



Electrostatic interactions of arbitrarily dispersed multicoated elliptic cylinders

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ABSTRACT

We propose a theoretical framework for evaluation of electrostatic potentials in an unbounded isotropic matrix containing a number of arbitrarily dispersed elliptic cylinders subjected to a remotely prescribed potential field. The inclusions could be homogeneous or confocally multicoated, and may have different sizes, aspect ratios and different conductivities. The approach is based on a multipole expansion formalism, together with a construction of consistency conditions and translation operators. This procedure generalizes the approach of the classic work of Rayleigh [1] for a periodic array of circular disks or spheres to an arbitrarily dispersion of elliptic cylinders. We combine the methods of complex potentials with a re-expansion formulae and the generalized Rayleigh's formulation to obtain a complete solution of the many-inclusion problem. We show that the coefficients of field expansions can be written in the form of an infinite set of linear algebraic equations. Numerical results are presented for several configurations. We further apply the obtained field solutions to determine the effective conductivity of the composite.

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1. Introduction

The transport phenomena of a heterogeneous medium consisting of many inclusions embedded in a host matrix are of fundamental theoretical interest, and also play an important role in optimal designs of industrial products. Among numerous aspects on micromechanics, most of the studies focused on the overall properties of composite materials. In contrast, the study on the field solutions appears to be a relatively rarely explored area due to its complexity. However, local fields of a composite medium are also of theoretical and technological interests in interpreting various physical phenomena such as the breakdown phenomenon [2] and in higher-order estimates of the overall properties [3]. Further, much work has assumed that the inclusions are circular or spherical, and that each phase possesses a constant property. In real systems, however, inclusions are not perfect circles or spheres, and there are situations in which the inclusions are inhomogeneous. For example, in some applications, to reduce heat or stress concentration along the interface, interphase layers between the inclusions and the matrix are often introduced to act as thermal barriers. Such interface layer may be continuous graded transition in composition or in fine, discrete step, across the interface between the two dissimilar materials. Materials with spatially varying properties, referred to as functionally graded materials, have found merits in quite a few engineering applications [4]. The main characteristic of spatially varying materials is the tailoring of graded composition to satisfy particular engineering applications. For a good review of research on functionally graded materials, the reader is referred to Suresh [4] and Hirai [5].

The present paper is concerned with the electrostatic potential field of an unbounded isotropic medium containing arbitrarily dispersed graded elliptic cylinders. Mathematically, the frameworks of electrostatics, dielectrics, magnetism, thermal

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conduction, diffusion or fluid transport, and anti-plane elasticity are all equivalent ([6, p. 19]). The solutions of any one physical problem can be readily applicable to others. Here we use the terminology of electrostatics throughout the paper simply for convenience. We consider the case that the elliptic fibers have different sizes, aspect ratios with different conductivities, and are arbitrarily positioned in the unbounded space or in a periodic array. Throughout the formulation we allow that the cylinders are homogeneous elliptic fibers or confocally multicoated ellipses. The framework of this study is based on the concept of multipole expansion formalism, together with a construction of consistency conditions and translation operators. The procedure generalizes the approach of the classic work of Rayleigh [1] for periodic arrays of circular cylinders and spheres to a medium containing many multicoated elliptic cylinders that can be arbitrarily positioned. We have employed the methods of complex potentials, a newly derived re-expansion formulae by Kushch et al. [7], and the generalized Rayleigh's formulae [8,9] to calculate the field potential and the effective conductivity of the considered composite.

The Rayleigh's formulation has been used in previous work of Nicrovici and McPhedran [10] and Yardley et al. [11] to determine the effective transport property of a rectangular array of elliptic cylinders. Yardley et al. [12] employed integral transforms to obtain rapidly convergent series for one-dimensional elliptical lattice sums, and used these lattice sums to determine the transport property of composites constructed from multiple layers of elliptical cylinders. Parallel studies have been carried out for elasticity by other methods. Meisner and Kouris [13] used the Papkovitch–Neuber displacement approach to study the interaction of two elliptic inclusions under a far-field biaxial tension. Kushch et al. [7] combined standard Muskhelishvili's representation of general solution in terms of complex potentials, the superposition principle and re-expansion formulae to obtain a complete solution of the many-inclusion problem. This analytical approach has been further developed and applied for studying the elastic behavior of a half plane containing a finite array of elliptic inclusions [14], and for the local stress and the effective elastic properties of unidirectional fiber reinforced composite with anisotropic phases [15]. For composites reinforced by discrete graded elliptic fibers, Ru et al. [16] showed that a three-phase elliptic inclusion under uniform remote stress and eigenstrain in anti-plane shear admits internal uniform stress field provided that the interfaces are two confocal ellipses. Ting et al. [17] provided a theoretical framework showing how to design a neutral cylinder with any number of coatings or with graded shear moduli in a cross-section under torsion. Other works included that of Chen [18] who considered the anti-plane shear of composites with a confocally multicoated elliptical inclusion. To our knowledge, the subject of composites with arbitrarily dispersed multicoated elliptic cylinders has not been examined in the literature before.

