J. Math. Anal. Appl. 365 (2010) 584–589

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Convergence of the Klein–Gordon equation to the wave map equation with magnetic field

Kung-Chien Wu

*Department of Applied Mathematics, Center of Mathematical Modeling and Scientific Computing, National Chiao Tung University, Hsinchu 30010, Taiwan*

ARTICLE INFO

*Article history:*
Received 22 September 2009
Available online 26 November 2009
Submitted by J. Xiao

*Keywords:*
Klein–Gordon equation
Magnetic field
Wave map equation
Nonrelativistic-semiclassical limit

ABSTRACT

This paper is devoted to the proof of the convergence from the modulated cubic nonlinear defocusing Klein–Gordon equation with magnetic field to the wave map equation. More precisely, we discuss the nonrelativistic-semiclassical limit of the modulated cubic nonlinear Klein–Gordon equation with magnetic field where the Planck's constant $\hbar = \varepsilon$ and the speed of light $c$ are related by $c = \varepsilon^{-\alpha}$ for some $\alpha \geqslant 1$. When $\alpha = 1$ the limit wave function satisfies the wave map with one extra term coming from the magnetic field. However, $\alpha > 1$, the effect of the magnetic filed disappears and the limit is the typical wave map equation only.



## 1. Introduction

The main concern of this paper is the cubic nonlinear Klein–Gordon equation in the presence of a magnetic field with vector potential $A$,

$$\frac{\hbar^2}{2mc^2}\partial_t^2\Psi + \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}A\right)^2\Psi + \frac{mc^2}{2}\Psi + \left(|\Psi|^2 - 1\right)\Psi = 0, \tag{1.1}$$

where $m$ is mass, $e$ is electron charge, $c$ is the speed of light and $\hbar$ is the Planck's constant. Here $\Psi(x,t)$ is a complex-valued function over a spatial domain $\Omega \subset \mathbb{R}^n$. Since the Planck's constant $\hbar$ has dimension of action $[\hbar] = [\text{energy}] \times [\text{time}] = [\text{action}]$ and $eA$ has dimension of energy $[eA] = [\text{energy}]$, it is easy to check that (1.1) is dimensional balance. Furthermore, we notice that $mc^2t$ and the Planck's constant $\hbar$ have the same dimension of action, $[mc^2t] = [\hbar] = [\text{action}]$, and we may consider the modulated wave function [9]

$$\psi(x,t) = \Psi(x,t)\exp\left(imc^2t/\hbar\right),$$

where the factor $\exp(imc^2t/\hbar)$ describes the oscillations of the wave function, then $\psi$ satisfies the modulated cubic nonlinear Klein–Gordon equation

$$i\hbar\partial_t\psi + \frac{1}{2m}\left(\hbar\nabla - i\frac{e}{c}A\right)^2\psi - \left(|\psi|^2 - 1\right)\psi = \frac{\hbar^2}{2mc^2}\partial_t^2\psi. \tag{1.2}$$

In this paper we will only discuss the nonrelativistic-semiclassical limit, so the Planck's constant $\hbar$ and the speed of light $c$ are chosen such that $\hbar = \varepsilon$ and $c = \varepsilon^{-\alpha}$ for some $\alpha \geqslant 1$, $0 < \varepsilon \ll 1$. Also after proper rescaling, we also assume the unit mass $m = 1$ and unit charge $e = 1$, and (1.2) is rewritten as

*E-mail address:* kjwu.am94g@nctu.edu.tw.

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2009.11.022

$$i\varepsilon\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{2+2\alpha}\partial_t^2\psi^\varepsilon + \frac{\varepsilon^2}{2}\left(\nabla - i\varepsilon^{\alpha-1}A\right)^2\psi - \left(|\psi^\varepsilon|^2 - 1\right)\psi^\varepsilon = 0, \tag{1.3}$$

where the superscript $\varepsilon$ in the wave function $\psi$ indicates the $\varepsilon$-dependence.

