CHAPTER 3

UNIFORMLY DISTRIBUTED SIMPLEX SET

In this chapter, a specific simplex set, called the uniformly distributed simplex set, is introduced. Section 3.1 presents the definition of the uniformly distributed simplex set. Then, a conventional construction is revealed to obtain the suitable uniformly distributed simplex set in Section 3.2. In Section 3.3, a new construction different to the conventional method is proposed to obtain the uniformly distributed simplex set.

3.1 Definition of Uniformly Distributed Simplex Set

In this section, the definition of the uniformly distributed simplex set is presented and some important properties resulted by the uniformly distributed simplex set is also derived. In the beginning, let's recall the definition of the simplex set S_U satisfying the following two conditions:

(C1) $det\,\boldsymbol{U}_i \neq 0$, $i = 1,2,\cdots,m+1$, where $\boldsymbol{U}_i$ is a square matrix containing m column vectors without $\boldsymbol{u}^i$, expressed as

$$\boldsymbol{U}_i = \begin{cases} \begin{bmatrix} \boldsymbol{u}^2 & \boldsymbol{u}^3 & \cdots & \boldsymbol{u}^{m+1} \end{bmatrix} & i = 1 \\ \begin{bmatrix} \boldsymbol{u}^1 & \cdots & \boldsymbol{u}^{i-1} & \boldsymbol{u}^{i+1} & \cdots & \boldsymbol{u}^{m+1} \end{bmatrix} & i = 2,3,\cdots,m \\ \begin{bmatrix} \boldsymbol{u}^1 & \boldsymbol{u}^2 & \cdots & \boldsymbol{u}^m \end{bmatrix} & i = m+1 \end{cases} \tag{3.1}$$

(C2) $\sum_{i=1}^{m+1} \psi_i \boldsymbol{u}^i = 0$ where $\psi_i > 0$ and $\sum_{i=1}^{m+1} \psi_i = 1$

where $\boldsymbol{u}^i = \begin{bmatrix} u_1^i & u_2^i & \cdots & u_m^i \end{bmatrix}^T \in \Re^m$ for $i = 1,2,\cdots,m+1$. Note that $\boldsymbol{u}^i$ represents one of the simplex set $S_U = \{\boldsymbol{u}^1, \boldsymbol{u}^2, \cdots, \boldsymbol{u}^{m+1}\}$. Based on the simplex set satisfying (C1) and

(C2), the space $\Re^{m}$ can be divided into m+1 disjointed open sub-regions and any vector $\boldsymbol{v} \in \Re^{m}$ can be expressed in a form as $\boldsymbol{v} = \sum_{j \neq i, j=1}^{m+1} \phi_j \boldsymbol{u}^j, \phi_j > 0$.

In order to obtain more beneficial properties, two extra conditions for S_U are added, which are

$$\text{(C3)}\quad \left(\boldsymbol{u}^i\right)^T \boldsymbol{u}^i = \left\|\boldsymbol{u}^i\right\|^2 = 1, \quad i = 1,2,\cdots,m+1,$$

$$\text{(C4)}\quad \left(\boldsymbol{u}^i\right)^T \boldsymbol{u}^j = -\frac{1}{m}, \quad i, j = 1,2,\cdots,m+1, \quad i \neq j.$$

Note that all the vectors' magnitudes are equal to 1 and all the inner products between any two different vectors are equal to –1/m. In fact, the similar definitions (C1)-(C4) is also adopted in the field of communication [46]. Attracted to the feature of the unit magnitude, the set satisfying (C1)-(C4) is called the regular simplex codes in communication literatures [47]. In the viewpoint of simplex sliding-mode control (SSMC), it pays more attention to the feature that all the vectors $\boldsymbol{u}^i$ in S_U are uniformly distributed in $\Re^{m}$. As a result, the set S_U satisfying (C1)-(C4) is called the uniformly distributed simplex set in this dissertation.

