n}+1$), a discussion similar to that in Section 3.2.3 leads to

$$I_{3,h,\tilde{n}+1} = -\frac{4\pi iD^{w-2\tilde{n}-2}}{Nn}.$$

Finally, the condition $dD + cNh = 1$ means that c is the multiplicative inverse of Nh modulo D.

Lemma 3.5. *The contribution from $S_{3,h}$ is*

$$c_n^{-1}I_{3,h,m} = (-1)^{n-1}(2i)^{w+1}\binom{w}{m}^{-1}\binom{\tilde{n}}{m}D^{w-2m}N^{\tilde{n}-m}\frac{B_{\tilde{n}-m+1}(\{-\overline{N}\overline{h}/D\})}{\tilde{n}-m+1}$$

and

$$c_n^{-1} F_{3,h}(X) = (-1)^{n-1}(2i)^{w+1} N^{\tilde{n}} D^w X^w \frac{B_{\tilde{n}+1}(\{-\overline{N}\bar{h}/D\} - 1/D^2 NX)}{\tilde{n}+1},$$

where $\overline{N}$ and $\bar{h}$ denote the multiplicative inverses of N and h modulo D, respectively.

3.5. *Case $(c,d) = \pm(D,-h)$ (with $N|D$)*

This case is similar to the previous case. Choose integers a and b with $ah + bD = -1$. Then the matrices in $\mathrm{SL}(2,\mathbb{Z})$ with a lower row $\pm(D,-h)$ are

$$\pm\begin{pmatrix} a+kD & b-kh \\ D & -h \end{pmatrix}.$$

Thus,

$$S_{4,h}(\tau) = 2\sum_{k\in\mathbb{Z}} \frac{1}{((a+kD)\tau - 1/D)^{\tilde{n}+1}(D\tau)^{n+1}}.$$

Arguing similarly as in the previous section, we obtain the following evaluation.

Lemma 3.6. *We have*

$$c_n^{-1} I_{4,h,m} = (2i)^{w+1}\binom{w}{m}^{-1}\binom{n}{m} D^{w-2m} \frac{B_{n-m+1}(\{-\bar{h}/D\})}{n-m+1}$$

and

$$c_n^{-1} F_{4,h} = (2i)^{w+1} D^w X^w \frac{B_{n+1}(\{-\bar{h}/D\} - 1/D^2 X)}{n+1},$$

where $\bar{h}$ is the multiplicative inverse of h modulo D.

3.6. *Case $ac(ah/D+b)(ch/D+d) < 0$*

From (3) we know that

$$\int_0^{i\infty} S_{5,h}(z)(X-z)^w\,dz + (-1)^{n-1}\int_0^{i\infty} S_{5,-h}(z)(X+z)^w\,dz = \int_{-i\infty}^{i\infty} S_{5,h}(z)(X-z)^w\,dz. \tag{4}$$

Considering Eq. (3), we assume $h > 0$ and evaluate the integral above.

There are two cases $ac > 0$ and $ac < 0$. In the former case, because $ad - bc = 1$, we must have $-d/c < h/D < -b/a$. In the latter case we have $-b/a < h/D < -d/c$ instead. Also, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ contributes to $S_{5,h}$, then so does $\left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right)$. It follows that

$$S_{5,h}(z) = 2 \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N), a,c>0 \\ -d/c<h/D<-b/a}} \frac{1}{(az + ah/D + b)^{\tilde{n}+1}(cz + ch/D + d)^{n+1}}$$
$$+ 2 \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N), a>0,c<0 \\ -b/a<h/D<-d/c}} \frac{1}{(az + ah/D + b)^{\tilde{n}+1}(cz + ch/D + d)^{n+1}}.$$

Now the condition $ad - bc = 1$ implies that $-b/a$ and $-d/c$ are Farey neighbors. Then a general property of the Farey fractions says that in order for a fraction $h/D > 0$ to be sandwiched between $-b/a$ and $-d/c$, h and D must be of the form $h = k|b| + \ell|d|$ and $D = k|a| + \ell|c|$ for some positive integers k and ℓ. This in particular shows that the number of terms in the sum $S_{5,h}$ is finite.