When there is no magnetic field, i.e., $A = 0$, the singular limits including semiclassical, nonrelativistic and nonrelativistic-semiclassical limits of the Cauchy problem for the modulated defocusing nonlinear Klein–Gordon equation were established in [7], where the charge-energy inequality and the convergence of the relative and nonrelative linear momentums plays the most important role. Motivated by [7] and as an extension, we study the nonrelativistic-semiclassical limit of the modulated Klein–Gordon equation with magnetic field. The asymptotic behavior of the modulated cubic nonlinear Klein–Gordon equation with magnetic field depends on the scale of the light speed. The hydrodynamical structures and the formal analysis is referred to [6]. If $\alpha = 1$, then the limit equation is the wave map equation with one extra term coming from the magnetic field, and the associated phase function satisfies the wave equation with magnetic field. On the other hand, if $\alpha > 1$, then the limit equation is the typical wave map equation and the associated phase function satisfies the wave equation. We can conclude that the magnetic effect occurs only when $\alpha = 1$.

The singular limits of the nonlinear Klein–Gordon equation and the related equations has received considerable attention in the last three decades. But most of the researches are focused on the nonrelativistic limit. In particular, Machihara, Nakanishi and Ozawa [9] gave a very complete answer of the Cauchy problem for the modulated Klein–Gordon equation, they proved that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space. Let us also mention in [11] Masmaudi and Nakanishi show that the solutions for the nonlinear Klein–Gordon equation can be described by using a system of two coupled nonlinear Schrödinger equations as the speed of light $c$ tends to infinity.

In addition to the introduction, the paper is organized as follows. We state the existence and the main theorems in Section 2. The hydrodynamical structures and the associated charge and energy equations are also derived which play the key roles in the proof of the singular limit. Unlike the Schrödinger type equations [2,3,5], the charge is not positive definite and the energy is not conserved for Eq. (1.3), so we have to construct the charge-energy inequality (see Theorem 2.1) as the main estimate to obtain compactness results. In Section 3, we prove the main theorem. Since we only have $L_t^\infty L_x^2$ bound for $\varepsilon^\alpha\partial_t\psi^\varepsilon$, thus we need more argument (compare with [2,3]) to obtain the strong convergence (see Lemma 3.1). Finally, the density fluctuation is not definite, we use equation of charge to construct the exact limit function of density fluctuation (see (3.12), (3.13)).

**Notation.** In this paper, $L^p(\Omega)$ $(p \geqslant 1)$ denotes the classical Lebesgue space with norm $\|f\|_p = (\int_\Omega |f|^p\,dx)^{1/p}$, the Sobolev space of functions with all its $k$-th partial derivatives in $L^2(\Omega)$ will be denoted by $H^k(\Omega)$, and its dual space is $H^{-k}(\Omega)$. We use $\langle f, g\rangle = \int_\Omega fg\,dx$ to denote the standard inner product on the Hilbert space $L^2(\Omega)$. Given any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$ and $p \geqslant 1$, the space of measurable functions $u = u(t)$ from $[0, T]$ into $\mathbb{X}$ such that $\|u\|_{\mathbb{X}} \in L^p([0,T])$ will be denoted $L^p([0,T];\mathbb{X})$. And $C([0,T]; w\text{-}H^k(\Omega))$ will denote the space of continuous function from $[0,T]$ into $w\text{-}H^k(\Omega)$. Finally, we abbreviate "$\leqslant C$" to "$\lesssim$", where $C$ is a positive constant depending only on fixed parameter.

## 2. Main result

The modulated cubic nonlinear Klein–Gordon equation (1.3) can be re-written as

$$i\partial_t\psi^\varepsilon - \frac{1}{2}\varepsilon^{1+2\alpha}\partial_t^2\psi^\varepsilon + \frac{\varepsilon}{2}\left(\nabla - i\varepsilon^{\alpha-1}A\right)^2\psi - \left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right)\psi^\varepsilon = 0, \tag{2.1}$$

and the initial conditions are supplemented by

$$\psi^\varepsilon(x,0) = \psi_0^\varepsilon(x), \qquad \partial_t\psi^\varepsilon(x,0) = \psi_1^\varepsilon(x). \tag{2.2}$$

To avoid the complications at the boundary, we concentrate below on the case where $x \in \Omega = \mathbb{T}^n$, the $n$-dimensional torus. Notice that the 4th term $\frac{|\psi^\varepsilon|^2-1}{\varepsilon}$ of (2.1) can be served as the density fluctuation of the sound wave which is similar to the acoustic wave as discussed in the low Mach number limit of the compressible fluid [1,4,8,10].