In virtue of the proposed uniformly distributed simplex set which satisfies (C1)-(C4), one significant equation is derived as below. First, let's introduce an $m \times m$ square matrix $\boldsymbol{V}_i$ as below:

$$\boldsymbol{V}_i = \left[\boldsymbol{u}^i \quad \cdots \quad \boldsymbol{u}^i \quad \boldsymbol{u}^i \quad \cdots \quad \boldsymbol{u}^i\right], \quad i = 1,2,\cdots,m+1 \tag{3.2}$$

where $\boldsymbol{u}^i = \left[u_1^i \quad u_2^i \quad \cdots \quad u_m^i\right]^T \in \Re^{m}$ means one of the uniformly distributed simplex set $S_U = \{\boldsymbol{u}^1, \boldsymbol{u}^2, \cdots, \boldsymbol{u}^{m+1}\}$. Based on (C3) and (C4), it can be achieved by direct calculation that

$$\left[\frac{m}{m+1}(\boldsymbol{U}_i - \boldsymbol{V}_i)^T\right]\boldsymbol{U}_i = \boldsymbol{I}_{m\times m},\ i = 1,2,\ldots,m+1 \tag{3.3}$$

where $\boldsymbol{U}_i$ has been defined in (3.1) and $\boldsymbol{I}_{m\times m}$ is the $m\times m$ identity matrix. The truth of (3.3) implies $\frac{m}{m+1}(\boldsymbol{U}_i - \boldsymbol{V}_i)^T$ is the inverse matrix of $\boldsymbol{U}_i$ and will be employed to design the simplex sliding-mode control in Chapter 4. As to the selection of the uniformly distributed simplex set, two construction procedures will be proposed as follows.

3.2 Conventional Construction

For the space $\Re^{\boldsymbol{m}}$, a conventional construction for the uniformly distributed simplex set $S_U = \{\boldsymbol{u}^1, \boldsymbol{u}^2, \cdots, \boldsymbol{u}^{m+1}\}$ is introduced with the form as [48]

$$\begin{aligned}
\boldsymbol{u}^1 &= \frac{1}{\sqrt{m}}[\,1 \quad 1 \quad 1 \quad \cdots \quad 1\,] \\
\boldsymbol{u}^2 &= \frac{1}{\sqrt{m}}[-a \quad b \quad b \quad \cdots \quad b] \\
\boldsymbol{u}^3 &= \frac{1}{\sqrt{m}}[b \quad -a \quad b \quad \cdots \quad b] \\
&\ \ \vdots \\
\boldsymbol{u}^{\boldsymbol{m}+1} &= \frac{1}{\sqrt{m}}[b \quad b \quad b \quad \cdots \quad -a]
\end{aligned} \tag{3.4}$$

where $\boldsymbol{u}^{\boldsymbol{i}}$ for $3 \le i \le m+1$ is a cyclic shift of $\boldsymbol{u}^2$ and the corresponding coefficients a and b are selected as

$$(a,b)=\left(\frac{1+(m-1)\sqrt{m+1}}{m},\frac{-1+\sqrt{m+1}}{m}\right)$$
$$\text{or}\left(\frac{1-(m-1)\sqrt{m+1}}{m},\frac{-1-\sqrt{m+1}}{m}\right) \tag{3.5}$$

In the following, it will verify the set of (3.4) and (3.5) satisfies (C1)-(C4).

First, the candidate set is chosen as (3.4). In order to satisfy (C3), it leads to the following equation:

$$(\boldsymbol{u}^{i},\boldsymbol{u}^{i})=\frac{1}{m}\left(a^{2}+(m-1)b^{2}\right)=1,\quad \text{for } 2\le i\le m+1 \tag{3.6}$$

Similarly, the fact of (C4) yields the two equations as

$$(\boldsymbol{u}^{1},\boldsymbol{u}^{i})=\frac{1}{m}\left(-a+(m-1)b\right)=-\frac{1}{m},\quad \text{for } 2\le i\le m+1 \tag{3.7}$$

$$(\boldsymbol{u}^{i},\boldsymbol{u}^{j})=\frac{1}{m}\left(-2ab+(m-2)b^{2}\right)=-\frac{1}{m},\quad \text{for } 2\le i,j\le m+1, i\ne j \tag{3.8}$$