In the case $a, c > 0$, the integers b and d are non-positive. Thus, $D = ka + \ell c$ and $h = -kb - \ell d$. We have

$$\frac{ah}{D} + b = \frac{a(-kb - \ell d)}{ka + \ell c} + b = -\frac{\ell}{ka + \ell c} = -\frac{\ell}{D}.$$

Likewise, we have

$$\frac{ch}{D} + d = \frac{k}{D}.$$

In the case $a > 0$, $c < 0$, we have $b \leqslant 0$ and $d \geqslant 0$. Thus, $D = ka - \ell c$, $h = -kb + \ell d$, and

$$\frac{ah}{D} + b = \frac{\ell}{D}, \qquad \frac{ch}{D} + d = \frac{k}{D}.$$

Then $S_{5,h}$ becomes

$$S_{5,h}(z) = 2 \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N), a,c,k,\ell>0 \\ D=ka+\ell c, h=-kb-\ell d}} \frac{1}{(az - \ell/D)^{\tilde{n}+1}(cz + k/D)^{n+1}}$$
$$+ 2 \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N), a,k,\ell>0,c<0 \\ D=ka-\ell c, h=-kb+\ell d}} \frac{1}{(az + \ell D)^{\tilde{n}+1}(cz + k/D)^{n+1}}. \tag{5}$$

Now let us recall a formula from [8].

Lemma 3.7. *Let a, b, c, d be real numbers such that $abcd < 0$. Then*

$$\int_{-i\infty}^{i\infty} \frac{(X - \tau)^w}{(a\tau + b)^{\tilde{n}+1}(c\tau + d)^{n+1}}\, d\tau = \frac{(-1)^n 2\pi i}{(ad - bc)^{w+1}} \binom{w}{n} \operatorname{sgn}(ab)(aX + b)^n (cX + d)^{\tilde{n}}.$$

Proof. See [8, p. 341]. □

Applying this lemma to (5), we obtain the following formula for (4).

Lemma 3.8. *We have*

$$c_n^{-1}F_{5,h}(X) = -(2i)^{w+1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N), a,c,k,\ell>0 \\ D=ka+\ell c, h=-kb-\ell d}} (aX-\ell/D)^n (cX+k/D)^{\tilde{n}}$$
$$+ (2i)^{w+1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N), a,k,\ell>0, c<0 \\ D=ka-\ell c, h=-kb+\ell d}} (aX+\ell/D)^n (cX+k/D)^{\tilde{n}}.$$

3.7. Case $ac(ah/D+b)(ch/D+d)>0$

The evaluation of the terms with $ac(ah/D+b)(ch/D+d)>0$ follows the argument in [8, pp. 13–16]. Here we only provide a sketch.

Firstly, we have $S_{6,-h}(z) = (-1)^{n-1}S_{6,h}(-z)$, so that

$$I_{6,h,m} = \int_{-i\infty}^{i\infty} S_{6,h}(z) z^m \, dz.$$

We can show that

$$\sum_{ac(ah/D+b)(ch/D+d)>0} \frac{1}{|(az+ah/D+b)^{\tilde{n}+1}(cz+ch/D+d)^{n+1}|}$$
$$\ll \begin{cases} 1/|z|, & \text{if } |z| \ll 1, \\ 1/|z|^{w+1}, & \text{if } |z| \gg 1. \end{cases}$$

3.7.1. Case $0<m<w$

If m is not 0 or w, we may change the order of integration of summation. In this case, since the two poles of $1/(az+ah/D+b)^{\tilde{n}+1}(cz+ch/D+d)^{n+1}$ lie on the same side of the imaginary axis, we have

$$I_{6,h,m} = 0.$$

3.7.2. Case $m=w$

Here we consider the contribution from $S_{6,h}(z)$ in the case $m=w$. The key point to be observed here is that $I_{6,h,w}$ takes the same value for all h with $(h,D)=1$, so that when we combine everything into a formula for $r_\chi(R_n)(X)$, the contributions from $S_{6,h}(z)$ cancel out each other.