The plan of the paper is as follows. In Section 2, we outline the framework of the electrostatic problem of an unbounded medium containing a number of homogeneous isotropic elliptic cylinders. The potential field is expanded versus various local elliptic coordinates with origins positioned at each inclusion's centroid. The key step is to link the potentials with the remote applied field. This is accomplished by making use of Green's second identity in the matrix domain. In Section 3 we consider the case of multicoated elliptic cylinders. We propose a recurrence procedure analogous to a method originally devised for a multicoated circular cylinder [17]. We show that a (2×2) array alone can mathematically simulate the effects of multiple coatings. In Section 4, we determine the effective conductivity from the obtained field solutions, and show that the overall property solely depends on one coefficient B_1 among the infinite number of expansion coefficients. We show in Section 5 that coefficients of the field expansions is governed by an infinite set of linear algebraic equation. Numerical results are presented for several configurations, and the effects of multiple coatings is exemplified numerically. Lastly, some concluding remarks are made in Section 6.

2. Multiple elliptic cylinders

2.1. Problem statement

We consider an infinite medium Ω containing N arbitrarily distributed elliptic cylinders. The domain of the p th elliptic cylinder is represented by V_p , $p = 1, 2, \dots, N$, each of which has the conductivity tensor \mathbf{k}_p . The remaining part, the matrix Ω_m , is isotropic with conductivity $k_m \mathbf{I}$. Let us introduce a Cartesian coordinate system (x, y) , or symbolically as \mathbf{x} , positioned at a selected point O of the plane (Fig. 1). The centroids of the p th elliptic cylinders are designated as O_p , with $O_p x_p$ and $O_p y_p$ axes directed along the major and minor axes of the ellipse. Each of ellipse has the major and minor semi-axis, $l_x^{(p)}$ and $l_y^{(p)}$, and the inter-foci distance is $2d_p$, where $d_p^2 = l_x^{(p)2} - l_y^{(p)2}$. The ellipses are all well separated so that any two inclusions will not get in touch with each other. Along the remote boundary of the matrix, an external potential field is prescribed. Under steady-state condition, the potential field φ in the medium is governed by

$$\nabla \cdot [\mathbf{k}(\mathbf{x}) \nabla \varphi] = 0 \quad \text{in } \Omega \quad (2.1)$$

with

$$\mathbf{k}(\mathbf{x}) = \begin{cases} k_m \mathbf{I} & \text{if } \mathbf{x} \in \Omega_m, \\ \mathbf{k}_p & \text{if } \mathbf{x} \in V_p. \end{cases} \quad (2.2)$$

Here \mathbf{k}_p may vary with \mathbf{x} , and \mathbf{I} is the identity matrix. For an isotropic phase, the governing field (2.1) for the potential is simply the Laplace equation. In the analysis we assume that the inclusion and the matrix are perfectly bonded at their interfaces ∂V_p . This means that the electrostatic potential and the normal component of electric current are continuous across the interfaces

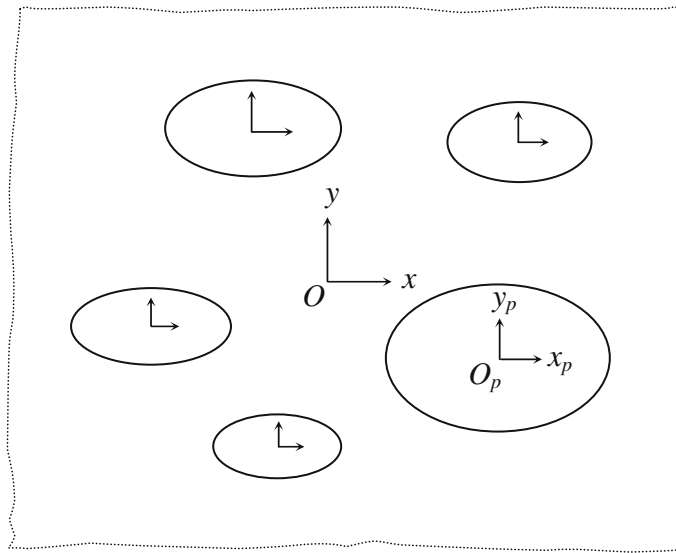


Fig. 1. A schematic representation of the composite.

$$\begin{aligned} \varphi_i^{(p)} \Big|_{\partial V_p} &= \varphi_m^{(p)} \Big|_{\partial V_p}, \\ \mathbf{k}_p \nabla \varphi_i^{(p)} \cdot \mathbf{n}_p \Big|_{\partial V_p} &= k_m \nabla \varphi_m^{(p)} \cdot \mathbf{n}_p \Big|_{\partial V_p}, \end{aligned} \tag{2.3}$$

where $\varphi_i^{(p)}$ and $\varphi_m^{(p)}$ are the potentials in the p th inclusion and matrix, respectively, and \mathbf{n}_p is the unit outward normal of the interface ∂V_p , pointing from the inclusion to the matrix.