To solve the equations (3.6)-(3.8), it is easy to obtain the coefficient values of a and b as

$$(a,b)=\left(\frac{1+(m-1)\sqrt{m+1}}{m},\frac{-1+\sqrt{m+1}}{m}\right)$$
$$\text{or}\left(\frac{1-(m-1)\sqrt{m+1}}{m},\frac{-1-\sqrt{m+1}}{m}\right) \tag{3.9}$$

On the other hand, the sum of each column in (3.4) is equal to $1-a+(m-1)b$. Obviously, (C2) is guaranteed since (3.7) leads to $-a+(m-1)b=-1$. Further, since the candidate set of (3.4) and (3.5) satisfies (C3) and (C4), the truth of (3.3) implies $\frac{m}{m+1}(\boldsymbol{U}_i-\boldsymbol{V}_i)^T$ is the inverse matrix of $\boldsymbol{U}_i$. In other words, $\boldsymbol{U}_i$ is invertible and thus (C1) is guaranteed. Therefore, it is verified that the candidate set of (3.4) and (3.5) is really a uniformly distributed simplex set since it satisfies the four conditions (C1)-(C4). For illustration, an example is shown as

Example 3.1

For the case of m=2, the uniformly distributed simplex set selected by (3.4) and (3.5) is

$$\begin{aligned} \boldsymbol{u}^1 &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} \\ \boldsymbol{u}^2 &= \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{-1-\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2}\end{bmatrix} = \begin{bmatrix}\frac{-1-\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}}\end{bmatrix} \\ \boldsymbol{u}^3 &= \frac{1}{\sqrt{2}}\begin{bmatrix}\frac{-1+\sqrt{3}}{2} & \frac{-1-\sqrt{3}}{2}\end{bmatrix} = \begin{bmatrix}\frac{-1+\sqrt{3}}{2\sqrt{2}} & \frac{-1-\sqrt{3}}{2\sqrt{2}}\end{bmatrix} \end{aligned} \tag{3.10}$$

Note that the coefficient values of a and b are assigned as

$$\begin{aligned} (a,b) &= \left(\frac{1+(m-1)\sqrt{m+1}}{m}, \frac{-1+\sqrt{m+1}}{m}\right) \\ &= \left(\frac{1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right) \end{aligned}$$

In Figure 3.1, it is shown that the vectors $\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{u}^3$ of the uniformly distributed simplex set (3.10) are uniformly separated the space $\Re^2$.

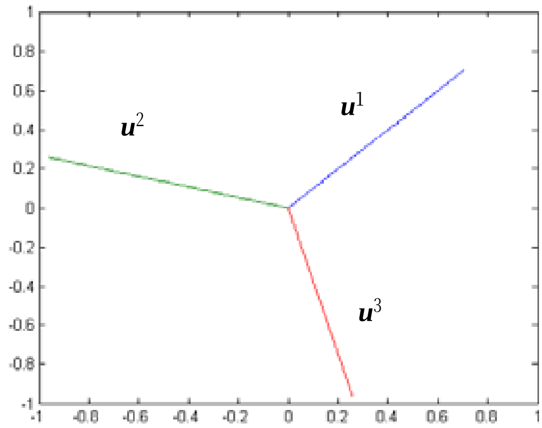

Figure 3.1 Uniformly distributed simplex set constructed by the conventional method for the case of $m = 2$

3.3 New Construction

In this section, it will propose a novel construction of the uniformly disturbance simplex set. To start with, let the following m+1 vectors form a candidate of the uniformly disturbance simplex set:

$$
\begin{array}{ll}
\boldsymbol{u}^1 & = [\begin{array}{cccccccc} \beta_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}]^T \\
\boldsymbol{u}^2 & = [\begin{array}{cccccccc} \alpha_1 & \beta_2 & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}]^T \\
\boldsymbol{u}^3 & = [\begin{array}{cccccccc} \alpha_1 & \alpha_2 & \beta_3 & 0 & 0 & 0 & \cdots & 0 \end{array}]^T \\
 & \vdots \\
\boldsymbol{u}^k & = [\begin{array}{ccccccc} \alpha_1 & \alpha_2 & \cdots & \alpha_{k-1} & \beta_k & 0 & \cdots & 0 \end{array}]^T \\
 & \vdots \\
\boldsymbol{u}^{m-1} & = [\begin{array}{cccccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{m-2} & \beta_{m-1} & 0 \end{array}]^T \\
\boldsymbol{u}^m & = [\begin{array}{cccccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{m-2} & \alpha_{m-1} & \beta_m \end{array}]^T \\
\boldsymbol{u}^{m+1} & = [\begin{array}{cccccccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{m-2} & \alpha_{m-1} & \alpha_m \end{array}]^T
\end{array}
\tag{3.11}
$$

where α_k and β_k, for k=1,2,…,m, will be obtained systematically as the following procedures:

Step I. Let $\beta_1 = 1$ and k=1.

Step II. Calculate $\alpha_k = -\dfrac{\beta_k}{m-(k-1)}$, which is less than 0.

Step III. Calculate $\beta_{k+1} = \sqrt{1-\sum_{j=1}^{k} \alpha_j^2}$, which is greater than 0.

Step IV. Increase k by 1.

Step V. If $k<m$ then go to Step II.

Step VI. Let $\alpha_m = -\beta_m$ and stop.

From Step II and VI, it leads to

$$
\beta_k + (m-(k-1))\alpha_k = 0, \qquad k=1,2,\ldots,m \tag{3.12}
$$

which results in $\sum_{i=1}^{m+1} \boldsymbol{u}^{i} = 0$ by direct calculation. Consequently, $\sum_{i=1}^{m+1} \psi_i \boldsymbol{u}^{i} = 0$ with $\psi_i = 1/(m+1)$ for i=1,2,…,m+1, which guarantees (C2). Furthermore, it can be found from Step III that

$$\sum_{j=1}^{k} \alpha_j^2 + \beta_{k+1}^2 = 1, \qquad k=1,2,\ldots,m-1, \tag{3.13}$$

Then, for k=m–1 in (3.13) and $\alpha_m = -\beta_m$ in Step VI, it yields

$$\sum_{j=1}^{m-1} \alpha_j^2 + \beta_m^2 = \sum_{j=1}^{m-1} \alpha_j^2 + \alpha_m^2 = \sum_{j=1}^{m} \alpha_j^2 = 1 \tag{3.14}$$

Clearly, (C3) is guaranteed since (3.13) and (3.14) lead to $\left\| \boldsymbol{u}^{i} \right\|^2 = 1$ for $i = 1,2,\cdots,m+1$.

Before checking (C4), let's derive an important relationship between α_i and α_{i-1}. For k=i–1 and k=i–2 in (3.13), it is easy to find that $\sum_{j=1}^{i-1} \alpha_j^2 + \beta_i^2 = \sum_{j=1}^{i-2} \alpha_j^2 + \beta_{i-1}^2 = 1$, and then

$$\beta_i^2 = \beta_{i-1}^2 - \alpha_{i-1}^2 = (\beta_{i-1} - \alpha_{i-1})(\beta_{i-1} + \alpha_{i-1}) \tag{3.15}$$

Since $\beta_i = -(m-(i-1))\alpha_i$ and $\beta_{i-1} = -(m-(i-2))\alpha_{i-1}$ from (3.12), substituting β_i and β_{i-1} into (3.15) leads to

$$\begin{aligned}(m-(i-1))^2 \alpha_i^2 &= ((m-(i-2))\alpha_{i-1} - \alpha_{i-1})((m-(i-2))\alpha_{i-1} + \alpha_{i-1}) \\ &= (m-(i-1))(m-(i-3))\alpha_{i-1}^2\end{aligned} \tag{3.16}$$

i.e.,

$$(m-(i-1))\alpha_i^2 = (m-(i-3))\alpha_{i-1}^2 \tag{3.17}$$

Obviously, if further multiplying $(m-(i-2))$ on both sides of (3.17), it yields

$$(m-(i-2))(m-(i-1))\alpha_i^2 = (m-(i-3))(m-(i-2))\alpha_{i-1}^2 \tag{3.18}$$

which result in

$$
\begin{aligned}
(m-(i-2))(m-(i-1))\alpha_i^2 &= (m-(i-3))(m-(i-2))\alpha_{i-1}^2 \\
&= (m-(i-4))(m-(i-3))\alpha_{i-2}^2 \\
&\quad \vdots \\
&= m(m-1)\alpha_2^2 \\
&= (m+1)m\,\alpha_1^2
\end{aligned}
\tag{3.19}
$$

