CHAPTER 5

UNIFORMLY DISTRIBUTED SIMPLEX SLIDING-MODE CONTROL WITH GREY PREDICTION

This chapter proposes a novel design method of the uniformly distributed simplex sliding-mode control (UDSSMC) combined with grey prediction, which is used to predict the matched disturbances. Section 5.1 gives an introduction and Section 5.2 shows the system description. In Section 5.3, the grey prediction is briefly introduced. Then, the UDSSMC incorporated with grey prediction is developed in Section 5.4 and a numeric example is shown in Section 5.5 for demonstration.

5.1 Introduction

From the previous chapters, it is explicitly shown that the developed uniformly distributed simplex sliding-mode control (UDSSMC) is an effective control algorithm for systems suffering from matched disturbances under the condition their upper bounds are given. However, it is not an easy work to precisely estimate these upper bounds in practice. In order to get rid of the matched disturbances, their upper bounds are often over-estimated or too conservative. As a result, the control algorithm has to increase largely and maybe unreasonable. In case the upper bounds are not well estimated, the control algorithm could be inadequate to suppress the matched disturbances, which makes the control unsuccessful. Obviously, it is important to effectively predict the matched disturbances when the UDSSMC is applied.

To tackle this problem, we will propose a novel design method of the UDSSMC combined with grey prediction, which is used to predict the matched disturbances. By means of grey prediction, the matched disturbances can be suppressed without any prior information concerning their upper bounds. In this chapter, the system description is given in Section 5.2 and the grey prediction is briefly introduced in Section 5.3. Then, the UDSSMC incorporated with grey prediction is developed in Section 5.4. Finally, a numeric example is shown in Section 5.5 to demonstrate the advantage of the UDSSMC incorporated with grey prediction.

5.2 System Description

In general, a linear time-invariant system suffering from matched disturbance is described as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{d}(\boldsymbol{x},t) \tag{5.1}$$

where $\boldsymbol{x} \in \Re^n$ is the system state, $\boldsymbol{u} \in \Re^{\boldsymbol{m}}$ is the system input, and $\boldsymbol{d}(\boldsymbol{x},t) \in \Re^m$ represents the matched disturbance. In addition, the pair ($\boldsymbol{A}$,$\boldsymbol{B}$) is assumed to be controllable and $\boldsymbol{B}$ is of full rank. The matched disturbance, represented by $\boldsymbol{d}(\boldsymbol{x},t) = [d_1(\boldsymbol{x},t) \quad d_2(\boldsymbol{x},t) \quad \cdots \quad d_{\boldsymbol{r}}(\boldsymbol{x},t)]^{\boldsymbol{T}} \in \Re^{\boldsymbol{m}}$, is considered to be of the form [52]

$$\boldsymbol{d}(\boldsymbol{x},t) = \Delta\boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{f}(t) \tag{5.2}$$

which possesses state-dependent uncertainty $\Delta\boldsymbol{A}(t)\boldsymbol{x}$ and external unknown input $\boldsymbol{f}(t)$. Since both $\Delta\boldsymbol{A}(t)$ and $\boldsymbol{f}(t)$ are not obtainable, thus the matched disturbance $\boldsymbol{d}(x,t)$

can not be eliminated by simply letting $\boldsymbol{u}=-\boldsymbol{d}(\boldsymbol{x},t)$. To tackle such problem, investigators have successfully developed several robust control technologies, such as adaptive control, H_∞ control, and sliding-mode control, which could effectively suppress the matched disturbance $\boldsymbol{d}(x,t)$.

Similar to the sliding-mode control, the UDSSMC could completely get rid of the matched disturbance when the upper bounds of $\|\Delta\boldsymbol{A}(t)\|$ and $\|\boldsymbol{f}(t)\|$ are given. Unfortunately, appropriate upper bounds of $\|\Delta\boldsymbol{A}(t)\|$ and $\|\boldsymbol{f}(t)\|$ are often hard to estimate. In fact, these upper bounds could not be under-estimated; otherwise the system will be out of control. In general, they are over-estimated and even too conservative, which often result in unreasonable tremendous control inputs. To improve the above drawbacks, this chapter employs the grey prediction technology for the matched disturbance $\boldsymbol{d}(x,t)$, which will be briefly introduced in the next section.

5.3 Grey Prediction

This section first shows three basic operations related to grey prediction and then gives the famous GM(1,1) model, which will be adopted in the sliding-mode control.

Consider a positive data sequence $y^{(0)}(k)>0$ for $k=1,2,\cdots,p$ where p is chosen as $p\geq 4$ [33]. The grey prediction technology is employed to establish a mathematic model that can properly represent this positive data sequence and, most importantly, can well predict the data coming after $y^{(0)}(p)$, denoted as $\hat{y}^{(0)}(p+q)$

where $q = 1,2,\ldots$. Three basic operations are required in the process of the data sequence, named as the accumulated generating operation, the inverse accumulated operation, and the mean operation.