2.2. Field expansions and field identities

We first consider the case that the cylinders are homogeneous and isotropic with conductivity $k_p \mathbf{I}$. In this situation, φ is harmonic. Therefore we can construct the analytic function $\Phi(z) = \varphi(z) + i\psi(z)$, of the complex variable $z = x + iy$, where ψ is the conjugate harmonic function, related to φ by Cauchy–Riemann equations

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.4}$$

The shape of the cross section of the cylinders defines the elliptic coordinates ($\mu > 0, -\pi < \theta \leq \pi$) ([19, p. 17]),

$$z = d \cosh w = d \cosh (\mu + i\theta), \tag{2.5}$$

are the most appropriate system for the solution of Laplace’s equation. With the above, Eq. (2.3) is recast as

$$\begin{aligned} \left(\Phi_i^{(p)} + \overline{\Phi_i^{(p)}} \right) \Big|_{\partial V_p} &= \left(\Phi_m^{(p)} + \overline{\Phi_m^{(p)}} \right) \Big|_{\partial V_p}, \\ k_p \frac{\partial}{\partial \mu_p} \left(\Phi_i^{(p)} - \overline{\Phi_i^{(p)}} \right) \Big|_{\partial V_p} &= k_m \frac{\partial}{\partial \mu_p} \left(\Phi_m^{(p)} - \overline{\Phi_m^{(p)}} \right) \Big|_{\partial V_p}. \end{aligned} \tag{2.6}$$

Following Yardley et al. [11], we expand the potential field for the p th elliptic cylinder and its surrounding matrix with respect to its centroid O_p as

$$\Phi_i^{(p)}(z) = \sum_{n=-\infty}^{\infty} C_n^{(p)} e^{-nw_p}, \quad C_n^{(p)} = C_{-n}^{(p)} \tag{2.7}$$

for the inclusion and

$$\Phi_m^{(p)}(z) = \sum_{n=-\infty}^{\infty} A_n^{(p)} e^{-nw_p} + \sum_{n=1}^{\infty} B_n^{(p)} e^{-nw_p}, \quad A_n^{(p)} = A_{-n}^{(p)} \tag{2.8}$$

for the matrix. Here $w_p = \mu_p + i\theta_p$ is the local elliptic coordinate centered at the origin of the p th ellipse, and the subscripts i and m denote the inclusion and matrix, respectively. The coefficients $A_n^{(p)} = A_{nR}^{(p)} + iA_{nI}^{(p)}$, $B_n^{(p)} = B_{nR}^{(p)} + iB_{nI}^{(p)}$ and $C_n^{(p)} = C_{nR}^{(p)} + iC_{nI}^{(p)}$ are some complex unknowns to be determined. The superscripts p appearing in (2.7) and (2.8) are used to designate the

fields that are expanded with respect to the p th ellipse centroid. Based on the orthogonality properties of harmonic functions, the interface conditions (2.6) yield

$$A_{n,A}^{(p)} = T_{An,A}^{(p)} B_{n,A}^{(p)}, \quad C_{n,A}^{(p)} = T_{Cn,A}^{(p)} B_{n,A}^{(p)}, \tag{2.9}$$

where A represents R , the real part, or I , the imaginary part of the coefficients and

$$\begin{aligned} T_{AnR}^{(p)} &= \frac{k_p \sinh na_p + k_m \cosh na_p}{(k_m - k_p) \sinh 2na_p} e^{-na_p}, \\ T_{AnI}^{(p)} &= \frac{k_p \cosh na_p + k_m \sinh na_p}{(k_p - k_m) \sinh 2na_p} e^{-na_p}, \\ T_{CnR}^{(p)} &= \frac{k_m}{(k_m - k_p) \sinh 2na_p}, \\ T_{CnI}^{(p)} &= \frac{k_m}{(k_p - k_m) \sinh 2na_p}. \end{aligned} \tag{2.10}$$

To proceed, we apply the Green's second identity ([20, p. 62]) to the matrix domain Ω_m (Fig. 2). This will give

$$\int_{\Omega_m} [G(\mathbf{x}; \mathbf{x}') \nabla'^2 \Phi_m(\mathbf{x}') - \Phi_m(\mathbf{x}') \nabla'^2 G(\mathbf{x}; \mathbf{x}')] dA' = \int_{\partial\Omega_m} [G(\mathbf{x}; \mathbf{x}') \nabla' \Phi_m(\mathbf{x}') - \Phi_m(\mathbf{x}') \nabla' G(\mathbf{x}; \mathbf{x}')] \cdot \mathbf{n}' ds', \tag{2.11}$$

where the prime ' denotes the operation in reference to the \mathbf{x}' coordinate, \mathbf{n}' is the outward unit normal to the matrix's boundary $\partial\Omega_m$, dA' represents the area element for the \mathbf{x}' coordinate, ds' is the differential arc length. Here $G(\mathbf{x}; \mathbf{x}')$ is the free-space Green's function for Laplace operator satisfying $\nabla'^2 G(\mathbf{x}; \mathbf{x}') = -\delta(\mathbf{x} - \mathbf{x}')$, where $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac-delta function. Following the procedure in Kuo and Chen [9] and the addition theorem of the Green's function for elliptic coordinates ([21, p.1202])

$$G(\mathbf{x}; \mathbf{x}') = -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}'| = -\frac{1}{2\pi} \left[\ln \frac{d}{2} + w - 2 \sum_{n=1}^{\infty} \frac{e^{-nw}}{n} \cosh(nw') \right], \tag{2.12}$$

for $|w| > |w'|$, it can be shown that Eq. (2.11) yields the consistency equation

$$\Phi_m(z) = -Ez + \sum_{l=1}^N \sum_{n=1}^{\infty} B_n^{(l)} e^{-nw_l}. \tag{2.13}$$