We define some important physical quantities associated with the modulated cubic nonlinear Klein–Gordon equation (2.1); Schrödinger part charge $\rho_S^\varepsilon$, relativistic part charge $\rho_K^\varepsilon$, momentum (current) $J^\varepsilon$ and energy $e^\varepsilon$ as follows:

$$\rho_S^\varepsilon = \left|\psi^\varepsilon\right|^2, \qquad \rho_K^\varepsilon = \frac{i}{2}\varepsilon^{1+2\alpha}\left(\psi^\varepsilon\partial_t\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\partial_t\psi^\varepsilon\right),$$

$$J^\varepsilon = \frac{i}{2}\varepsilon\left(\psi^\varepsilon\nabla\overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\right),$$

$$e^\varepsilon = \frac{1}{2}\left|\varepsilon^\alpha\partial_t\psi^\varepsilon\right|^2 + \frac{1}{2}\left|\left(\nabla - i\varepsilon^{\alpha-1}A\right)\psi^\varepsilon\right|^2 + \frac{1}{2}\left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right)^2.$$

The equations of charge and energy associated with (2.1) are given below.

(A) Equation of charge

$$\frac{\partial}{\partial t}\big(\rho_S^\varepsilon - \rho_K^\varepsilon\big) + \nabla\cdot\big(J^\varepsilon - \varepsilon^\alpha \rho_S^\varepsilon A\big) = 0. \tag{2.3}$$

(B) Equation of energy

$$\frac{d}{dt}\int_{\mathbb{T}^n} e^\varepsilon(\cdot,t)\,dx = -\varepsilon^{-2+\alpha}\int_{\mathbb{T}^n} \partial_t A\cdot\big(J^\varepsilon - \varepsilon^\alpha \rho_S^\varepsilon A\big)\,dx. \tag{2.4}$$

For the model (2.1)–(2.2) we have the following existence result.

**Theorem 2.1.** *Let $\alpha \geqslant 1$, $A \in C^1([0,\infty)\times\mathbb{T}^n)$, $T > 0$ and $0 < \varepsilon \ll 1$. Given $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$ and $\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon} \in L^2(\mathbb{T}^n)$, then there exists a function $\psi^\varepsilon$ such that*

$$\psi^\varepsilon \in L^\infty\big([0,T]; H^1\big(\mathbb{T}^n\big)\big)\cap C\big([0,T]; L^2\big(\mathbb{T}^n\big)\big),$$
$$\partial_t\psi^\varepsilon \in L^\infty\big([0,T]; L^2\big(\mathbb{T}^n\big)\big)\cap C\big([0,T]; H^{-1}\big(\mathbb{T}^n\big)\big),$$
$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \in L^\infty\big([0,T]; L^2\big(\mathbb{T}^n\big)\big),$$

*and it satisfies the weak formulation of* (2.1) *given by*

$$\begin{aligned} 0 = {} & i\big\langle \psi^\varepsilon(\cdot,t_2) - \psi^\varepsilon(\cdot,t_1), \varphi\big\rangle - \frac{\varepsilon^{1+2\alpha}}{2}\big\langle \partial_t\psi^\varepsilon(\cdot,t_2) - \partial_t\psi^\varepsilon(\cdot,t_1), \varphi\big\rangle \\ & + \frac{\varepsilon}{2}\int_{t_1}^{t_2}\big\langle\big(\nabla - i\varepsilon^{\alpha-1}A\big)\psi^\varepsilon(\cdot,\tau), \big(\nabla + i\varepsilon^{\alpha-1}A\big)\varphi\big\rangle\,d\tau - \int_{t_1}^{t_2}\bigg\langle\bigg(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\bigg)\psi^\varepsilon(\cdot,\tau), \varphi\bigg\rangle\,d\tau, \end{aligned} \tag{2.5}$$

*for every $[t_1,t_2]\subset[0,T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Moreover, it satisfies the charge-energy inequality*