The accumulated generating operation is defined as

$$y^{(1)}(k) = \sum_{l=1}^{k} y^{(0)}(l), \qquad k = 1,2,\ldots,p \tag{5.3}$$

which accumulates the data sequence $y^{(0)}(k)$. The inverse accumulated generating operation is defined as

$$\begin{aligned} &y^{(0)}(1) = y^{(1)}(1) \\ &y^{(0)}(k) = y^{(1)}(k) - y^{(1)}(k-1), \qquad k = 2,3,\cdots,p \end{aligned} \tag{5.4}$$

which is an inverse process of the accumulated generating operation (5.3). As for the mean operation, it simply takes the average value of $y^{(1)}(k)$ and $y^{(1)}(k-1)$, i.e.,

$$z^{(1)}(k) = \frac{1}{2}\left[y^{(1)}(k) + y^{(1)}(k-1)\right], \qquad k = 2,3,\cdots,p \tag{5.5}$$

With the above three basic operations, the famous grey model GM(1,1) is constructed to suitably represent the positive sequence $y^{(0)}(k)$ as below [33]:

$$y^{(0)}(k) + az^{(1)}(k) = b, \qquad k = 2,\cdots,p \tag{5.6}$$

where $z^{(1)}(k)$ is the mean operation as given in (5.5). Both a and b are constants to be determined, where a is called the developing coefficient and b is treated as the grey input. It is noticed that the establishment of GM(1,1) model is mainly based on an ideal model which imitates the first-order differential equation as follows

$$\frac{dy^{(1)}(t)}{dt} + ay^{(1)}(t) = b \tag{5.7}$$

Clearly, $y^{(0)}(k)$ in (5.6) is similar to the first term $\frac{dy^{(1)}(t)}{dt}$ in (5.7) and called the grey derivative. Rewriting (5.6) into a matrix form leads to

$$\boldsymbol{y} = \boldsymbol{F} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \tag{5.8}$$

where

$$\boldsymbol{y} = \begin{bmatrix} y^{(0)}(2) \\ \vdots \\ y^{(0)}(p) \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} -z^{(1)}(2) & 1 \\ \vdots & \vdots \\ -z^{(1)}(p) & 1 \end{bmatrix}$$

According to the least square method, a and b can be solved as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left(\boldsymbol{F}^{T}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^{T}\boldsymbol{y} \tag{5.9}$$

The GM(1,1) model is then obtained and the predicted values of $y^{(1)}(p+q)$ is achieved as [33]

$$\hat{y}^{(1)}(p+q) = \left(y^{(0)}(1) - \frac{b}{a}\right)e^{-\boldsymbol{a}(p+q-1)} + \frac{b}{a} \tag{5.10}$$

where q=1,2,…, called the predictive step. Note that (5.10) is derived in a way by setting t=p+q to the solution of (5.7) with initial condition $y^{(0)}(1)$ at t=1. It is obvious that for the first predictive step, i.e. q=1, we have

$$\hat{y}^{(1)}(p+1) = \left(y^{(0)}(1) - \frac{b}{a}\right)e^{-\boldsymbol{a}p} + \frac{b}{a} \tag{5.11}$$

Further using the inverse accumulated generating operation (5.4) yields

$$\hat{y}^{(0)}(p+1) = \left(1 - e^{\boldsymbol{a}}\right) \cdot \left(y^{(0)}(1) - \frac{b}{a}\right) \cdot e^{-\boldsymbol{a}p} \tag{5.12}$$

Clearly, this is the first predictive value of the data sequence $y^{(0)}(k),\ k = 1,2,\cdots,p$.

It should be noticed that the grey prediction model GM(1,1) introduced above is only suitable for positive data sequences. If a sequence with negative data is processed, it should be modified into a positive data sequence first. The most common way is adding a bias, such as

$$bias = \left| \min_{\boldsymbol{k}=1}^{p} y^{(0)}(k) \right| \tag{5.13}$$

to the original data sequence and then a new positive sequence is formed and expressed as

$$y_{\boldsymbol{m}}^{(0)}(k) = y^{(0)}(k) + bias \tag{5.14}$$

Now, its GM(1,1) model could be directly established by the procedure from (5.6) to (5.12) and the first predictive value is

$$\hat{y}_m^{(0)}(p+1) = \left(1 - e^{\boldsymbol{a}}\right) \cdot \left(y_m^{(0)}(1) - \frac{b}{a} \right) \cdot e^{-\boldsymbol{a} p} \tag{5.15}$$

After taking away the bias, the first predictive value of the original sequence $y^{(0)}(k)$ is then found as

$$\hat{y}^{(0)}(p+1) = \hat{y}_{\boldsymbol{m}}^{(0)}(p+1) - bias \tag{5.16}$$

which will be employed to predict the matched disturbance $\boldsymbol{d}(\mathbf{x},t)$ in (5.2), since a disturbance is intrinsically unknown and, of course, not necessary to be positive.

5.4 A novel UDSSMC incorporated with Grey Prediction

5.4.1 Design of The New UDSSMC Controller

Consider the multi-input linear time-invariant system described in (5.1), which encounters matched disturbance $\boldsymbol{d}(\boldsymbol{x},t)$. From the design of the UDSSMC described in Chatper 4, it is known that the UDSSMC could theoretically suppress the matched disturbance whose prior information concerning the upper bounds of the matched disturbance, denoted as $\|\boldsymbol{d}(\boldsymbol{x},t)\|_{max}$ or $|d_i(\boldsymbol{x},t)|_{max}$, i=1,2,…,m, are required. However, appropriate upper bound $\|\boldsymbol{d}(\boldsymbol{x},t)\|_{max}$ and $|d_i(\boldsymbol{x},t)|_{max}$ are not so easy to estimate. They are often over-estimated and thus make the control inputs increased largely, sometimes even unreasonably. To avoid such problem, the grey prediction is used to directly predict the matched disturbance $\boldsymbol{d}(\boldsymbol{x},t)$.