Here $E = E_R + iE_I$ is the uniform intensity applied in the far field. This consistency equation connects each of the multipole expansions and the potential induced from the external applied field. It states that the matrix potential is composed of two parts: the induced potential from the applied field and the influence of each inclusion. It is seen that the terms in the field expansion (2.13) are expressed in different local coordinates. We can now shift the origin of the expansions to a fixed point, say Z_p , the centroid of the p th ellipse, by expanding the term e^{-mw_l} as [7]

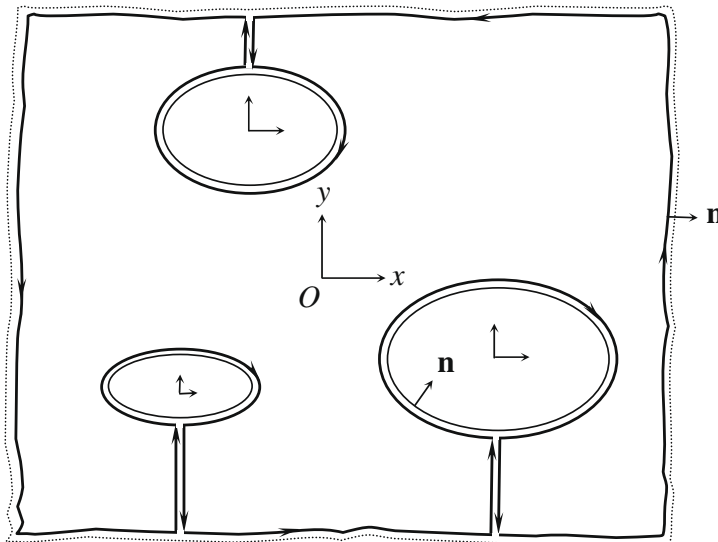


Fig. 2. A schematic representation of the contour of integration.

$$e^{-nw_l} = \sum_{m=-\infty}^{\infty} \eta_{nm}^{lp} e^{-mw_p} \quad (2.14)$$

with

$$\eta_{nm}^{lp} = (-1)^m n \left(\frac{d_l}{d_p} \right)^n \sum_{s=0}^{\infty} v_{lp}^{-(n+m+2s)} \sum_{t=0}^s \frac{(-1)^{s-1}}{(s-t)!} \left(\frac{d_p}{d_{lp}} \right)^{m+2t} M_{nmt}(d_l, d_p) \frac{(n+m+t+s-1)!}{(s-t)!}, \quad (2.15)$$

where $d_{lp} \equiv d_l + d_p$, $v_{lp} \equiv Z_{lp}/d_{lp} + \sqrt{(Z_{lp}/d_{lp})^2 - 1}$ and

$$M_{nmt}(d_l, d_p) = \sum_{k=0}^t \frac{(d_l/d_p)^{2k}}{k!(t-k)!(k+n)!(m+t-k)!}. \quad (2.16)$$

Introducing (2.14) into (2.13), we have the expansion

$$\Phi_m^{(p)}(z) = -EZ_p + \sum_{n=-\infty}^{\infty} (B_n^{(p)} + b_n^{(p)}) e^{-nw_p}, \quad (2.17)$$

where

$$b_n^{(p)} = -E \frac{d_p}{2} \delta_{n,\pm 1} + \sum_{l \neq p}^N \sum_{m=1}^{\infty} B_m^{(l)} \eta_{mn}^{lp} \quad (2.18)$$

valid for the domain within an ellipse centered in Z_p with inter-foci distance $2d_{lp}$ and passing the pole of l th elliptic coordinate systems closest to Z_p . Further, since z lies in the matrix domain, Eqs. (2.17) and (2.8) should be identical. This will provide the condition

$$\sum_{n=-\infty}^{\infty} A_n^{(p)} e^{-nw_p} = -EZ_p + \sum_{n=-\infty}^{\infty} b_n^{(p)} e^{-nw_p}. \quad (2.19)$$

Taking the real part and the imaginary part of (2.19), we obtain

$$A_{nR}^{(p)} = (-E_R \text{Re} Z_p + E_I \text{Im} Z_p) \delta_{n,0} + b_{nR}^{(p)} \quad (2.20)$$

and

$$A_{nI}^{(p)} = (-E_I \text{Re} Z_p - E_R \text{Im} Z_p) \delta_{n,0} + b_{nI}^{(p)}. \quad (2.21)$$

These two coupled conditions (2.20) and (2.21), can be viewed as generalized Rayleigh's identities in distinct with those originally derived for periodic circular and spherical inclusions [1]. The above Eqs. (2.20) and (2.21) constitute a set of infinite linear algebraic equations, which, together with (2.9), suffice to determine the expansion coefficients $A_n^{(p)}$, $B_n^{(p)}$, $C_n^{(p)}$ and hence the potential. Here we make one further remark.