We have

$$I_{6,h,w} = \lim_{\epsilon\to 0} \sum_{ac(ah/D+b)(ch/D+d)>0} \int_{-i/\epsilon}^{i\epsilon} \frac{z^w \, dz}{(az+ah/D+b)^{\tilde{n}+1}(cz+ch/D+d)^{n+1}}$$
$$= -\lim_{\epsilon\to 0} \sum \left(\int_{i/\epsilon}^{i\infty} + \int_{-i\infty}^{-i/\epsilon} \right) \frac{z^w \, dz}{(az+ah/D+b)^{\tilde{n}+1}(cz+ch/D+d)^{n+1}}.$$

Making a change of variable $z \mapsto i/(\epsilon t)$, we obtain

$$I_{6,h,w} = i \lim_{\epsilon \to 0} \epsilon \sum \int_{-1}^{1} \frac{dt}{(a - i\epsilon t(ah/D + b))^{\tilde{n}+1}(c - i\epsilon t(ch/D + d))^{n+1}}.$$

For a given pair of integers a and c, we fix integers b_0 and d_0 such that $ad_0 - b_0c = 1$. The other integers b and d satisfying $ad - bc = 1$ are $b = b_0 + ak$ and $d = d_0 + ck$ for $k \in \mathbb{Z}$. Then

$$I_{6,h,w} = i \lim_{\epsilon \to 0} \epsilon \sum_{a,c} \frac{1}{a^{\tilde{n}+1}c^{n+1}} \sum_{u \in \mathbb{Z}+b_0/a+h/D} \int_{-1}^{1} \frac{dt}{(1 - i\epsilon tu)^{\tilde{n}+1}(1 - i\epsilon t(u + 1/ac))^{n+1}}.$$

(Note that there might be some integers k such that $ac(ah/D + b_0 + ak)(ch/D + d_0 + ck)$ is not positive. However, it should be clear that the contribution of these terms to the above sum is not significant.) Observe that

$$\left| \frac{1}{(1 - i\epsilon t(u + 1/ac))^{n+1}} - \frac{1}{(1 - i\epsilon tu)^{n+1}} \right| \ll \frac{\epsilon}{|ac|} \frac{|t|}{|1 - i\epsilon tu|^{n+2}}.$$

Since

$$\frac{1}{|1 - i\epsilon tu|^{n+2}} \ll \begin{cases} 1, & \text{if } |u| \ll 1/|\epsilon t|, \\ 1/|\epsilon tu|^{n+2}, & \text{if } |u| \gg 1/|\epsilon t|, \end{cases}$$

we have

$$\sum_{u \in \mathbb{Z}+b_0/a+h/D} \left| \frac{1}{(1 - i\epsilon t(u + 1/ac))^{n+1}} - \frac{1}{(1 - i\epsilon tu)^{n+1}} \right| \ll 1.$$

Hence

$$\begin{aligned} I_{6,h,w} &= i \lim_{\epsilon \to 0} \epsilon \sum_{a,c} \frac{1}{a^{\tilde{n}+1}c^{n+1}} \sum_{u \in \mathbb{Z}+b_0/a+h/D} \int_{-1}^{1} \frac{dt}{(1 - i\epsilon tu)^{w+2}} \\ &= i \sum_{a,c} \frac{1}{a^{\tilde{n}+1}c^{n+1}} \int_{-\infty}^{\infty} \int_{-1}^{1} \frac{dt}{(1 - itu)^{w+2}} \, du. \end{aligned}$$

(Note that at this point it is already enough to complete the proof of Theorem 1 since the last expression is independent of h so that the contributions from all h will cancel out each other. For the sake of completeness, we will carry through the computation.) Following the computation in [8, p. 341], we find that

$$\int_{-\infty}^{\infty} \int_{-1}^{1} \frac{dt}{(1 - ixt)^{w+2}} \, dx = \frac{2\pi}{w+1},$$

and

$$I_{6,h,w} = \frac{2\pi i}{w+1} \sum_{(a,c)=1,N|c,a,c\neq 0} \frac{1}{a^{\tilde{n}+1}c^{n+1}}.$$

The sum is equal to 0 if n is even, and is equal to

$$I_{6,h,w} = \frac{8\pi i}{w+1} \frac{\zeta(n+1)\zeta(\tilde{n}+1)}{\zeta(w+2)N^{n+1}} \prod_{p|N} \frac{1-p^{-(\tilde{n}+1)}}{1-p^{-(w+2)}}$$

if n is odd. In either case, we have

$$I_{6,h,w} = -\frac{4\pi i}{N^{n+1}} \binom{w}{n} \frac{w+2}{B_{w+2}} \frac{B_{n+1}}{n+1} \frac{B_{\tilde{n}+1}}{\tilde{n}+1} \prod_{p|N} \frac{1-p^{-\tilde{n}-1}}{1-p^{-w-2}}, \tag{6}$$

where p runs through all prime divisors of N. This settles the case $m = w$.