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^n} \big|\psi^\varepsilon\big|^2 + \big|\varepsilon^\alpha\partial_t\psi^\varepsilon\big|^2 + \big|\big(\nabla - i\varepsilon^{\alpha-1}A\big)\psi^\varepsilon\big|^2 + \bigg(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\bigg)^2\,dx \leqslant C, \tag{2.6}$$

*where the constant $C = C(\psi_0^\varepsilon, \psi_1^\varepsilon, T, \|\partial_t A\|_{L^\infty([0,T]\times\mathbb{T}^n)})$.*

The proof of Theorem 2.1 is based on the charge and energy equations (2.3) and (2.4), which is similar to the defocusing cubic nonlinear Klein–Gordon equation without magnetic field as given in the appendix of [7] with modification. Therefore, we omit the detail.

The limiting behavior of (2.1)–(2.2) depends on the scale of light speed. If $\alpha = 1$, then the limit equation is the wave map equation with one extra term coming from the magnetic field,

$$\partial_t^2\psi - \Delta\psi = \big[\big|\nabla\psi\big|^2 - |\partial_t\psi|^2 - i\nabla\cdot A\big]\psi, \quad |\psi| = 1 \text{ a.e.} \tag{2.7}$$

We can write the wave function $\psi = e^{i\theta}$, then the phase function $\theta$ satisfies the wave equation with magnetic field

$$\partial_t^2\theta - \Delta\theta = -\nabla\cdot A. \tag{2.8}$$

However, when $\alpha > 1$, the limit equation is the typical wave map equation

$$\partial_t^2\psi - \Delta\psi = \big[|\nabla\psi|^2 - |\partial_t\psi|^2\big]\psi, \quad |\psi| = 1 \text{ a.e.} \tag{2.9}$$

If we write the wave function $\psi = e^{i\theta}$, then the phase function $\theta$ satisfies the wave equation

$$\partial_t^2\theta = \Delta\theta. \tag{2.10}$$

Now, we state the main theorem in this paper.

**Theorem 2.2.** *Let $\alpha \geqslant 1$, $A \in C^1([0,\infty)\times\mathbb{T}^n)$, $(\psi_0^\varepsilon, \psi_1^\varepsilon) \in H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, $|\psi_0^\varepsilon| = 1$ a.e., and let $\psi^\varepsilon$ be the corresponding weak solution of* (2.1)–(2.2)*. Assume that*

$$\big(\psi_0^\varepsilon, \psi_1^\varepsilon\big) \to (\psi_0, 0) \quad \text{in } H^1\big(\mathbb{T}^n\big)\oplus L^2\big(\mathbb{T}^n\big), \quad |\psi_0| = 1 \text{ a.e.}$$

*The weak limit $\psi$ solves* (2.7) *if $\alpha = 1$ and* (2.9) *if $\alpha > 1$ with initial condition $(\psi(0), \partial_t\psi(0)) = (\psi_0, 0)$. And the phase function $\theta$ solves* (2.8) *if $\alpha = 1$ and* (2.10) *if $\alpha > 1$ with initial condition $(\theta(0), \partial_t\theta(0)) = (\arg\psi_0, 0)$.*

## 3. Proof of the main theorem

The main proof of the singular limit is based on the charge-energy inequality (2.6) from which we deduce that

$$\{\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in } L^\infty\big([0,T];H^1\big(\mathbb{T}^n\big)\big), \tag{3.1}$$

$$\{\varepsilon^\alpha \partial_t \psi^\varepsilon\}_\varepsilon \quad \text{is bounded in } L^\infty\big([0,T];L^2\big(\mathbb{T}^n\big)\big), \tag{3.2}$$

$$\left\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\right\}_\varepsilon \quad \text{is bounded in } L^\infty\big([0,T];L^2\big(\mathbb{T}^n\big)\big). \tag{3.3}$$

It follows from (3.1) that there exists a subsequence still denoted by $\{\psi^\varepsilon\}_\varepsilon$ and a function $\psi \in L^\infty([0,T];H^1(\mathbb{T}^n))$ such that

$$\psi^\varepsilon \rightharpoonup \psi \quad \text{weakly} * \text{in } L^\infty\big([0,T];H^1\big(\mathbb{T}^n\big)\big). \tag{3.4}$$

Next, from (3.3), we have

$$\big|\psi^\varepsilon\big|^2 \to 1 \quad \text{a.e. and strongly in } L^2\big(\mathbb{T}^n\big). \tag{3.5}$$

Note that (3.3) only shows that $\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\}_\varepsilon$ is a weakly relative compact set in $L^\infty([0,T];L^2(\mathbb{T}^n))$. Thus to overcome the difficulty caused by nonlinearity, i.e., the 4th term on the left-hand side of (2.1), we have to prove $\psi^\varepsilon \to \psi$ strongly in $C([0,T];L^2(\mathbb{T}^n))$.