The first step of the UDSSMC design is to choose an appropriate sliding vector. Let the sliding vector $\boldsymbol{\sigma} \in \Re^{\boldsymbol{m}}$ be of the form as

$$\boldsymbol{\sigma} = (\boldsymbol{CB})^{-1}\boldsymbol{Cx} \tag{5.17}$$

where $\boldsymbol{\sigma} = [\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_{\boldsymbol{m}}]^T$. Note that $\boldsymbol{C} \in \Re^{m \times n}$ is a coefficient matrix and $(\boldsymbol{CB})^{-1}$ must exist. For the selection of the sliding vector, the UDSSMC adopts the virtual eigenvalue method proposed by Chang and Chen [49], which is revealed in Section 4.2.

Next, let’s start to design the control algorithm of the UDSSMC combined with grey prediction. First, further taking the first derivative of (5.17) yields

$$\dot{\boldsymbol{\sigma}} = (\boldsymbol{CB})^{-1}\boldsymbol{C}\dot{\boldsymbol{x}} = (\boldsymbol{CB})^{-1}\boldsymbol{CAx} + \boldsymbol{u} + \boldsymbol{d}(\boldsymbol{x},t) \tag{5.18}$$

Based on the design of the UDSSMC described in Chapter 4, the control algorithm is selected as

$$\boldsymbol{u} = -\boldsymbol{Kx} + \bar{\boldsymbol{u}} \tag{5.19}$$

with

$$\begin{aligned}
&\text{In the approaching mode:}\\
&\bar{\boldsymbol{u}} = \begin{cases} \left(\sqrt{\dfrac{2m}{m+1}} \cdot \left\|\boldsymbol{d}(\boldsymbol{x},t)\right\|_{max} + \varepsilon\right)\boldsymbol{u}^{i}, \quad \text{for } \boldsymbol{\sigma} \in \boldsymbol{\Sigma}_i & \text{if } dist_{\min} \geq \xi_1 \\ \text{Unchanged} & \text{if } dist_{\min} < \xi_1 \end{cases}\\
&\text{In the sliding mode:}\\
&\bar{\boldsymbol{u}} = \begin{cases} \left(\sqrt{\dfrac{2m}{m+1}} \cdot \left\|\boldsymbol{d}(\boldsymbol{x},t)\right\|_{max} + \varepsilon\right)\boldsymbol{u}^{i}, \quad \text{for } \boldsymbol{\sigma} \in \boldsymbol{\Sigma}_i & \text{if } \left\|\boldsymbol{\sigma}\right\| \geq \xi_2 \\ \text{Unchanged} & \text{if } \left\|\boldsymbol{\sigma}\right\| < \xi_2 \end{cases}
\end{aligned} \tag{5.20}$$

where $dist_{min}$ is defined as (4.39), $\boldsymbol{u}^i$ for i=1,2,…,m+1 represent the uniformly distributed simplex set described in Chapter 3, and $\boldsymbol{\Sigma}_i$ are the disjointed open sub-regions given in (2.4). Based on the derivation of Section 4.3, it is verified that the UDSSMC algorithm (5.19) and (5.20) can efficiently suppress the matched disturbance. However, for the implementation of (5.20), it is necessary to obtain the prior information concerning the upper bounds of the matched disturbance. To tackle this problem, the control algorithm (5.19) is modified as

$$\boldsymbol{u} = -\boldsymbol{Kx} + \bar{\boldsymbol{u}} - \hat{\boldsymbol{d}}(qT) \quad \text{for} \quad t \in [qT,(q+1)T) \tag{5.21}$$

where T is the prediction period and $\hat{\boldsymbol{d}}(qT)$ is the predicted value of $\boldsymbol{d}(\boldsymbol{x},t)$ at t=qT.

Besides, (5.20) for $t \in [qT,(q+1)T)$ becomes

$$\begin{aligned}
&\text{In the approaching mode:}\\
&\bar{\boldsymbol{u}} = \begin{cases} \left(\sqrt{\dfrac{2m}{m+1}} \cdot \left\| \boldsymbol{d}(\boldsymbol{x},t) - \hat{\boldsymbol{d}}(qT) \right\|_{max} + \varepsilon \right) \boldsymbol{u}^{\boldsymbol{i}}, & \text{for } \boldsymbol{\sigma} \in \boldsymbol{\Sigma}_i \quad \text{if } dist_{\min} \geq \xi_1 \\ \text{Unchanged} & \text{if } dist_{\min} < \xi_1 \end{cases}\\
&\text{In the sliding mode:}\\
&\bar{\boldsymbol{u}} = \begin{cases} \left(\sqrt{\dfrac{2m}{m+1}} \cdot \left\| \boldsymbol{d}(\boldsymbol{x},t) - \hat{\boldsymbol{d}}(qT) \right\|_{max} + \varepsilon \right) \boldsymbol{u}^{\boldsymbol{i}}, & \text{for } \boldsymbol{\sigma} \in \boldsymbol{\Sigma}_i \quad \text{if } \|\boldsymbol{\sigma}\| \geq \xi_2 \\ \text{Unchanged} & \text{if } \|\boldsymbol{\sigma}\| < \xi_2 \end{cases}
\end{aligned} \tag{5.22}$$

Note that the value of $\hat{\boldsymbol{d}}(qT)$ will be obtained by employing the grey prediction, which will be explained later. Similar to the stability proof described in Section 4.3, it is easy to show that the control algorithm (5.21) and (5.22) can guarantees the system trajectory to reach the sliding mode $\boldsymbol{\sigma} = 0$ in a finite time.