3.7.3. Case $m = 0$

We first make a change of variable $z \to -1/ND^2 z$ and obtain

$$I_{6,h,0} = \int_{-i\infty}^{i\infty} \sum \frac{(-1)^n N^{\tilde{n}} D^w z^w \, dz}{(a/D - N(ah+bD)z)^{\tilde{n}+1}(-c/ND + (ch+dD)z)^{n+1}}.$$

Now choose integers u and v such that $Du - Nhv = 1$. For each $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_0(N)$, we set

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & h \\ Nv & D \end{pmatrix}.$$

We check that the condition $ac(ah/D + b)(ch/D + d) > 0$ holds if and only if the matrix

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} \delta & -\gamma/N \\ -\beta N & \alpha \end{pmatrix}$$

satisfies

$$a'c'\big(a'v/D + b'\big)\big(c'v/D + d'\big) > 0.$$

In fact, we have

$$a' = ch + dD, \qquad c' = -N(ah + bD), \qquad a'v/D + b' = -c/DN, \qquad c'v/D + d' = a/D,$$

and as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ goes through every element in $\Gamma_0(N)$ satisfying $ac(ah/D + b)(ch/D + d) > 0$, the corresponding $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ goes through elements in $\Gamma_0(N)$ satisfying $a'c'(a'v/D + b')(c'v/D + d') > 0$. It follows that

$$I_{6,h,0} = \int_{-i\infty}^{i\infty} \sum_{a',b',c',d'} \frac{(-1)^n N^{\tilde{n}} D^w z^w \, dz}{(a'z + a'v/D + b')^{n+1}(c'z + c'v/D + d')^{\tilde{n}+1}}.$$

By (6), this is equal to

$$\frac{4\pi i D^w}{N}\binom{w}{n}\frac{w+2}{B_{w+2}}\frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1}\prod_{p|N}\frac{1-p^{-n-1}}{1-p^{-w-2}}.$$

Remark 3.9. Note that the argument above can be extended to show that

$$r_{m,h/D}(R_n)=(-1)^{n+m}N^{\tilde{n}-m}D^{\tilde{m}-m}r_{\tilde{m},v/D}(R_{\tilde{n}})$$

for $0\leqslant m\leqslant w$ and $1\leqslant n\leqslant w-1$. Here the integer v is the multiplicative inverse of $-Nh$ modulo D since u and v satisfy $Du-Nhv=1$.

3.7.4. Summary for the case $ac(ah/D+b)(ch/D+d)>0$

Combining the computations above, we arrive at the following conclusion.

Lemma 3.10. *We have, for all h with $(h,D)=1$,*

$$\begin{aligned}c_n^{-1}F_{6,h}(X)&=(-1)^n(2i)^{w+1}\frac{w+2}{B_{w+2}}\frac{B_{n+1}}{n+1}\frac{B_{\tilde{n}+1}}{\tilde{n}+1}\\&\quad\times\left(X^w\frac{D^w}{N}\prod_{p|N}\frac{1-p^{-n-1}}{1-p^{-w-2}}-\frac{1}{N^{n+1}}\prod_{p|N}\frac{1-p^{-\tilde{n}-1}}{1-p^{-w-2}}\right),\end{aligned}$$

where the products run over all prime divisors p of N.

3.8. Proof of Theorem 1

This is just a summarization of Lemmas 3.3–3.6, 3.8, and 3.10.

3.9. Proof of Theorem 2

Since $\{f_1,\dots,f_s\}$ is an orthogonal basis, we have

$$R_n=\sum_{i=1}^{s}\frac{\langle R_n,f_i\rangle}{\langle f_i,f_i\rangle}f_i. \tag{7}$$

Now applying $r_{m,\chi}$ to both sides of (7), we obtain

$$\begin{aligned}r_{m,\chi}(R_n)&=\sum_{i=1}^{s}\frac{\langle R_n,f_i\rangle}{\langle f_i,f_i\rangle}r_{m,\chi}(f_i)=\sum_{i=1}^{s}\frac{1}{\langle f_i,f_i\rangle}\overline{r_n(f_i)}r_{m,\chi}(f_i)\\&=\frac{n!}{(2\pi i)^{n+1}}\frac{m!}{(-2\pi i)^{m+1}}\sum_{i=1}^{s}\frac{1}{\langle f_i,f_i\rangle}\overline{L(f_i,n+1)}L(f_i,\chi,m+1).\end{aligned}$$

Then Theorem 2 follows from Theorem 1.

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