**Lemma 3.1.** *Under the hypothesis of Theorem* 2.2, *the sequence* $\{\psi^\varepsilon\}_\varepsilon$ *is a relatively compact set in* $C([0,T];w\text{-}H^1(\mathbb{T}^n))$, *thus there exists a function* $\psi \in C([0,T];w\text{-}H^1(\mathbb{T}^n))$ *such that*

$$\psi^\varepsilon \to \psi \quad \text{in } C\big([0,T];w\text{-}H^1\big(\mathbb{T}^n\big)\big) \text{ as } \varepsilon \to 0. \tag{3.6}$$

*Furthermore,* $\{\psi^\varepsilon\}_\varepsilon$ *is a relatively compact set in* $C([0,T];L^2(\mathbb{T}^n))$ *endowed with its strong topology and*

$$\psi^\varepsilon \to \psi \quad \text{in } C\big([0,T];L^2\big(\mathbb{T}^n\big)\big) \text{ as } \varepsilon \to 0. \tag{3.7}$$

**Proof.** We appeal to the Arzela–Ascoli theorem which states that the sequence $\{\psi^\varepsilon\}_\varepsilon$ is a relatively compact set in $C([0,T];w\text{-}H^1(\mathbb{T}^n))$ if and only if

(1) $\{\psi^\varepsilon(t)\}$ is a relatively compact set in $w\text{-}H^1(\mathbb{T}^n)$ for all $t \geqslant 0$;
(2) $\{\psi^\varepsilon\}$ is equicontinuous in $C([0,T];w\text{-}H^1(\mathbb{T}^n))$, i.e., for every $\varphi \in H^{-1}(\mathbb{T}^n)$ the sequence $\{\langle\psi^\varepsilon,\varphi\rangle\}_\varepsilon$ is equicontinuous in the space $C([0,T])$.

Since $\{\psi^\varepsilon(t)\}_\varepsilon$ is uniformly bounded in $H^1(\mathbb{T}^n)$, thus $\{\psi^\varepsilon(t)\}_\varepsilon$ is a relatively compact set in $w\text{-}H^1(\mathbb{T}^n)$ for every $t > 0$. In order to establish condition (2), let $B \subset C_c^\infty(\mathbb{T}^n)$ be an enumerable set which is dense in $H^{-1}$, then for any $\rho \in B$, we have

$$i\big\langle\psi^\varepsilon(\cdot,t_2)-\psi^\varepsilon(\cdot,t_1),\rho\big\rangle = \frac{\varepsilon^{1+2\alpha}}{2}\big\langle\partial_t\psi^\varepsilon(\cdot,t_2)-\partial_t\psi^\varepsilon(\cdot,t_1),\rho\big\rangle + \frac{\varepsilon}{2}\int_{t_1}^{t_2}\big\langle\big(\nabla - i\varepsilon^{\alpha-1}A\big)\psi^\varepsilon(\cdot,\tau),\big(\nabla + i\varepsilon^{\alpha-1}A\big)\rho\big\rangle\,d\tau$$
$$+\int_{t_1}^{t_2}\left\langle\left(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\right)\psi^\varepsilon(\cdot,\tau),\rho\right\rangle d\tau,$$

hence

$$\big|\big\langle\psi^\varepsilon(\cdot,t_2)-\psi^\varepsilon(\cdot,t_1),\rho\big\rangle\big| \lesssim \varepsilon^{1+\alpha}\|\rho\|_{L^2(\mathbb{T}^n)} + |t_2-t_1|\big(\|\rho\|_{H^1(\mathbb{T}^n)}+\|\rho\|_{L^\infty(\mathbb{T}^n)}\big).$$