Now, let's compare the new controller (5.21) and (5.22) with the conventional one (5.19) and (5.20). Obviously, the term $\hat{\boldsymbol{d}}(qT)$ in (5.21) is not adopted in the conventional UDSSMC control algorithm (5.19). On the other hand, the upper bound value $\|\boldsymbol{d}(\boldsymbol{x},t)\|_{max}$ in (5.20) should be changed into $\left\|\boldsymbol{d}(\boldsymbol{x},t) - \hat{\boldsymbol{d}}(qT)\right\|_{max}$. In other words, the design of in (5.22) no longer depends on the upper bound $\|\boldsymbol{d}(\boldsymbol{x},t)\|_{max}$ unlike the conventional UDSSMC control algorithm. In fact, it depends on $\left\|\boldsymbol{d}(\boldsymbol{x},t) - \hat{\boldsymbol{d}}(qT)\right\|_{max}$, the maximum value of the difference between $\boldsymbol{d}(\boldsymbol{x},t)$ and $\hat{\boldsymbol{d}}(qT)$ for $t \in [qT,(q+1)T)$. It is clear that the magnitude of (5.22) could be chosen to be a small value if $\boldsymbol{d}(\boldsymbol{x},t)$ at t=qT is well predicted by $\hat{\boldsymbol{d}}(qT)$.

5.4.2 Grey Prediction for Disturbances

Next, let's concentrate on the grey prediction for $\hat{\boldsymbol{d}}(qT)$, used in (5.21) for $t \in [qT, (q+1)T)$. First, each component of $\hat{\boldsymbol{d}}(qT)$ can be expressed as $\hat{d}_i(qT), i = 1, \cdots, m$. From the grey prediction introduced in Section 5.3, for a sequence data $y^{(0)}(k)$, where $k = 1, 2, \cdots, p$ and $p \geq 4$, it can obtain its first predictive value $\hat{y}^{(0)}(p+1)$ as given in (5.16). Clearly, if $d_i(\boldsymbol{x}, t)$ at $t=(q-p)T,(q-p+1)T,\ldots,(q-1)T$, can be calculated from other measurable variables, then they will form a data sequence as

$$d_i(\boldsymbol{x}((q-j)T), (q-j)T) \quad \text{for} \quad j = p, (p-1), \ldots, 2, 1 \tag{5.23}$$

For convenience, the above expression is changed into

$$d_i(\boldsymbol{x}((q-p-1+k)T), (q-p-1+k)T) \quad \text{for} \quad k = 1, 2, \cdots, p \tag{5.24}$$

Let $y^{(0)}(k) = d_i(\boldsymbol{x}((q-p-1+k)T), (q-p-1+k)T)$, $k = 1, 2, \cdots, p$, be the data sequence to be processed. Following the grey prediction procedure, the first predictive value $\hat{y}^{(0)}(p+1)$ in (5.16) is achieved and then is assigned as the value for $d_i(\boldsymbol{x}, t)$ at $t=qT$, denoted as $\hat{d}_i(\boldsymbol{x}(qT), qT)$ or as $\hat{d}_i(qT)$ for simplicity. This is the term used in the control algorithm (5.22) for $t \in [qT, (q+1)T)$.

Then, let's show the way how to calculate the sequence of $d_i(\boldsymbol{x}, t)$ at $t=(q-p)T,(q-p+1)T,\ldots,(q-1)T$. Substituting (5.21) into (5.18), it leads to

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= (\boldsymbol{CB})^{-1}\boldsymbol{CAx} - \boldsymbol{Kx} + \bar{\boldsymbol{u}} - \hat{\boldsymbol{d}}(qT) + \boldsymbol{d}(\boldsymbol{x}, t) \\ &= (\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} + \bar{\boldsymbol{u}} - \hat{\boldsymbol{d}}(qT) + \boldsymbol{d}(\boldsymbol{x}, t) \end{aligned}, \quad \text{for} \quad t \in [qT, (q+1)T) \tag{5.25}$$

Further changing (5.25) into the component form as

$$\dot{\sigma}_i = \left[(\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}\right]_i + \bar{u}_i - \hat{d}_i(qT) + d_i(\boldsymbol{x},t), \quad \text{for } t \in [qT,(q+1)T) \tag{5.26}$$

where $i = 1,\cdots,m$. And, $\left[(\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}\right]_i$ represents the i-th row vector of $(\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}$. Clearly, the value of $d_i(\boldsymbol{x},t)$ at $t=qT$ could be derived as

$$d_i(\boldsymbol{x}(qT),qT) = \dot{\sigma}_i(qT) - \left[(\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}(qT)\right]_i - \bar{u}_i(qT) + \hat{d}_i(qT) \tag{5.27}$$