Remark. The essential step of the formulation is to establish the generalized Rayleigh's identities, (2.20) and (2.21). We observe, however, that the derivation of the identities does not involve the inclusion properties. In other words, these identities can be applicable to inclusions with inhomogeneous constituents provided that the admissible fields in the inclusions and the transition relations, similar to (2.9), can be constructed.

2.3. Periodic array

Next we consider the case that the multiple cylinders are distributed in a periodic array. The regular array of composite is one of the special cases of the multiple inclusion composite. This periodic configuration implies that we can simply focus on a unit cell which in turn greatly reduces the calculations but the lattice sums, sums over all sites of an infinite perfect lattice of potential function, will then be involved. In this section efforts will be made on the differences arose from the symmetric conditions accompanied with the periodic configuration.

For convenience, let us introduce a Cartesian coordinate system (x, y) positioned at center O of one of the ellipses (Fig. 3). The sides of the rectangular cell parallel to the x and y coordinates are, respectively, denoted by α and β , and the elliptic cylinders are of the same orientation, elliptic radius $\mu = a$ and inter-foci distance $2d$. Let us apply a uniform intensity E along the positive x -axis, i.e. $\varphi_{\text{ext}}(\mathbf{x}) = -Ex$. Under steady state conditions, the potential field inside the unit cell Ω is governed by $\nabla \cdot [\mathbf{k}(\mathbf{x}) \nabla \varphi] = 0$ in Ω , where $\mathbf{k}(\mathbf{x}) = k_i \mathbf{I}$ if $\mathbf{x} \in V$ and $\mathbf{k}(\mathbf{x}) = k_m \mathbf{I}$ if $\mathbf{x} \in \Omega \setminus V$. Here k_i is the conductivity of the inclusion, and V is the area occupied by the elliptic cylinder. At the interface ∂V , the inclusion and the matrix are perfectly bonded

$$\varphi_i|_{\partial V} = \varphi_m|_{\partial V}, \quad k_i \nabla \varphi_i \cdot \mathbf{n}|_{\partial V} = k_m \nabla \varphi_m \cdot \mathbf{n}|_{\partial V}, \quad (2.22)$$

where \mathbf{n} is the unit normal of the interface ∂V . This periodic configuration implies that $x = \pm\alpha/2$ are equipotential lines and $y = \pm\beta/2$ are lines of flow. That is, for the unit cell Ω , the following boundary conditions prevail:

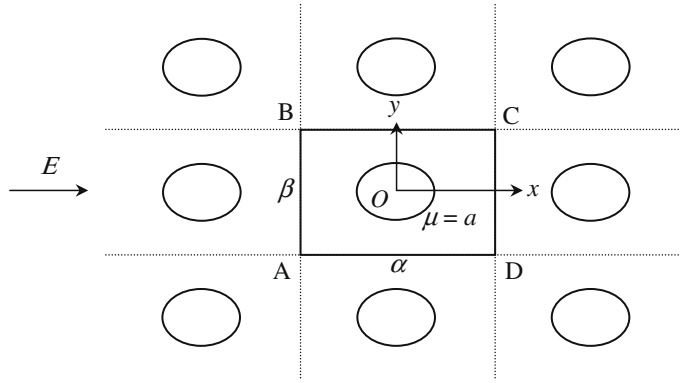


Fig. 3. A schematic representation of a unit cell.

$$\varphi|_{x=-\alpha/2} = \frac{1}{2}E\alpha, \quad \varphi|_{x=\alpha/2} = -\frac{1}{2}E\alpha, \quad \frac{\partial\varphi}{\partial y}\Big|_{y=\pm\beta/2} = 0. \tag{2.23}$$

Following the formulation for disordered systems, expansions of the form (2.7) and (2.8) are valid for each unit cell, and by applying the boundary conditions on the surfaces of the inclusions, we obtain relations of the form (2.9). Imposing the generalized Rayleigh's identities (2.20) and (2.21) for the present periodic case, we deduce

$$A_{nA} = b_{nA}, \quad A = R, I \tag{2.24}$$

with

$$b_n = -E\frac{d}{2}\delta_{n,\pm 1} + \sum_{m=1}^{\infty} B_m \sum_{l \neq 0}^{\infty} \eta_{mn}^{lo}, \tag{2.25}$$

$$\sum_{l \neq 0}^{\infty} \eta_{mn}^{lo} = (-1)^m n \sum_{t=0}^{\infty} \left(\frac{d}{2}\right)^{n+m+2t} M_{nmt}(d, d)(n+m+2t-1)! S_{n+m+2t}, \tag{2.26}$$

where $M_{nmt}(\cdot)$ is defined in (2.16), the quantities

$$S_m = \sum_{l \neq 0} Z_l^{-m}, \tag{2.27}$$

are the lattice sums characterizing the geometry of the periodic structure, and Z_l is the centroid of the l th cylinder when measured in the complex plane centered at the central point O . The index l runs over all cylinders' centers underlying the periodic array except the central one. Previous studies [1] have reported that the sum S_2 is conditionally convergent and its value depends upon the shape of the exterior boundary of the array. A list of S_2 for different values of α/β can be found in Ref. [10]. Eq. (2.24) constitutes an infinite set of linear algebraic equations which, by appropriate truncation at a finite order M , can be resolved numerically.

