Classical Lagrangian theory with radiative reaction: Extension of the Rohrlich two-field formalism to include monopoles

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We give an action integral $L(x, X, A_\mu, a_\mu)$. The stability of L against the variations $\delta A_\mu(\xi)$, $\delta a_\mu(\xi)$, $\delta x_\mu(\theta)$, and $\delta X_\mu(\tau)$ gives us the coupled Maxwell equations and the Lorentz-Dirac equation for the positron and monopole.

I. INTRODUCTION

It was Dirac who introduced the concept of magnetic monopole[1] in 1931. Then he came back in 1948 to the question[2] of the classical action principle. In Dirac's formulation, it was necessary to introduce the concept of strings attached to monopoles, because his vector potential for a static magnetic monopole is singular along a semi-infinite line in three-space. Recently, Wu and Yang[3] were able to formulate this classical problem without introducing the Dirac strings. The key points in Wu and Yang's formulation are (1) dividing the space-time into many overlapping regions and (2) introducing the potential A_μ in each region such that one of the Maxwell equations becomes a kinematic equation. But in their formulation, Wu and Yang did not take the classical radiative reaction into consideration, so that the equations they obtained from the variation of the action integral can only be regarded as formal equations and possess no finite solution.

It is the purpose of this paper to find a classical Lagrangian theory of positrons and magnetic monopoles including the radiative reaction. The essential point that permits a solution of this problem is the realization[4] that one is dealing with not only one electromagnetic field, but with two such fields. One satisfies the homogeneous Maxwell equations. The other satisfies the inhomogeneous Maxwell equations. They are mathematically and physically entirely different. Although the separation of the total electromagnetic field into the above-mentioned two parts is not unique it can be made unambiguous[4] by the proper boundary conditions.

We will write down the action integral in Sec. II and study the Euler-Lagrange equations for this action integral in Sec. III. Finally we will give some concluding remarks.

II. THE ACTION INTEGRAL

We will use the same notations as in Ref. 3, in which ξ^μ are the space-time coordinates, $x^\mu(\theta)$ is the world line of a positron with electric charge e, and $X^\mu(\tau)$ is the world line of a magnetic monopole with magnetic charge g. Here θ and τ are, respectively, the proper times of positron and monopole. The metric used is $\eta_{\mu\nu} = (-1, 1, 1, 1)$, the relation between the electromagnetic field $F_{\mu\nu}$ and its dual $\bar{F}_{\mu\nu}$ is

$$\bar{F}_{\mu\nu} = -\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}, \quad F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{F}^{\alpha\beta}, \tag{1}$$

where $\epsilon^{0123} = -1$ and $\epsilon^{\mu\nu\alpha\beta}$ = complete antisymmetric tensor.

The action integral L is

$$L = -m\int d\theta - M\int d\tau - \frac{1}{16\pi}\int F_{\mu\nu}(\xi)F^{\mu\nu}(\xi)d^4\xi - \frac{1}{8\pi}\int F_{\mu\nu}(\xi)h_+^{\mu\nu}(\xi)d^4\xi - \frac{1}{8\pi}\int F_{\mu\nu}(\xi)f_+^{\mu\nu}(\xi)d^4\xi - \frac{1}{8\pi}\int h_{\mu\nu+}(\xi)f_+^{\mu\nu}(\xi)d^4\xi + e\int A_\mu(x(\theta))dx^\mu + e\int b_\mu(x(\theta))dx^\mu, \tag{2}$$

where

$$F_{\mu\nu}(\xi) = \frac{\partial A_\mu(\xi)}{\partial\xi^\nu} - \frac{\partial A_\nu(\xi)}{\partial\xi^\mu}, \tag{3}$$

$$h_{\mu\nu+}(\xi) = \frac{\partial a_\mu(\xi)}{\partial\xi^\nu} - \frac{\partial a_\nu(\xi)}{\partial\xi^\mu}, \tag{4}$$