Thus for any $\varepsilon_0 > 0$, we can choose $\delta = \varepsilon_0$ such that if $|t_2-t_1| < \delta$ and $\varepsilon^{1+\alpha} < \varepsilon_0$, then

$$\big|\big\langle\psi^\varepsilon(\cdot,t_2)-\psi^\varepsilon(\cdot,t_1),\rho\big\rangle\big| \lesssim \varepsilon_0.$$

Moreover, by density argument we can prove

$$\big|\big\langle\psi^\varepsilon(\cdot,t_2)-\psi^\varepsilon(\cdot,t_1),\varphi\big\rangle\big| \lesssim \varepsilon_0,$$

for all $\varphi \in H^{-1}(\mathbb{T}^n)$. Thus $\{\psi^\varepsilon\}_\varepsilon$ is equicontinuous in $C([0,T];w\text{-}H^1(\mathbb{T}^n))$ for $\varepsilon$ smaller, this prove (3.6). The second statement follows immediately by Rellich lemma which states that $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ compactly, i.e., $w\text{-}H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ continuously. This completes the proof of Lemma 3.1. □

The quantity $\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon}$ is bounded in $L^\infty([0,T];L^2(\mathbb{T}^n))$, and hence it converges weakly $*$ to some function $w \in L^\infty([0,T];L^2(\mathbb{T}^n))$. To find the explicit form of $w$, we define two functions $W(\psi^\varepsilon)$ and $Z(\psi^\varepsilon)$ respectively by

$$W(\psi^\varepsilon) = \frac{i}{2}\big(\psi^\varepsilon \nabla \overline{\psi^\varepsilon} - \overline{\psi^\varepsilon}\nabla\psi^\varepsilon\big), \qquad Z(\psi^\varepsilon) = \frac{i}{2}\varepsilon^{2\alpha}\big(\overline{\psi^\varepsilon}\partial_t\psi^\varepsilon - \psi^\varepsilon\partial_t\overline{\psi^\varepsilon}\big).$$

We rewrite the equation of charge (2.3) as

$$\frac{\partial}{\partial t}\left[\frac{|\psi^\varepsilon|^2-1}{\varepsilon} + Z(\psi^\varepsilon)\right] + \nabla\cdot\big[W(\psi^\varepsilon) - \varepsilon^{\alpha-1}\big|\psi^\varepsilon\big|^2 A\big] = 0, \tag{3.8}$$

then integrating (3.8) with respect to $t$ and using the initial condition $|\psi_0^\varepsilon|^2 = 1$, we have

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} = -Z(\psi^\varepsilon) + Z\big(\psi^\varepsilon(x,0)\big) - \int_0^t \nabla\cdot\big[W(\psi^\varepsilon) - \varepsilon^{\alpha-1}\big|\psi^\varepsilon\big|^2 A\big]\,d\tau. \tag{3.9}$$

Thus to obtain the compactness of the sequence $\{\frac{|\psi^\varepsilon(x,t)|^2-1}{\varepsilon}\}_\varepsilon$, we have to treat the compactness of the right-hand side of (3.9). By (3.1)–(3.2), we have $Z(\psi^\varepsilon) \rightharpoonup 0$, $Z(\psi^\varepsilon(x,0)) \rightharpoonup 0$ in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. In order to treat the integral part, we need the following lemma.

**Lemma 3.2.** *Assume the hypothesis of Theorem* 2.2, *then*

$$\int_0^t \nabla\cdot\big(\big|\psi^\varepsilon\big|^2 A\big)\,d\tau \rightharpoonup \int_0^t \nabla\cdot A\,d\tau, \tag{3.10}$$

$$\int_0^t \nabla\cdot\big(\psi^\varepsilon\nabla\overline{\psi^\varepsilon}\big)\,d\tau \rightharpoonup \int_0^t \nabla\cdot(\psi\nabla\overline{\psi})\,d\tau \tag{3.11}$$

*in* $\mathcal{D}'((0,T)\times\mathbb{T}^n)$.