That means the data sequence (5.23) can be calculated as

$$\begin{aligned} &d_i(\boldsymbol{x}((q-j)T),(q-j)T) \\ &= \dot{\sigma}_i((q-j)T) - \left[(\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}((q-j)T)\right]_i - \bar{u}_i((q-j)T) + \hat{d}_i((q-j)T) \end{aligned} \tag{5.28}$$

where $j=p,(p-1),\ldots,2,1$. Unfortunately, the term $\dot{\sigma}_i((q-j)T)$ is not measurable; instead, it is approximated as the following simplest way

$$\dot{\sigma}_i((q-j)T) \approx \frac{\sigma_i((q-j+1)T) - \sigma_i((q-j)T)}{T} \tag{5.29}$$

for $j=p,(p-1),\ldots,3,2$. Note that the above approximation is not applicable to the case of $j=1$, i.e. $\dot{\sigma}_i((q-1)T)$, since $\dot{\sigma}_i((q-1)T) \approx \dfrac{\sigma_i(qT) - \sigma_i((q-1)T)}{T}$, approximated by (5.29), requires the term $\sigma_i(qT)$ at $t=qT$. In other words, the calculation of $d_i(\boldsymbol{x}((q-1)T),(q-1)T)$ also requires the term $\sigma_i(qT)$ at $t=qT$. That means the establishment of the data sequence (5.28) has to use the term $\sigma_i(qT)$ at $t=qT$ if (5.29) is employed. It is noticed that the first predictive value $\hat{d}_i(qT)$ obtained from the data sequence (5.28) will be adopted in the control algorithm (5.21) for $t \in [qT,(q+1)T)$, which includes the moment that $\sigma_i(qT)$ is attained at $t=qT$. Obviously, it is impossible to adopt a value $\hat{d}_i(qT)$ at $t=qT$, which should be calculated from a value $\sigma_i(qT)$ also at the same moment $t=qT$. In order to avoid such situation, the term

$\dot{\sigma}_i((q-1)T)$ is approximated as $\dfrac{\sigma_i((q-0.5)T)-\sigma_i((q-1)T)}{0.5T}$. Here, we assume that the predictive value $\hat{d}_i(qT)$ can be achieved within $t\in((q-0.5)T, qT)$. Hence, the sequence (5.23) is obtained approximately as

$$
\begin{aligned}
& d_i(\boldsymbol{x}((q-j)T),(q-j)T)\approx \\
& \begin{cases}
\dfrac{\sigma_i((q-j+0.5)T)-\sigma_i((q-j)T)}{0.5T}-\left[(\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}((q-j)T)\right]_i \\
-\bar{u}_i((q-j)T)+\hat{d}_i((q-j)T), \quad \text{for } j=1 \\
\dfrac{\sigma_i((q-j+1)T)-\sigma_i((q-j)T)}{T}-\left[(\boldsymbol{CB})^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{x}((q-j)T)\right]_i \\
-\bar{u}_i((q-j)T)+\hat{d}_i((q-j)T), \quad \text{for } j=2,3,\ldots,\mathrm{p}
\end{cases}
\end{aligned}
\tag{5.30}
$$

which is used for the grey prediction of $\hat{d}_i(qT)$. Next, a numeric example will be simulated to demonstrate the usefulness of the new UDSSMC algorithm combined with grey prediction.

5.5 Simulation Results

Consider a linear time-invariant system (5.1) suffering from the matched disturbance, with the following numeric data:

$$
\boldsymbol{A}=\begin{bmatrix} -0.0506 & 0 & -1 & 0.2380 \\ -0.7374 & -1.3345 & 0.3696 & 0 \\ 0.01 & 0.1074 & -0.3320 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

and $\boldsymbol{B}=\begin{bmatrix} 0.0409 & 0 \\ 1.2714 & -20.3106 \\ -2.0625 & 1.3350 \\ 0 & 0 \end{bmatrix}$

The control input and system state are respectively represented by $\boldsymbol{u}=[u_1 \quad u_2]^T$ and $\boldsymbol{x}=[x_1 \quad x_2 \quad x_3 \quad x_4]^T$. The matched disturbance $\boldsymbol{d}(\boldsymbol{x},t)=[d_1(\boldsymbol{x},t) \quad d_2(\boldsymbol{x},t)]^T$ is assumed as

$$\boldsymbol{d}(\boldsymbol{x},t)=\begin{bmatrix}10\,sin(2t)+2\,cos(0.5t)\times(x_1+x_2)\\ 10\,cos(3t)+2\,sin(t)\times(x_3+x_4)\end{bmatrix} \tag{5.31}$$

Apparently, the upper bound of the matched disturbance are obtained as

$$\|\boldsymbol{d}(\boldsymbol{x},t)\|\le\delta_{max}(t)=20+2|x_1+x_2|+2|x_3+x_4| \tag{5.32}$$