and

$$\begin{aligned} f_{\mu\nu+}(\xi) &= \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{f}_+^{\alpha\beta}(\xi), \\ \bar{f}_+^{\alpha\beta}(\xi) &= \frac{\partial\bar{b}^\alpha(\xi)}{\partial\xi_\beta} - \frac{\partial\bar{b}^\beta}{\partial\xi_\alpha} \\ &= \tfrac{1}{2}(\bar{f}_{\text{ret}}^{\alpha\beta} + \bar{f}_{\text{adv}}^{\alpha\beta}), \\ \bar{b}^\alpha(\xi) &= g\int \dot{X}^\alpha(\tau)\delta((\xi - X)^2)d\tau \end{aligned} \tag{5}$$

so that

$$f_{+}^{\mu\nu}{}_{,\nu}(\xi) = 0,$$

$$\bar{f}_{+}^{\mu\nu}{}_{,\nu}(\xi) = -4\pi g\int \dot{X}^\mu\delta^4(\xi - X)d\tau.$$

The last term of the action integral L requires some explanations. As in Ref. 3, the world lines $x(\theta)$ and $X(\tau)$ are constrained to be timelike. Furthermore, they must not cross. That is, $\vec{x}(t) - \vec{X}(t) \neq 0$ for all t. The reasons can be found in Ref. 3. Now, since world lines $x(\theta)$ and $X(\tau)$ do not cross, we can always find a three-dimensional surface S which divides the space-time into two regions E_1 and E_2, where E_1 contains the positron world line only and E_2 contains the monopole world line only. The field $b_\mu(\xi)$, which appears in the last term of L, satisfies the equation

$$\frac{\partial b_\mu(\xi)}{\partial \xi^\nu} - \frac{\partial b_\nu(\xi)}{\partial \xi^\mu} = f_{\mu\nu+}(\xi), \quad \xi \in E_1. \tag{6}$$

Since in region E_1, $\bar{f}_{+}^{\mu\nu}{}_{,\nu}(\xi) = 0$, and region E_1 is a simple connected region, so that there exists a function $b_\mu(\xi)$ satisfying Eq. (6). We only need to know that $b_\mu(\xi)$ exists and do not need to write down the explicit form of $b_\mu(\xi)$. The independent dynamical variables for the action integral L are $x_\mu(\theta)$, $X_\mu(\tau)$, $A_\mu(\xi)$, and $a_\mu(\xi)$.

III. STABILITY OF THE ACTION INTEGRAL L

It is easy to show that the stability of L against the variations δA_μ, δa_μ, and $\delta x_\mu(\theta)$ gives us, respectively,

$$[F^{\mu\nu}(\xi) + h_+^{\mu\nu}(\xi) + f_+^{\mu\nu}(\xi)]_{,\nu} = -4\pi e \int \dot{x}^\mu(\theta)\delta^4(\xi - x)d\theta, \tag{7}$$

$$[F^{\mu\nu}(\xi) + f_+^{\mu\nu}(\xi)]_{,\nu} = 0, \tag{8}$$

$$m\ddot{x}^\mu(\theta) = -eF^{\mu\nu}(x)\dot{x}_\nu(\theta) - ef_+^{\mu\nu}(x)\dot{x}_\nu(\theta). \tag{9}$$

Since $f_+^{\mu\nu}{}_{,\nu}(\xi) = 0$, Eqs. (7) and (8) become

$$h_+^{\mu\nu}{}_{,\nu}(\xi) = -4\pi e \int \dot{x}^\mu(\theta)\delta^4(\xi - x)d\theta \tag{10}$$

and

$$F^{\mu\nu}{}_{,\nu}(\xi) = 0. \tag{11}$$

Using the boundary conditions in Ref. 4 to separate the total electromagnetic field into free part and singular part, we will get

