QUASI-SIMILARITY OF WEAK CONTRACTIONS

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ABSTRACT. Let T be a completely nonunitary (c.n.u.) weak contraction (in the sense of Sz.-Nagy and Foiaş). We show that T is quasi-similar to the direct sum of its C_0 part and C_{11} part. As a corollary, two c.n.u. weak contractions are quasi-similar to each other if and only if their C_0 parts and C_{11} parts are quasi-similar to each other, respectively. We also completely determine when c.n.u. weak contractions and C_0 contractions are quasi-similar to normal operators.

Recall that a contraction T on the Hilbert space H is called a weak contraction if its spectrum $\sigma(T)$ does not fill the open unit disc D and $1 - T^*T$ is of finite trace. Contained in this class are all contractions T with finite defect index $d_T \equiv \dim \operatorname{rank}(1 - T^*T)^{1/2}$ and with $\sigma(T) \neq \bar{D}$ (cf. [9, p. 323]).

Assume that T is a weak contraction which is also completely nonunitary (c.n.u.), that is, T has no nontrivial reducing subspace on which T is a unitary operator. For such a contraction, Sz.-Nagy and Foiaş obtained a C_0-C_{11} decomposition and then found a variety of invariant subspaces which furnish its spectral decomposition (cf. [9, Chapter VIII]). In this note we are going to supplement other interesting properties of such contractions. We show that a c.n.u. weak contraction is quasi-similar to the direct sum of its C_0 part and C_{11} part. Although the proof is not difficult, some of its interesting applications justify the elaboration here. An immediate corollary is that two such contractions are quasi-similar to each other if and only if their C_0 parts are quasi-similar and their C_{11} parts are quasi-similar to each other. This is, in turn, used to show that two quasi-similar weak contractions have equal spectra. Another interesting consequence is that a c.n.u. weak contraction is quasi-similar to a normal operator if and only if its C_0 part is. The latter can be shown to be equivalent to the condition that its minimal function is a Blaschke product with simple zeros, thus completely settling the question when a c.n.u. weak contraction is quasi-similar to a normal operator.

Before we start to prove our main theorem, we provide some background work for our notations and terminology. The main reference is [9].

Let T be an arbitrary contraction on H. Let $H_0 = \{h \in H: T^n h \to 0\}$, $H_0' = \{h \in H: T^{*n} h \to 0\}$, $H_1' = H \ominus H_0$ and $H_1 = H \ominus H_0'$. Note that H_0 and H_0' are invariant for T and T^*, respectively. Consider the triangulations of T with respect to the orthogonal decompositions $H = H_0 \oplus H_1'$ and $H = H_1 \oplus H_0'$:

$$T = \begin{bmatrix} T_0 & X \\ 0 & T_1' \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & Y \\ 0 & T_0' \end{bmatrix}.$$

The triangulations are of type

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix},$$

respectively (cf. [9, p. 73]). Recall that a contraction T is of class $C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n h \to 0$ (resp. $T^{*n} h \to 0$) as $n \to \infty$ for all h and T is of class $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n h \nrightarrow 0$ (resp. $T^{*n} h \nrightarrow 0$) as $n \to \infty$ for all $h \neq 0$. T is of class C_{00} if $T \in C_{0\cdot} \cap C_{\cdot 0}$ and of class C_{11} if $T \in C_{1\cdot} \cap C_{\cdot 1}$. A c.n.u. contraction T is said to be of class C_0 if there exists a nonzero function $u \in H^\infty$ such that $u(T) = 0$. In this case we can choose u to be a minimal inner function in the sense that u is an inner function such that $u(T) = 0$ and u divides (in H^∞) every other function $v \in H^\infty$ for which $v(T) = 0$. Such a function is called a minimal function for T and is denoted by m_T. If T is a c.n.u. weak contraction, then in the previous triangulations T_0 is of class C_0 and T_1 is of class C_{11}, called the C_0 part and the C_{11} part of T (cf. [9, p. 331]). Note that in this case we have $H_0 \vee H_1 = H$ and $H_0 \cap H_1 = \{0\}$ (cf. [9, p. 332]). For arbitrary operators T_1, T_2 on H_1, H_2, respectively, $T_1 \prec T_2$ denotes that T_1 is a quasi-affine transform of T_2, that is, there exists a linear one-to-one and continuous transformation S from H_1 onto a dense linear manifold in H_2 (called quasi-affinity) such that $S T_1 = T_2 S$. T_1 and T_2 are quasi-similar if $T_1 \prec T_2$ and $T_2 \prec T_1$.