3. Confocally multicoated elliptic cylinders

From the previous remark, we now consider that the inclusions are confocally multicoated elliptic cylinders with the outer radius $a_p^{(1)}$, $p = 1, 2, \dots, N$, where N is the number of inclusions. For convenience, let us denote the matrix as matrix 0, with isotropic conductivity k_0 . The multicoated cylinder consists of a core, with radius $\mu_p = a_p^{(\mathcal{M})}$, surrounded by $(\mathcal{M} - 1)$ layers of coating. The j th layer of the coatings occupies the annulus $V_p^{(j)} : a_p^{(j+1)} \leq r_p \leq a_p^{(j)}$, $j = 1, 2, \dots, \mathcal{M}$, in which $V_p = V_p^{(1)} \cup V_p^{(2)} \cup \dots \cup V_p^{(\mathcal{M})}$. Here the innermost core is solid so that we have $a_p^{(\mathcal{M}+1)} = 0$. We assume that the conductivity of j th constituent layer of the p th multicoated cylinder is $k_p^{(j)}$.

By separation of variables, the admissible potentials in each constituent layer of the multicoated inclusion can be expressed as

$$\Phi^{(pj)} = \sum_{n=-\infty}^{\infty} A_n^{(pj)} e^{-nw_p} + \sum_{n=1}^{\infty} B_n^{(pj)} e^{-nw_p}, \quad A_n^{(pj)} = A_{-n}^{(pj)}, \tag{3.1}$$

where $A_n^{(pj)}$ and $B_n^{(pj)}$ are unknown complex constants to be determined. Note that the potential at $\mu_p \rightarrow 0$ should be finite and thus we can set

$$B_n^{(p,\mathcal{M})} = 0. \tag{3.2}$$

For the isotropic matrix, the electrostatic potential simply reduces to

$$\Phi^{(p,0)} = \sum_{n=-\infty}^{\infty} A_n^{(p,0)} e^{-nw_p} + \sum_{n=1}^{\infty} B_n^{(p,0)} e^{-nw_p}, \quad A_n^{(p,0)} = A_{-n}^{(p,0)}. \quad (3.3)$$

As the interfaces are perfectly bonded, the potential and the normal component of electrical current are continuous across the interfaces,

$$\begin{aligned} \left(\Phi_i^{(p,j-1)} + \overline{\Phi_i^{(p,j-1)}} \right) \Big|_{\partial V_p^{(j)}} &= \left(\Phi_m^{(p,j)} + \overline{\Phi_m^{(p,j)}} \right) \Big|_{\partial V_p^{(j)}}, \\ k_p^{(j-1)} \frac{\partial}{\partial \mu_p} \left(\Phi_i^{(p,j-1)} - \overline{\Phi_i^{(p,j-1)}} \right) \Big|_{\partial V_p^{(j)}} &= k_p^{(j)} \frac{\partial}{\partial \mu_p} \left(\Phi_i^{(p,j)} - \overline{\Phi_i^{(p,j)}} \right) \Big|_{\partial V_p^{(j)}}. \end{aligned} \quad (3.4)$$

This continuity condition (3.4) will lead to

$$\begin{pmatrix} A_{nA}^{(p,j-1)} \\ B_{nA}^{(p,j-1)} \end{pmatrix} = \mathbf{K}_{nA}^{(p,j)} \begin{pmatrix} A_{nA}^{(p,j)} \\ B_{nA}^{(p,j)} \end{pmatrix}, \quad A = R, I, \quad (3.5)$$

where

$$\begin{aligned} \mathbf{K}_{nR}^{(p,j)} &\equiv \begin{pmatrix} 2 \cosh na_p^{(j)} & e^{-na_p^{(j)}} \\ -2 \frac{k_p^{(j-1)}}{k_p^{(j)}} \sinh na_p^{(j)} & \frac{k_p^{(j-1)}}{k_p^{(j)}} e^{-na_p^{(j)}} \end{pmatrix}^{-1} \begin{pmatrix} 2 \cosh na_p^{(j)} & e^{-na_p^{(j)}} \\ -2 \sinh na_p^{(j)} & e^{-na_p^{(j)}} \end{pmatrix}, \\ \mathbf{K}_{nI}^{(p,j)} &\equiv \begin{pmatrix} -2 \sinh na_p^{(j)} & e^{-na_p^{(j)}} \\ 2 \frac{k_p^{(j-1)}}{k_p^{(j)}} \cosh na_p^{(j)} & \frac{k_p^{(j-1)}}{k_p^{(j)}} e^{-na_p^{(j)}} \end{pmatrix}^{-1} \begin{pmatrix} -2 \sinh na_p^{(j)} & e^{-na_p^{(j)}} \\ 2 \cosh na_p^{(j)} & e^{-na_p^{(j)}} \end{pmatrix}. \end{aligned} \quad (3.6)$$