**Proof.** For (3.10), using integration by part, Fubini theorem, Lebesgue dominated convergence theorem and (3.5), we conclude that

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\int_0^t \nabla\cdot\big[\big(\big|\psi^\varepsilon(x,\tau)\big|^2-1\big)A(x,\tau)\big]\,d\tau\,\varphi(x)\,dx\,dt = -\int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}^n}\big[\big(\big|\psi^\varepsilon(x,\tau)\big|^2-1\big)A(x,\tau)\big]\cdot\nabla\varphi(x)\,dx\,d\tau\,dt \to 0.$$

For (3.11), we observe that $\nabla\psi^\varepsilon \in L^\infty([0,T];L^2(\mathbb{T}^n))$ implies $\nabla\psi^\varepsilon \in L^2([0,T]\times\mathbb{T}^n)$ and $\nabla\psi^\varepsilon$ converges weakly to $\nabla\psi$ in $L^2([0,T]\times\mathbb{T}^n)$, similar to (3.10), we conclude that

$$\begin{aligned}
&-\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\int_0^t \nabla\cdot\big[\psi^\varepsilon(x,\tau)\nabla\overline{\psi^\varepsilon}(x,\tau) - \psi(x,\tau)\nabla\overline{\psi}(x,\tau)\big]\,d\tau\,\varphi(x)\,dx\,dt \\
&\quad = \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}^n}\big[\psi^\varepsilon(x,\tau) - \psi(x,\tau)\big]\nabla\overline{\psi^\varepsilon}(x,\tau)\cdot\nabla\varphi(x)\,dx\,d\tau\,dt \\
&\qquad + \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}^n}\big[\nabla\overline{\psi^\varepsilon}(x,\tau) - \nabla\overline{\psi}(x,\tau)\big]\psi(x,\tau)\cdot\nabla\varphi(x)\,dx\,d\tau\,dt \to 0.
\end{aligned}$$

This completes the proof of Lemma 3.2. □

It follows from Lemma 3.2 that the limit function is given explicitly

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \rightharpoonup -\int_0^t \nabla\cdot\big(W(\psi) - A\big)\,d\tau \quad \text{if } \alpha = 1 \tag{3.12}$$

and

$$\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} \rightharpoonup -\int_0^t \nabla \cdot W(\psi)\, d\tau \quad \text{if } \alpha > 1 \tag{3.13}$$

in the sense of distribution.

For $\alpha = 1$, by (3.1), (3.2), (3.7) and (3.12), passing the limit to the weak formulation (2.5), the limit wave function $\psi$ satisfies

$$i\partial_t\psi + \left[\int_0^t \nabla \cdot \big(W(\psi) - A\big)\, d\tau\right]\psi = 0 \tag{3.14}$$

in the sense of distribution. Note $|\psi|^2 = 1$, we have $\overline{\psi}\nabla\psi + \psi\nabla\overline{\psi} = 0$, hence

$$\frac{1}{2}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) = \overline{\psi}\nabla\psi.$$

Differentiating (3.14) with respect to $t$, we have

$$\partial_t^2\psi - \big[\nabla \cdot (\overline{\psi}\nabla\psi - iA)\big]\psi - \frac{\partial_t\psi}{\psi}\partial_t\psi = 0,$$

or

$$\partial_t^2\psi - [\overline{\psi}\Delta\psi + \nabla\psi \cdot \nabla\overline{\psi} - i\nabla \cdot A]\psi + |\partial_t\psi|^2\psi = 0.$$

Therefore $\psi$ satisfies the wave map equation with magnetic field

$$\partial_t^2\psi - \Delta\psi = \big[|\nabla\psi|^2 - |\partial_t\psi|^2 - i\nabla \cdot A\big]\psi, \quad |\psi| = 1 \text{ a.e.}$$

Using the fact $|\psi| = 1$ and writing $\psi = e^{i\theta}$ shows

$$\partial_t^2\theta - \Delta\theta = -\nabla \cdot A.$$

i.e., $\theta$ is a distribution solution of wave equation with magnetic field.

The proof for $\alpha > 1$ is similar and we omit the detail.

## Acknowledgments

The author thanks Professor Chi-Kun Lin and Professor Tai-Ping Liu for stimulating discussion and encouragement. This work is partially supported by National Science Council of Taiwan under the grant NSC98-2115-M-009-004-MY3.

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