Our main theorem is the following:

THEOREM 1. *Let T be a c.n.u. weak contraction on H. Let T_0 and T_1 be the C_0 part and C_{11} part of T. Then T is quasi-similar to $T_0 \oplus T_1$.*

PROOF. Let $S: H_0 \oplus H_1 \to H$ be defined by $S(h_0 \oplus h_1) = h_0 + h_1$. Certainly T is a continuous linear transformation. Since $H_0 \vee H_1 = H$ and $H_0 \cap H_1 = \{0\}$, it is easily seen that S is a quasi-affinity such that $S(T_0 \oplus T_1) = TS$. Thus $T_0 \oplus T_1 \prec T$. Note that T^* is also a c.n.u. weak contraction and $T_0'^*$ and $T_1'^*$ are the C_0 and C_{11} parts of T^* (cf. [9, p. 332]). As above, we have $T_0'^* \oplus T_1'^* \prec T^*$. Hence $T \prec T_0' \oplus T_1'$, and $T_0 \oplus T_1 \prec T \prec T_0' \oplus T_1'$. Let V be the quasi-affinity from $H_0 \oplus H_1$ to $H_0' \oplus H_1'$ such that $V(T_0 \oplus T_1) = (T_0' \oplus T_1')V$. Since T_0 and T_1' are of class $C_{0\cdot}$ and $C_{1\cdot}$, respectively, it is easily seen that $VH_0 \subseteq H_0'$. Say,

$$V = \begin{bmatrix} V_0 & Z \\ 0 & V_1 \end{bmatrix}$$

is the corresponding triangulation. An easy calculation shows that $ZT_1 = T_0'Z$. Since T_1 is of class C_{11} and T_0' is of class C_{00}, we must have $Z = 0$ (cf. [4, Lemma 4.4]). Thus V_0 and V_1 are quasi-affinities satisfying $V_0T_0 = T_0'V_0$ and $V_1T_1 = T_1'V_1$. Hence $T_0 \prec T_0'$ and $T_1 \prec T_1'$. It follows from the uniqueness of the Jordan model for C_0 contractions that T_0 and T_0' are quasi-similar to each other (cf. [2]). To show that T_1 is quasi-similar to T_1', note that T_1 and T_1', being C_{11} contractions, are quasi-similar to unitary operators, say U_1 and U_1', respectively. We have $U_1 \prec U_1'$. By a theorem of Douglas [5], U_1 and U_1' are unitarily equivalent. Hence T_1 is quasi-similar to T_1', and T is quasi-similar to $T_0 \oplus T_1$.

An immediate corollary of Theorem 1 is

COROLLARY 1. *Let T_1 and T_2 be c.n.u. weak contractions. Then T_1 and T_2 are quasi-similar to each other if and only if their C_0 parts are quasi-similar and their C_{11} parts are quasi-similar to each other.*

PROOF. The sufficiency follows immediately from Theorem 1. The necessity can be proved by a similar argument as in Theorem 1.

In particular, for c.n.u. contractions with scalar-valued characteristic functions, we have

COROLLARY 2. *For $j = 1, 2$, let T_j be a c.n.u. contraction with the scalar-valued characteristic function $\psi_j \not\equiv 0$. Let $\psi_j = \psi_{ji}\psi_{je}$ be the canonical factorization into the product of its inner part ψ_{ji} and outer part ψ_{je}, and let $E_j = \{e^{it}: |\psi_j(e^{it})| < 1\}$. Let*

$$T_j = \begin{pmatrix} T_{j1} & X_j \\ 0 & T_{j2} \end{pmatrix}$$

be the triangulation of type

$$\begin{bmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{bmatrix}, \quad j = 1, 2.$$

Then the following are equivalent:
  (i) T_1 *is quasi-similar to* T_2;
  (ii) T_{11} *is quasi-similar to* T_{21} *and* T_{12} *is unitarily equivalent to* T_{22};
  (iii) $\psi_{1i} = \psi_{2i}$ *and* E_1 *and* E_2 *differ by a set of zero Lebesgue measure.*

PROOF. Since T_1 and T_2 are c.n.u. weak contractions, the equivalence of (i) and (ii) follows from Corollary 1. Note that T_{j1} is quasi-similar to the multiplication by e^{it} on the space $L^2(E_j)$ and T_{j2} is unitarily equivalent to the compression of the shift $S(\psi_{ji})$ on $H^2 \ominus \psi_{ji}H^2, j = 1, 2$. Thus the equivalence of (ii) and (iii) follows immediately.

