$$T(m,n) = \begin{bmatrix} M_0 & M_1 & \cdot & \cdot & \cdot & M_{n-1} \\ M_1 & M_2 & \cdot & \cdot & \cdot & M_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ M_{m-1} & M_m & \cdot & \cdot & \cdot & M_{m+n-2} \end{bmatrix}. \quad (2)$$

Alternatively, the matrix used in the paper[1] is defined as

$$H(m,n) = [H_1(m,n)H_2(m,n)\cdots H_p(m,n)] \quad (5)$$

where

$$H_i(m,n) = \begin{bmatrix} m_{i0} & m_{i1} & \cdots & m_{in-1} \\ m_{i1} & m_{i2} & \cdots & m_{in} \\ \cdot & \cdot & \cdots & \cdot \\ m_{i,m-1} & m_{i,m} & \cdots & m_{i,m+n-2} \end{bmatrix}. \quad (4)$$

Proof of Theorem 1:[1] Observe from the definition of M_k, (22), that $H(m,n)$ can be obtained from $T(m,n)$, (2), through column interchanges. This implies that

$$\rho[H(m,n)] = \rho[T(m,n)]. \quad (23)$$

This allows $T(m,n)$ to be used in place of $H(m,n)$ in the proof of this theorem.

Next, recall the well-known fact that

$$\rho[T(m,n)] = \rho[T(r,r)], \qquad m,n \geq r$$
$$= n$$

where r equals the degree of least common denominator of $G(s)$ and n equals the dimension of minimal realization. For the general case of multi-input multi-output systems, use of the foregoing lemma allows one to write

$$n \geq r. \quad (24)$$

The equality in (24) holds in the special case being considered since an explicit realization of order r may be obtained by letting

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{r-1} \end{bmatrix}$$

where

$$\gamma(s) = \sum_{i=0}^{r} b_i s^i, \qquad b_r = 1;$$

also

$$B = \begin{bmatrix} M_0 \\ M_1 \\ \cdot \\ \cdot \\ \cdot \\ M_{r-1} \end{bmatrix}$$

$$C = [1, 0, 0, \cdots].$$

By considering a multi-output single-input system and repeating the foregoing procedure, the second part of Theorem 1 can be proved.

Reference

[1] L. M. Silverman, "Realization of linear dynamical systems," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 554–567, Dec. 1971.

Author's Reply[2]

C.-T. CHEN

It seems that Gupta and Fairman have missed the main contribution of the paper. Theorem 1 is established without resorting to any result in irreducible realization. It is clear that, after establishing irreducible realization, Theorem 1 can then be reduced. It seems, however, that it is more logical to establish first properties of Hankel matrices and then to establish irreducible realization.

[2] Manuscript received August 28, 1972.
The author is with the National Chiao-Tung University, Hsinchu, Taiwan.