The first step of the UDSSMC design is to choose a sliding vector $\boldsymbol{\sigma}=(\boldsymbol{CB})^{-1}\boldsymbol{Cx}$ as given in (4.10). Based on the design procedure described in Section 4.2, the eigenvalues for $\boldsymbol{A}-\boldsymbol{BK}$ are assigned to be

$$\lambda_1=-1,\ \lambda_2=-2,\ \ \omega_1=\omega_2=-5 \tag{5.33}$$

where ω_1 and ω_2 are purposely set to be the same and negative. By the aid of MATLAB, the state-feedback gain $\boldsymbol{K}$ and the left eigenvectors of $\boldsymbol{A}-\boldsymbol{BK}$ corresponding to ω_1 and ω_2 could be calculated as

$$\boldsymbol{K}=\begin{bmatrix}2.3642 & -0.1194 & -2.7990 & 0.3789\\ 0.1622 & -0.2864 & -0.1894 & -0.4697\end{bmatrix}$$

and

$$(\boldsymbol{CB})^{-1}\boldsymbol{C}=\begin{bmatrix}0.4738 & -0.0326 & -0.4955 & 0.0532\\ 0.0252 & -0.0513 & -0.0311 & -0.0951\end{bmatrix}$$

According to the sliding vector design described in Section 4.2, it is evident that the system stability in the sliding mode is guaranteed since all the eigenvalues are allocated in the left half plane. Based on the new construction proposed in Section 3.3, the uniformly distributed simplex set for m=2 can be selected as

$$
\begin{aligned}
\boldsymbol{u}^1 &= \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\
\boldsymbol{u}^2 &= \begin{bmatrix} -1/2 & \sqrt{3}/2 \end{bmatrix}^T \\
\boldsymbol{u}^3 &= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \end{bmatrix}^T
\end{aligned} \tag{5.34}
$$

Further choose the parameters $\xi_1 = 0.1$, and $\xi_2 = 0.2$ for the smoothing strategy. Besides, let the predictive step be T=0.01 sec. To demonstrate the usefulness of grey prediction, these numeric values are simulated for the UDSSMC algorithm (5.19) and (5.20) without any prediction, named as "Controller 1", and the proposed UDSSMC algorithm (5.21) and (5.22) with grey prediction, named as "Controller 2". Therefore, the difference between Controller 1 and Controller 2 is just the term $\hat{d}_i(mT)$, which is set to be zero for Controller 1. For comparison, the magnitudes of (5.20) and (5.22) in Control 1 and Control 2 are purposely set as

$$
\bar{\boldsymbol{u}} = 2.5\boldsymbol{u}^i, \qquad \text{for } \boldsymbol{\sigma} = (\boldsymbol{CB})^{-1}\boldsymbol{Cx} \text{ is in } \Sigma_i \tag{5.35}
$$

Apparently, the upper bound (5.32) of the matched disturbance is larger than the magnitude of (5.35). Theoretically, Controller 1 is no longer suitable for this case since the values of (5.35) is not chosen large enough to overcome the upper bound of the matched disturbance. As a result, Figure 5.1 shows that Controller 1 fails to drive the system trajectory to the sliding mode. As for Controller 2, Figures 5.2 to 5.5 demonstrate the success of the UDSSMC combined with grey prediction. Figure 5.2 shows the time response of the sliding vector $\boldsymbol{\sigma} = [\sigma_1 \quad \sigma_2]^T$ and Figure 5.3 gives the trajectory of sliding vector in the $\boldsymbol{\sigma}$ space. It is clear that Controller 2 could steer the system trajectory into the sliding layer after a few predictive steps without any prior information of the upper bound of the matched disturbance. Figure 5.4 and Figure 5.5 are respectively the time response of the state variables and the control input. In Figure 5.4, it illustrates the system state variables all converge to $\boldsymbol{x}=\boldsymbol{0}$. Figure 5.5 is the time response of the control input. From simulation, it is obvious that the developed UDSSMC algorithm combined with grey prediction could successfully controls the system even though without any prior information of the upper bound of the matched disturbance.