For $\mathbf{K}_{nA}^{(p,1)}$, $k_p^{(0)} = k_0$. Now, repeated use of (3.5) will give

$$\begin{pmatrix} A_{nA}^{(p,0)} \\ B_{nA}^{(p,0)} \end{pmatrix} = \mathbf{K}_{pA}^{(j,n)} \begin{pmatrix} A_{nA}^{(p,j)} \\ B_{nA}^{(p,j)} \end{pmatrix}, \quad j = 1, 2, \dots, \mathcal{M}, \quad (3.7)$$

where

$$\mathbf{K}_{pA}^{(j,n)} \equiv \mathbf{K}_{nA}^{(p,1)} \mathbf{K}_{nA}^{(p,2)} \dots \mathbf{K}_{nA}^{(p,j)}. \quad (3.8)$$

According to (3.2), we have

$$\begin{pmatrix} A_{nA}^{(p,0)} \\ B_{nA}^{(p,0)} \end{pmatrix} = \mathbf{K}_{pA}^{(\mathcal{M},n)} \begin{pmatrix} A_{nA}^{(p,\mathcal{M})} \\ B_{nA}^{(p,\mathcal{M})} \end{pmatrix}, \quad (3.9)$$

which implies that

$$A_{nA}^{(p,0)} = \frac{[\mathbf{K}_{pA}^{(\mathcal{M},n)}]_{11}}{[\mathbf{K}_{pA}^{(\mathcal{M},n)}]_{21}} B_{nA}^{(p,0)}. \quad (3.10)$$

Here $[\mathbf{K}_{pA}^{(\mathcal{M},n)}]_{11}$ represents the (1, 1) entry of the matrix $\mathbf{K}_{pA}^{(\mathcal{M},n)}$ and similarly for $[\mathbf{K}_{pA}^{(\mathcal{M},n)}]_{21}$. The connection (3.10) is valid for any arbitrary number of coatings. The formulation implies that the effect of the multiple coatings can be incorporated through a recurrence procedure and is solely represented by a (2×2) array alone. This algebraic procedure was proposed by Ting et al. [17] in constructing a neutral multicoated cylinder in torsion and also by Chen [18] for a multicoated elliptic cylinder under anti-plane shear. Here the concept is extended to multicoated elliptic fibers. We mention that once we construct the admissible field (3.1) and the transition relation (3.10) in the inhomogeneous inclusions, we can follow the remaining generalized Rayleigh's framework proposed in previous section to determine the potential field distribution.

4. Effective conductivity

Our next task is to derive the effective conductivity from the potential fields of a periodic array composite. The major distinction with previous studies is that the inclusions are now elliptic and inhomogeneous. We start from the basic definition of the effective conductivity \mathbf{k}^* given by

$$\langle \mathbf{q} \rangle = \mathbf{k}^* \langle \mathbf{H} \rangle, \quad (4.1)$$

where \mathbf{q} is the electrical current, \mathbf{H} is the intensity, and the angular brackets denote the area averages over the unit cell Ω

$$\langle \mathbf{q} \rangle = \frac{1}{\Omega} \int_{\Omega} \mathbf{q} dA, \quad \langle \mathbf{H} \rangle = \frac{1}{\Omega} \int_{\Omega} \mathbf{H} dA. \tag{4.2}$$

According to Chen and Kuo [22], we can write

$$\langle q_x \rangle = \frac{1}{\Omega} \int_{\partial\Omega} \mathbf{x} \mathbf{q} \cdot \mathbf{n} ds \quad \text{and} \quad \langle H_x \rangle = E. \tag{4.3}$$

Here E is the uniform intensity applied along the positive x -axis. The effective conductivity k_{xx}^* is then found from the ratio $\langle q_x \rangle / E$. To find this ratio, we adopt Rayleigh’s approach which makes use of the Green’s second identity over the matrix region of a unit cell

$$\int_{\Omega_m} (u \nabla^2 v - v \nabla^2 u) dA = \int_{\partial\Omega_m} (u \nabla v - v \nabla u) \cdot \mathbf{n} ds. \tag{4.4}$$

Setting $u = x$ and taking v to be the potential in the region Ω_m , $\text{Re} \Phi_m$, Eqs. (4.4) and (4.3) provide