From (3.12) with $k=1$ and $\beta_1 = 1$ given in Step I, it can be found that

$$
\alpha_1 = -\frac{1}{m}\beta_1 = -\frac{1}{m} \tag{3.20}
$$

and thus (3.19) becomes

$$
(m-(i-2))(m-(i-1))\alpha_i^2 = \frac{m+1}{m}, \qquad i = 1,2,\ldots,m \tag{3.21}
$$

Now, let's focus on the calculation of $\left(\boldsymbol{u}^i\right)^T \boldsymbol{u}^{i+p}$, $i = 1,2,\cdots,m$, $p = 1,2, m-i+1$. From (3.11), (3.12), and (3.13), it leads to

$$
\begin{aligned}
\left(\boldsymbol{u}^i\right)^T \boldsymbol{u}^{i+p} &= \sum_{j=1}^{i-1}\alpha_j^2 + \beta_i\alpha_i \qquad i = 2,3,\ldots,m, \quad p = 1,2,\ldots,m-i+1 \\
&= 1-\beta_i^2 + \beta_i\alpha_i \\
&= 1-(\beta_i - \alpha_i)\beta_i \\
&= 1-(m-(i-2))(m-(i-1))\alpha_i^2
\end{aligned}
\tag{3.22}
$$

From (3.21), it can be shown that

$$
\left(\boldsymbol{u}^i\right)^T \boldsymbol{u}^{i+p} = -\frac{1}{m} \qquad i = 2,3,\ldots,m, \quad p = 1,2,\ldots,m-i+1 \tag{3.23}
$$

Clearly, (C4) is guaranteed. As to the condition (C1), let's utilize the square matrix $\boldsymbol{V}_i$ (3.2) again. Based on (C3) and (C4), the fact of (3.3) is also guaranteed. Therefore, (3.11) is really a set of uniformly distributed simplex vectors since it satisfies the four conditions (C1) to (C4).

Example 3.2

Following the procedure described in Section 3.3, the simplex set for the case of $m = 2$, selected by (3.11) is

$$\boldsymbol{u}^1 = [1 \quad 0]$$
$$\boldsymbol{u}^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \tag{3.24}$$
$$\boldsymbol{u}^3 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Figure 3.2 shows the vectors $\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{u}^3$ of the simplex set (3.24) in the space $\Re^2$. It is obvious that the vectors $\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{u}^3$ uniformly separate the space $\Re^2$ into three regions.

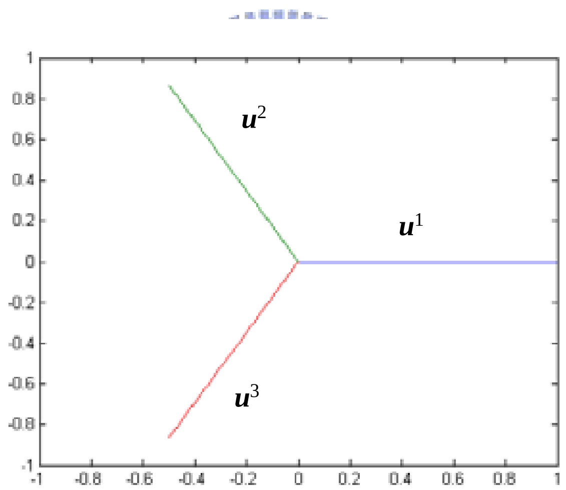

Figure 3.2 Uniformly distributed simplex set constructed by the novel method for the case of $m = 2$