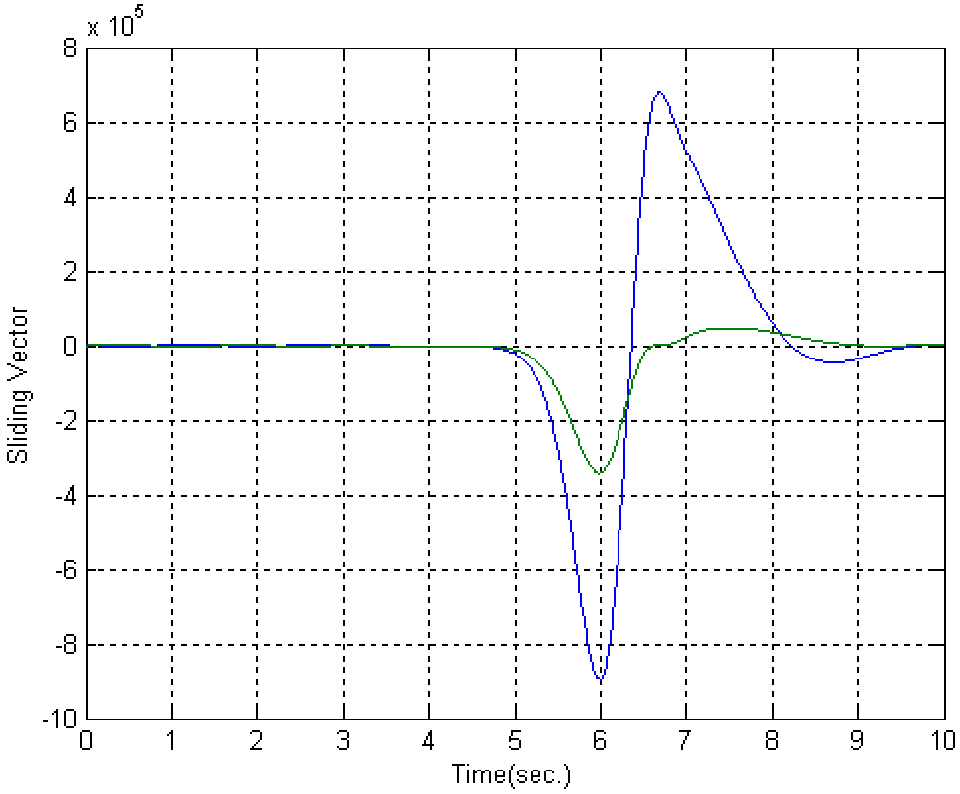

Figure 5.1 Time response of the sliding vector for Controller 1

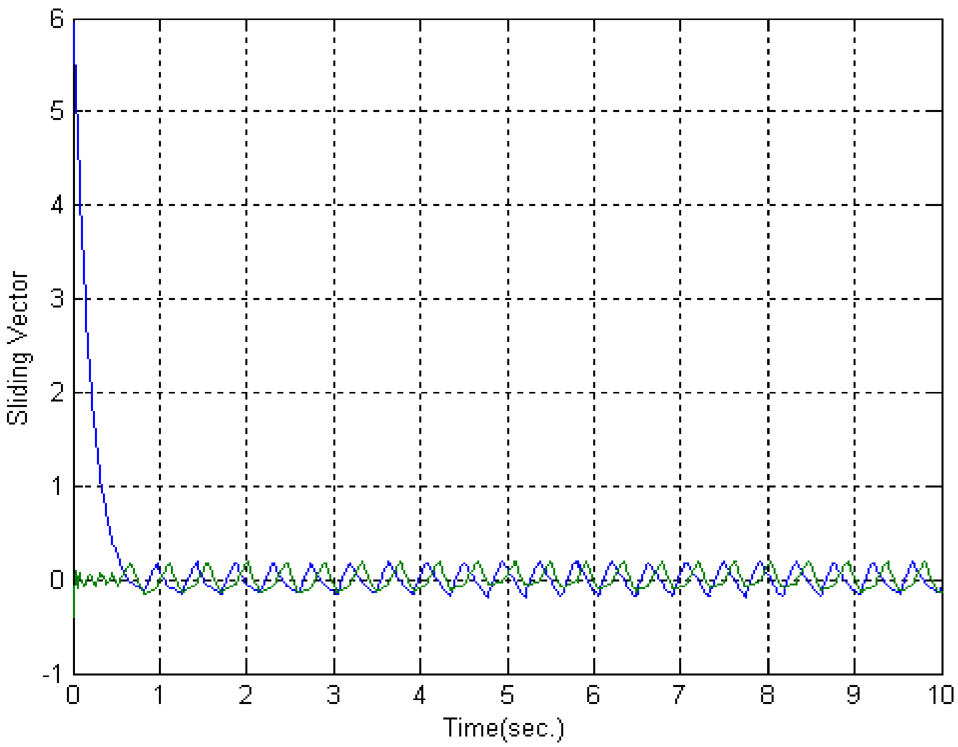

Figure 5.2 Time response of the sliding vector for Controller 2

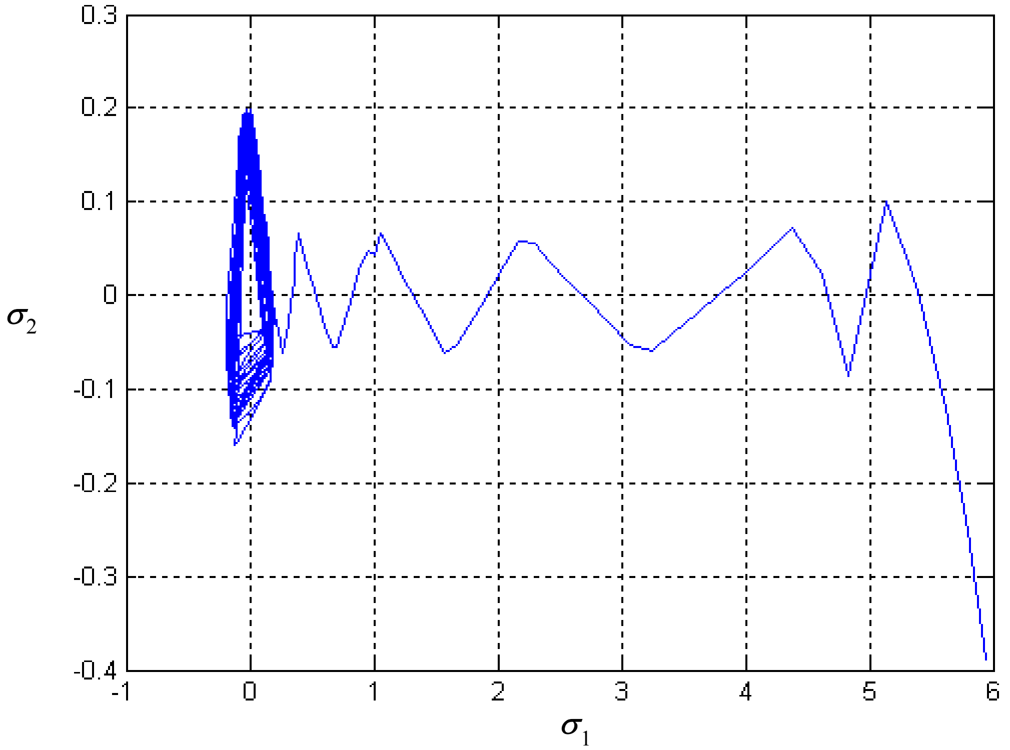

