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THE HYPERINVARIANT SUBSPACE LATTICE OF A CONTRACTION OF CLASS $C_{\cdot 0}$

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ABSTRACT. It is shown that if T is a $C_{\cdot 0}$ contraction with finite defect indices, then Hyperlat T is (lattice) generated by those subspaces which are either $\ker \psi(T)$ or $\overline{\operatorname{ran} \xi(T)}$, where ψ and ξ are scalar-valued inner functions.

For a bounded linear operator T on a complex Hilbert space H, Hyperlat T denotes the lattice of all hyperinvariant subspaces for T, that is, the lattice of those subspaces which are invariant for all operators commuting with T. Recently, Fillmore, Herrero and Longstaff **[1]** showed that on a finite-dimensional space H, Hyperlat T is (lattice) generated by those subspaces which are either $\ker p(T)$ or $\operatorname{ran} q(T)$, where p and q are polynomials. In this note we generalize this to the following

THEOREM. *Let T be a contraction of class $C_{\cdot 0}$ with finite defect indices acting on a separable Hilbert space. Then* Hyperlat T *is (lattice) generated by those subspaces which are either* $\ker \psi(T)$ *or* $\overline{\operatorname{ran} \xi(T)}$, *where ψ and ξ are scalar-valued inner functions.*

Recall that a contraction T ($\|T\| \leqslant 1$) is of class $C_{\cdot 0}$ if $T^{*n}x \to 0$ for all x. The *defect indices* of T are, by definition, $d_T = \operatorname{rank}(1 - T^*T)^{1/2}$ and $d_{T^*} = \operatorname{rank}(1 - TT^*)^{1/2}$. If T is of class $C_{\cdot 0}$, then $d_T \leqslant d_{T^*}$. For operators T, T' acting on H, H', respectively, $T \overset{\text{ci}}{\prec} T'$ means that there exists a family of operators $\{X_\alpha\}$ from H to H' such that (i) for each α, X_α is one-to-one, (ii) $\bigvee_\alpha X_\alpha H = H'$, and (iii) for each α, $X_\alpha T = T' X_\alpha$. If $T \overset{\text{ci}}{\prec} T'$ and $T' \overset{\text{ci}}{\prec} T$, then T, T' are said to be *completely injection-similar*, and this is denoted by $T \overset{\text{ci}}{\sim} T$. For contractions of class $C_{\cdot 0}$ and with $d_T = m < \infty$, $d_{T^*} = n < \infty$, there has been developed a *Jordan model* which is, in a certain sense, analogous to the Jordan model for finite matrices. More specifically, if T is such a contraction then it is completely injection-similar to a uniquely determined *Jordan operator* of the form

$$S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \oplus S_{n-m},$$

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where φ_i's are nonconstant inner functions satisfying $\varphi_{i-1}|\varphi_i$, $S(\varphi_i)$ denotes the operator on $H^2 \ominus \varphi_i H^2$ which is the compression of the multiplication by z to the space $H^2 \ominus \varphi_i H^2$, $i = 1, \ldots, k$, and S_{n-m} denotes the unilateral shift operator on H^2_{n-m}. For more details, the readers are referred to [3].

We first prove our theorem for the case when T is a Jordan operator.

LEMMA 1. *Let* $T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k)$ *be a Jordan operator. Then* Hyperlat T *is* (*lattice*) *generated by those subspaces which are either* $\ker \psi(T)$ *or* $\overline{\operatorname{ran} \xi(T)}$, *where* ψ *and* ξ *are scalar-valued inner functions.*

PROOF. Let $K \in$ Hyperlat T and for $i = 1, \ldots, k$, let $T_i = S(\varphi_i)$. Uchiyama [4] showed that K corresponds to a regular factorization

$$\begin{bmatrix} \varphi_1 & & 0 \\ & \ddots & \\ 0 & & \varphi_k \end{bmatrix} = \begin{bmatrix} \xi_1 & & 0 \\ & \ddots & \\ 0 & & \xi_k \end{bmatrix} \begin{bmatrix} \psi_1 & & 0 \\ & \ddots & \\ 0 & & \psi_k \end{bmatrix}$$

of the characteristic function

$$\begin{bmatrix} \varphi_1 & & 0 \\ & \ddots & \\ 0 & & \varphi_k \end{bmatrix}$$

of T, where ξ_i, ψ_i satisfy $\xi_{i-1}|\xi_i$, $\psi_{i-1}|\psi_i$, $i = 2, \ldots, k$. Also

$$K = \sum_{i=1}^{k} \oplus \left(\xi_i H^2 \ominus \varphi_i H^2\right).$$

We claim that $K = \bigvee_{i=1}^{k}[\ker \psi_i(T) \cap \overline{\operatorname{ran} \xi_i(T)}]$.

Since for each i, $\xi_i H^2 \ominus \varphi_i H^2 = \ker \psi_i(T_i) = \overline{\operatorname{ran} \xi_i(T_i)}$, one inclusion is trivial. To prove the other, fix j, $1 \leqslant j \leqslant k$, and let $x = \sum_{i=1}^{k} \oplus x_i$ be an element in $\ker \psi_j(T) \cap \overline{\operatorname{ran} \xi_j(T)}$. Let $\{y_n = \sum_{i=1}^{k} \oplus y_{in}\}$ be a sequence of vectors such that $\xi_j(T)y_n \to x$ in norm. Thus for each i, we have $\xi_j(T_i)y_{in} \to x_i$. For $i \leqslant j$, $\xi_i|\xi_j$, and therefore there is an inner ρ_i such that $\xi_j = \xi_i\rho_i$. Hence $\xi_i(T_i)\rho_i(T_i)y_{in} = \xi_j(T_i)y_{in} \to x_i$, which implies $x_i \in \overline{\operatorname{ran} \xi_i(T_i)} = \xi_i H^2 \ominus \varphi_i H^2$. On the other hand, for $j \leqslant i$, $\psi_j|\psi_i$, and therefore there is an inner ω_i such that $\psi_i = \omega_i\psi_j$. Hence $\psi_i(T_i)x_i = \omega_i(T_i)\psi_j(T_i)x_i = 0$, which implies $x_i \in \ker \psi_i(T_i) = \xi_i H^2 \ominus \varphi_i H^2$. It follows $x \in K$, completing the proof.