$$a^\mu(\xi) = e \int \dot{x}^\mu(\theta)\delta((\xi - x)^2)d\theta, \tag{12}$$

$$h_+^{\mu\nu}(\xi) = \frac{\partial a^\mu(\xi)}{\partial \xi_\nu} - \frac{\partial a^\nu(\xi)}{\partial \xi_\mu} = \tfrac{1}{2}[h_{\text{ret}}^{\mu\nu}(\xi) + h_{\text{adv}}^{\mu\nu}(\xi)], \tag{13}$$

and

$$F_{\text{total}}^{\mu\nu} = F^{\mu\nu} + f_+^{\mu\nu} + h_+^{\mu\nu} = F_{\text{in}}^{\mu\nu} + f_{\text{ret}}^{\mu\nu} + h_{\text{ret}}^{\mu\nu} = F_{\text{out}}^{\mu\nu} + f_{\text{adv}}^{\mu\nu} + h_{\text{adv}}^{\mu\nu}. \tag{14}$$

The notations in Eq. (14) are self-evident. From Eq. (14) we get

$$F^{\mu\nu} = \tfrac{1}{2}(F_{\text{in}}^{\mu\nu} + F_{\text{out}}^{\mu\nu}) = F_{\text{in}}^{\mu\nu} + \tfrac{1}{2}(f_{\text{ret}}^{\mu\nu} - f_{\text{adv}}^{\mu\nu}) + \tfrac{1}{2}(h_{\text{ret}}^{\mu\nu} - h_{\text{adv}}^{\mu\nu}). \tag{15}$$

Thus Eq. (9) becomes

$$m\ddot{x}^\mu(\theta) = -eF_{\text{in}}^{\mu\nu}\dot{x}_\nu(\theta) - e\tfrac{1}{2}(h_{\text{ret}}^{\mu\nu} - h_{\text{adv}}^{\mu\nu})\dot{x}_\nu - ef_{\text{ret}}^{\mu\nu}\dot{x}_\nu. \tag{16}$$

One can show[2] that $\tfrac{1}{2}(h_{\text{ret}}^{\mu\nu} - h_{\text{adv}}^{\mu\nu})$ is regular at the positron world line $x(\theta)$. This term is the radiative-reaction term which is generally accepted.

Now let us consider the stability of L against the variation $\delta X^\mu(\tau)$. The original form of the action integral L is not convenient to take the variation with respect to $X^\mu(\tau)$. We proceed as follows:

(a) Let L_0 be the extremum of L with respect to the variations $\delta A_\mu(\xi)$, $\delta a_\mu(\xi)$, and $\delta x_\mu(\theta)$. Thus $L_0(X,\dot{X})$ is equal to the value of L evaluated at those functions $A_\mu(\xi)$, $a_\mu(\xi)$, and $x_\mu(\theta)$ which satisfy the equations of motion, Eqs. (10), (11), and (16). Now consider the term

$$-\frac{1}{8\pi}\int F_{\mu\nu}(\xi)f_+^{\mu\nu}(\xi)d^4\xi = \frac{1}{8\pi}\int \bar{F}_{\mu\nu}(\xi)\bar{f}_+^{\mu\nu}(\xi)d^4\xi = \frac{1}{8\pi}\int(\bar{A}_{\mu,\nu} - \bar{A}_{\nu,\mu})\bar{f}_+^{\mu\nu}(\xi)d^4\xi = -\frac{1}{4\pi}\int \bar{A}_\mu \bar{f}_+^{\mu\nu}{}_{,\nu}(\xi)d^4\xi + \text{terms at infinity} = g\int \bar{A}_\mu(X)\dot{X}^\mu d\tau + \text{terms at infinity}. \tag{17}$$

Because $F^{\mu\nu}{}_{,\nu}(\xi) = 0$, there always exists $\bar{A}_\mu(\xi)$, such that

$$\bar{A}_{\mu,\nu}(\xi) - \bar{A}_{\nu,\mu}(\xi) = \bar{F}_{\mu\nu}(\xi).$$

Now consider the term

$$-\frac{1}{8\pi}\int h_{\mu\nu+}(\xi)f_{+}^{\mu\nu}(\xi)d^4\xi+e\int b_\mu(x)dx^\mu\,.$$