The equivalence of (i) and (iii) in Corollary 2 is compatible with the result of Kriete [7] that T_1 is similar to T_2 if and only if $\psi_1/\psi_2, \psi_2/\psi_1 \in H^\infty$ and E_1 and E_2 differ by a set of zero Lebesgue measure.

COROLLARY 3. *Let T_1 and T_2 be c.n.u. weak contractions. If T_1 and T_2 are*

quasi-similar to each other, then $\sigma(T_1) = \sigma(T_2)$.

PROOF. For $j = 1, 2$, let T_{j0} and T_{j1} be the C_0 part and C_{11} part of T_j. By Corollary 1, T_{10} and T_{11} are quasi-similar to T_{20} and T_{21}, respectively. Since the spectrum of a C_0 contraction is completely determined by its minimal function [9, p. 126], and T_{10} and T_{20} have the same minimal function, we have $\sigma(T_{10}) = \sigma(T_{20})$.

To show that $\sigma(T_{11}) = \sigma(T_{21})$, let U_j be the residual part of the minimal unitary dilation of $T_{j1}, j = 1, 2$ (cf. [9, p. 61]). Note that T_{j1} is quasi-similar to U_j and $\sigma(T_{j1})$ lies entirely on the unit circle (cf. [9, pp. 75, 328]). It follows that $\sigma(T_{j1}) = \sigma(U_j)$ (cf. [9, pp. 311–312]). By Douglas' theorem [5], U_1 and U_2 are quasi-similar implies they are unitarily equivalent. Thus $\sigma(T_{11}) = \sigma(U_1) = \sigma(U_2) = \sigma(T_{21})$. Since $\sigma(T_j) = \sigma(T_{j0}) \cup \sigma(T_{j1})$ [9, p. 332], we have $\sigma(T_1) = \sigma(T_2)$, completing the proof.

We remark that the proof can be modified to show that quasi-similar weak contractions (not necessarily c.n.u.) have equal spectra. This result is not new. It also follows from the facts that weak contractions are decomposable [6] and quasi-similar decomposable operators have equal spectra [3]. However, our proof seems more direct.

In the remaining part of this note we are concerned with the question when a c.n.u. weak contraction is quasi-similar to a normal operator. The next theorem reduces the problem to the C_0 part of the c.n.u. weak contraction.

THEOREM 2. *Let T be a c.n.u. weak contraction on H. Let*

$$T = \begin{bmatrix} T_0 & X \\ 0 & T_1' \end{bmatrix}$$

be the triangulation of type

$$\begin{bmatrix} C_{0\cdot} & * \\ 0 & C_{1\cdot} \end{bmatrix}$$

on the (orthogonal) decomposition $H = H_0 \oplus H_1'$. Then T is quasi-similar to a normal operator if and only if T_0 is.

PROOF. The sufficiency follows trivially from Theorem 1. To prove the necessity, we may assume that T is quasi-similar to a normal operator N on the space K with $\|N\| \leqslant \|T\| \leqslant 1$ (cf. [1, Proof of the sufficiency part of Theorem]). Let $K = K_1 \oplus K_2$ be the direct sum of reducing subspaces for N such that $N_1 \equiv N|K_1$ is c.n.u. and $N_2 \equiv N|K_2$ is unitary. Let S be the quasi-affinity from H to K such that $ST = NS$. Since T_0 is of class $C_{0\cdot}$ and N_2 is of class C_{11}, it is easily seen that $SH_0 \subseteq K_1$. Note that $\overline{SH_0}$ is an invariant subspace for N_1. Let $N_1' = N_1|\overline{SH_0}$. Then $S_1 \equiv S|H_0$ is a quasi-affinity from H_0 to $\overline{SH_0}$ satisfying $S_1 T_0 = N_1' S_1$. Since T_0 is of class C_0, so is N_1' (cf. [9, p. 125]). By the uniqueness of the Jordan model for C_0 contractions, we have T_0 is quasi-similar to N_1' (cf. [2]). Since N_1' is subnormal and $\sigma(N_1')$ has planar

area zero (cf. [**9**, p. 126]), it follows from Putnam's theorem [**8**] that N_1' is normal. This completes the proof.