We remark that in the preceding proof we actually showed that

$$K = \ker \psi_1(T) \vee \left[\bigvee_{i=2}^{k-1} \left(\ker \psi_i(T) \cap \overline{\operatorname{ran} \xi_i(T)}\right) \right] \vee \overline{\operatorname{ran} \xi_k(T)},$$

since for $j = 1, k$, we only used the assumptions $x \in \ker \psi_1(T)$ and $x \in \overline{\operatorname{ran} \xi_k(T)}$ to prove the assertion.

LEMMA 2. *Let* $T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \oplus S_{n-m}$ *be a Jordan operator.*

Then Hyperlat T *is* (*lattice*) *generated by those subspaces which are either* $\ker \psi(T)$ *or* $\overline{\operatorname{ran} \xi(T)}$, *where* ψ *and* ξ *are scalar-valued inner functions.*

PROOF. Let $S = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k)$ and $H = (H^2 \ominus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \ominus \varphi_k H^2)$. Uchiyama showed in **[5]** that the hyperinvariant subspaces of T must be of the form $K_1 \oplus K_2$, where $K_1 \subseteq H$, $K_2 \subseteq H^2_{n-m}$ are hyperinvariant for S, S_{n-m}, respectively, such that either $K_2 = 0$ or there exists an inner function φ such that $K_2 = \varphi H^2_{n-m}$ and $K_1 \supseteq \overline{\varphi(S)H}$. Note that for any inner function φ, $\ker \varphi(S_{n-m}) = 0$ and $\overline{\operatorname{ran} \varphi(S_{n-m})} = \varphi H^2_{n-m}$. Thus by Lemma 1 we can easily check that if $K_2 = 0$ then

$$K_1 \oplus K_2 = K_1 \oplus 0 = \bigvee_{i=1}^{k} \left[\ker \psi_i(T) \cap \overline{\operatorname{ran} \xi_i(T)}\,\right],$$

otherwise

$$K_1 \oplus K_2 = \overline{\operatorname{ran} \varphi(T)} \vee \left[\bigvee_{i=1}^{k} \left(\ker \psi_i(T) \cap \overline{\operatorname{ran} \xi_i(T)}\right)\right].$$

This proves our assertion.

PROOF OF THEOREM. Let T be completely injection-similar to its Jordan model $T' = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \oplus S_{n-m}$, and suppose that T and T' are acting on the spaces H and H', respectively. Note that the complete injection-similarity can be implemented by two suitably chosen operators $\{X_1, X_2\}$ from H to H' and two operators $\{Y_1, Y_2\}$ from H' to H (cf. **[2]** and **[3]**). Uchiyama **[5]** showed that in this case the induced mappings α: $K \to X_1 K \vee X_2 K$ and β: $K' \to Y_1 K' \vee Y_2 K'$ are (lattice) isomorphisms between Hyperlat T and Hyperlat T', which are inverses to each other. Thus in view of Lemmas 1 and 2 to complete the proof we have only to show that (i) $\beta(\ker \psi(T')) = \ker \psi(T)$ and (ii) $\beta(\overline{\operatorname{ran} \xi(T')}) = \overline{\operatorname{ran} \xi(T)}$ hold for arbitrary ψ, ξ in H^∞.

To prove (i), let $x = Y_1 y$, where $y \in \ker \psi(T')$. Since $\psi(T)x = \psi(T)Y_1 y = Y_1 \psi(T')y = 0$, we have $x \in \ker \psi(T)$. This shows that $Y_1 \ker \psi(T') \subseteq \ker \psi(T)$. Similarly, $Y_2 \ker \psi(T') \subseteq \ker \psi(T)$, and hence $\beta(\ker \psi(T')) \subseteq \ker \psi(T)$. In a similar fashion, we have $\alpha(\ker \psi(T)) \subseteq \ker \psi(T')$. Thus $\ker \psi(T) = \beta(\alpha(\ker \psi(T))) \subseteq \beta(\ker \psi(T'))$, which proves (i). (ii) can be proved analogously. This finishes the proof of the Theorem.

Fillmore, Herrero and Longstaff's result **[1]** follows as a corollary.

COROLLARY. *Let T be a linear transformation on a finite-dimensional space H. Then* Hyperlat T *is* (*lattice*) *generated by those subspaces which are either* $\ker p(T)$ *or* $\operatorname{ran} q(T)$, *where p and q are polynomials.*

PROOF. For $0 < \alpha < 1/\|T\|$, $S \equiv \alpha T$ is a strict contraction, hence a contraction of class $C_{\cdot 0}$. The Theorem implies that Hyperlat S = Hyperlat T is (lattice) generated by those subspaces which are either $\ker U$ or $\operatorname{ran} V$, where U, V are operators in $\{S\}'' = \{T\}''$, the double commutants of S and T. Our assertion follows from the fact that $\{T\}''$ consists of polynomials in T.

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