Figure 5.3 The trajectory of the sliding vector in the σ space for Controller 2

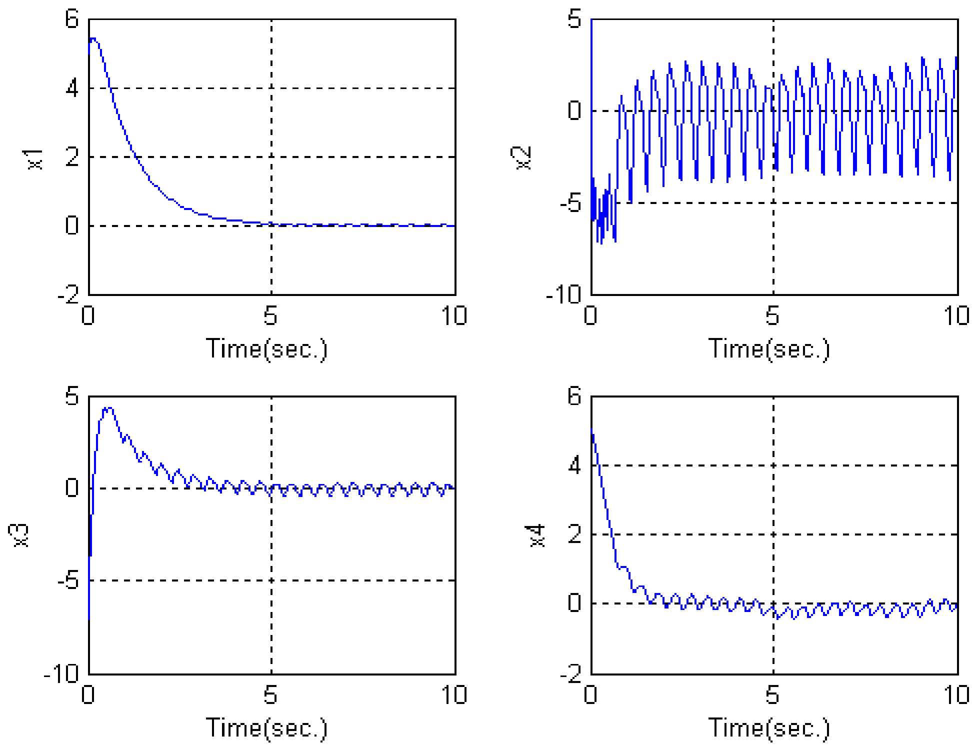

Figure 5.4 State variables x_1-x_4 for Controller 2

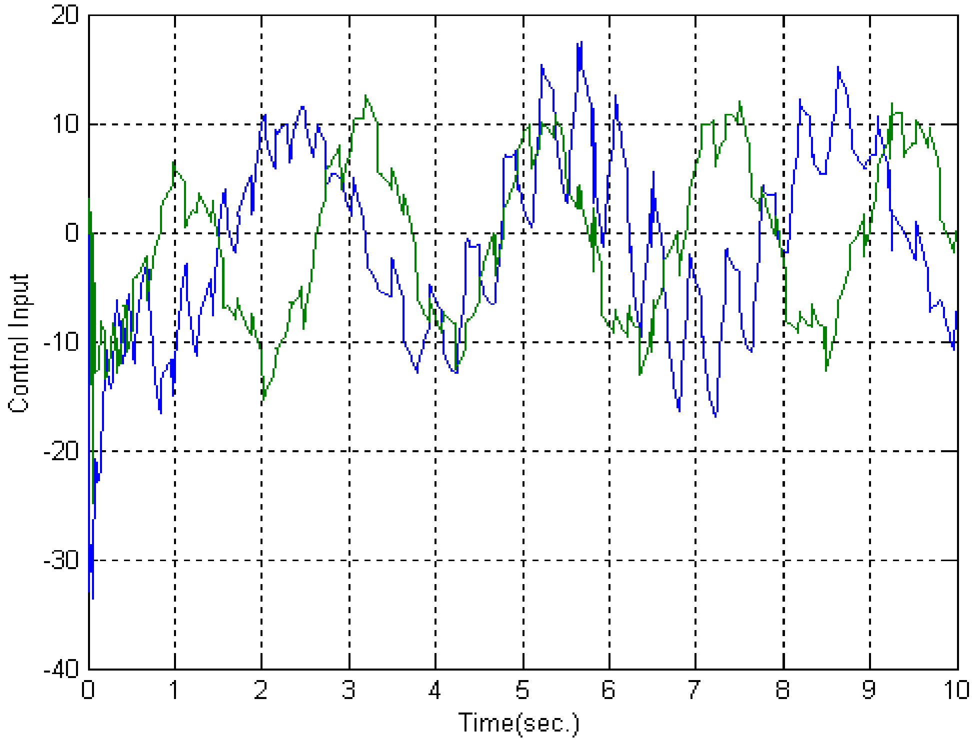

Figure 5.5 Control Inputs for Controller 2