Notice that Theorem 2 is compatible with the result that T is similar to a normal operator if and only if T_0 is similar to a normal operator and T_1' is similar to a unitary operator. This is true even for an arbitrary c.n.u. contraction (cf. [**10**, Theorem 3]).

Since the C_0 part of a c.n.u. weak contraction is a C_0 contraction, the next theorem furnishes the complete solution to the previously posed question.

THEOREM 3. *Let T be a C_0 contraction on the space H with the minimal function m_T. Then T is quasi-similar to a normal operator if and only if m_T is a Blaschke product with simple zeros.*

PROOF. *Necessity.* Let T be quasi-similar to the normal operator N on the space K and let S be the quasi-affinity from H to K such that $ST = NS$. As before we may assume that $\|N\| \leqslant \|T\| \leqslant 1$ (cf. [**1**]). Now we show that N must be c.n.u. Indeed, for any $k \in K$ and $\varepsilon > 0$, let $h \in H$ be such that $\|k - Sh\| < \varepsilon$. Since $ST^n h = N^n Sh \to 0$ as $n \to \infty$, we have $\|N^n Sh\| < \varepsilon$ for all $n \geqslant N_0$. Hence

$$\|N^n k\| \leqslant \|N^n k - N^n Sh\| + \|N^n Sh\| \leqslant \|N\|^n \|k - Sh\| + \|N^n Sh\|$$
$$< \varepsilon + \varepsilon = 2\varepsilon \quad \text{for all } n \geqslant N_0.$$

This shows that $N^n k \to 0$ for all $k \in K$ and hence N is c.n.u. Since N is quasi-similar to a C_0 contraction, N is also a C_0 contraction with the same minimal function $m_N = m_T$ (cf. [**9**, p. 125]). Let $m_T = Bs$, where

$$B(\lambda) = \prod_i \frac{\bar{\lambda}_i}{|\lambda_i|} \left(\frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda} \right)^{n_i}$$

is a Blaschke product and s is a singular function. Note that λ_i is a characteristic value of N with index n_i (cf. [**9**, p. 135]). Since N is a normal operator, $n_i = 1$ for all i. Let K_i be the corresponding eigenspace. Then $\bigvee_i K_i$ reduces N and the normal operator $N_1 \equiv N|(\bigvee_i K_i)^\perp$ has no eigenvalue. Hence the minimal function of the C_0 contraction N_1 must be s (cf. [**9**. p. 129]). It follows that $\sigma(N_1)$ is contained in the unit circle, and thus N_1 is a unitary operator. Since N is c.n.u., we must have $(\bigvee_i K_i)^\perp = \{0\}$ and $K = \bigvee_i K_i$. Hence $m_T = B$ is a Blaschke product with simple zeros (cf. [**9**, p. 135]).

Sufficiency. Assume that m_T is a Blaschke product with simple zeros, say,

$$m_T(\lambda) = \prod_i \frac{\bar{\lambda}_i}{|\lambda_i|} \frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda},$$

where the distinct λ_i's satisfy $|\lambda_i| < 1$ and $\Sigma_i(1 - |\lambda_i|) < \infty$. For each i let $H_i = \{h \in H: (T - \lambda_i)h = 0\}$. Then $T|H_i$ is a normal operator and the system $\{H_i\}_{i=1}^\infty$ of invariant subspaces satisfies

$$H = H_i \dotplus \bigvee_{j \neq i} H_j \quad \text{for each } i, \quad \text{and} \quad \bigcap_i \left(\bigvee_{j \geqslant i} H_j \right) = \{0\}$$

(cf. [9, pp. 135, 131]). That is, $\{H_i\}_{i=1}^{\infty}$ is a basic system of invariant subspaces for T. By a result of Apostol [1], T is quasi-similar to a normal operator, completing the proof.

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