As before, let S is a three-dimensional surface which divides the space-time into regions E_1 and E_2. Then

$$\begin{aligned}-\frac{1}{8\pi}\int h_{\mu\nu+}(\xi)f_{+}^{\mu\nu}(\xi)d^4\xi+e\int b_\mu(x)dx^\mu &= -\frac{1}{8\pi}\int_{E_1+E_2} h_{\mu\nu+}(\xi)f_{+}^{\mu\nu}(\xi)d^4\xi+e\int b_\mu(x)dx^\mu\\ &=\frac{1}{4\pi}\int_{E_1} h_{\mu\nu+}{}^{,\nu}(\xi)b^\mu d^4\xi-\frac{1}{4\pi}\int_{S_{E_1}} h_{\mu\nu+}b^\mu d\sigma^\nu(\xi)\\ &\quad+\frac{1}{4\pi}\int_{S_{E_2}}\bar{a}_\mu(\xi)\bar{f}_{+}^{\mu\nu}(\xi)d\sigma_\nu(\xi)-\frac{1}{4\pi}\int_{E_2}\bar{a}_\mu(\xi)\bar{f}_{+}^{\mu\nu}{}_{,\nu}(\xi)d^4\xi+e\int b_\mu(x)dx^\mu\\ &=g\int\bar{a}_\mu(X)dx^\mu+\text{surface term}+\text{terms at infinity.}\end{aligned}\tag{18}$$

Here the existence of $\bar{a}_\mu(\xi)$ is ensured by the condition

$$h^{\mu\nu}{}_{,\nu}(\xi)=0$$

in E_2.

One can show[3] that the two terms in Eq. (18) can be combined and are equal to terms at infinity. Thus

$$\begin{aligned}L_0=-M\int d\tau+g\int\bar{A}_\mu(X)dX^\mu+g\int\bar{a}_\mu(X)dX^\mu\\ +\text{terms which are irrelevant}\\ +\text{terms at infinity.}\end{aligned}$$

(b) The stability of L against the variation $\delta X(\tau)$ is the same as stability of L_0 against the variation $\delta X(\tau)$; thus we get

$$\begin{aligned}M\ddot{X}^\mu(\tau)&=-g\bar{F}^{\mu\nu}(X)\dot{X}_\nu-g\bar{h}_{+}^{\mu\nu}(X)\dot{X}_\nu\\ &=-g\bar{F}_{\text{in}}^{\mu\nu}(X)\dot{X}_\nu-\tfrac{1}{2}g(\bar{f}_{\text{ret}}^{\mu\nu}-\bar{f}_{\text{adv}}^{\mu\nu})\dot{X}_\nu\\ &\quad+g\bar{h}_{\text{ret}}^{\mu\nu}\dot{X}_\nu\,.\end{aligned}\tag{19}$$

In conclusion, we find that the stability of the action integral L against the variations δA_μ, δa_μ, δx, and δX gives, respectively, Eqs. (7), (8), (9), and (19).

IV. CONCLUDING REMARKS

(1). We get an action integral L. The stability of L against the variations of independent dynamical variables gives us the Maxwell equations and the equations of motion for the positron and monopole including the radiative reaction.

(2). The essential point that makes this possible is to separate the electromagnetic field into free part and singular part unambiguously. (See Ref. 4.)

(3). We need only to know the existence of $b_\mu(\xi)$ in E_1 and $\bar{a}_\mu(\xi)$ in E_2 and do not need to write down the explicit forms for $b_\mu(\xi)$ and $\bar{a}_\mu(\xi)$. But there are many solutions for $b_\mu(\xi)$ in E_1. Different solutions may give different values of the action integral L. In fact, L is a multivalued functional.[3]

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