$$\begin{aligned} 0 &= \text{Re} \int_{\partial\Omega_m} [x \nabla \Phi_m(\mathbf{x}) - \Phi_m(\mathbf{x}) \nabla x] \cdot \mathbf{n} ds \\ &= -\frac{1}{k_m} \int_{\partial\Omega} \mathbf{x} \mathbf{q} \cdot \mathbf{n} ds - \text{Re} \int_{\partial V} x \nabla \Phi_m(\mathbf{x}) \cdot \mathbf{n} ds - \text{Re} \left[\int_{\partial\Omega} \Phi_m(\mathbf{x}) \nabla x \cdot \mathbf{n} ds - \int_{\partial V} \Phi_m(\mathbf{x}) \nabla x \cdot \mathbf{n} ds \right] \\ &= -\frac{1}{k_m} \Omega \langle q_x \rangle - \text{Re} \int_{\partial V} x \nabla \Phi_m(\mathbf{x}) \cdot \mathbf{n} ds + \Omega \langle H_x \rangle + \text{Re} \int_{\partial V} \Phi_m(\mathbf{x}) \nabla x \cdot \mathbf{n} ds \\ &= -\frac{1}{k_m} \Omega \langle q_x \rangle + \Omega \langle H_x \rangle - \text{Re} \int_{\partial V} [x \nabla \Phi_m(\mathbf{x}) - \Phi_m(\mathbf{x}) \nabla x] \cdot \mathbf{n} ds. \end{aligned} \tag{4.5}$$

Upon substitution the correspondence potential expansion of the matrix Φ_m (2.8) and $x = \text{Re} \frac{d}{2} (e^w + e^{-w})$ leads to

$$\begin{aligned} \text{Re} \int_{\partial V} [x \nabla \Phi_m(\mathbf{x}) - \Phi_m(\mathbf{x}) \nabla x] \cdot \mathbf{n} ds &= \text{Re} \int_0^{2\pi} \left[x \frac{\partial \Phi_m(\mathbf{x})}{\partial \mu} - \Phi_m(\mathbf{x}) \frac{\partial x}{\partial \mu} \right]_{\mu=a} d\theta \\ &= \text{Re} \int_0^{2\pi} \frac{d}{2} (e^w + e^{-w}) \left\{ \sum_{n=1}^{\infty} n [A_n (e^{nw} - e^{-nw}) - B_n e^{-nw}] \right\}_{\mu=a} d\theta \\ &\quad - \text{Re} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} [A_n (e^{nw} + e^{-nw}) + B_n e^{-nw}] \right\} \frac{d}{2} (e^w - e^{-w})_{\mu=a} d\theta = -\pi d B_{1R}, \end{aligned} \tag{4.6}$$

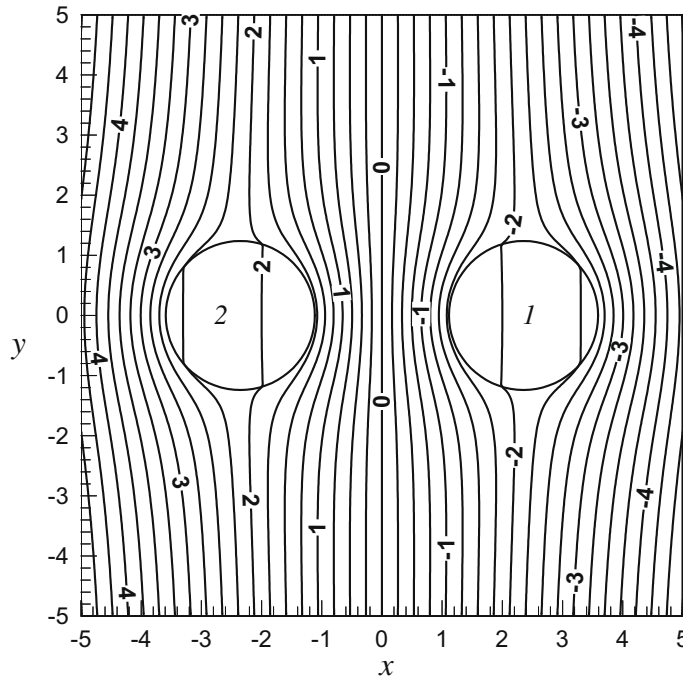


Fig. 4. Potential contours for a system under a unit intensity along the positive x -axis. Phase properties are $k_1/k_m = k_2/k_m = 10$, $I_x^{(1)} = I_x^{(2)} = I_y^{(1)} = I_y^{(2)} = 1.2388$. The centroids of the ellipses locate at $O_1 = (2.3526, 0)$ and $O_2 = (-2.3526, 0)$.

where B_{1R} is the potential expansion coefficient of the matrix. Finally, Eqs. (4.1), (4.5) and (4.6) give the effective conductivity

$$k_{xx}^* = \frac{\langle q_x \rangle}{\langle H_x \rangle} = k_m \left(1 + \frac{\pi d B_{1R}}{\alpha \beta E} \right). \tag{4.7}$$

5. Results and discussion

Here we present numerical results for the potential fields of an infinite medium containing a number of elliptic cylinders. We first write the Eqs. (2.20) and (2.21) in matrix forms. Upon a truncation at a finite order M , the infinite set of linear equations can be written as

$$\begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{I} & \mathbf{0} \\ \mathbf{M}_{41} & \mathbf{M}_{42} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{b}_R \\ \mathbf{b}_I \\ \mathbf{a}_{0R} \\ \mathbf{a}_{0I} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_R \\ \mathbf{u}_I \\ \mathbf{v}_R \\ \mathbf{v}_I \end{pmatrix}, \tag{5.1}$$

where

$$\mathbf{M}_{st} = \begin{pmatrix} \mathbf{m}_{st}^{11} & \mathbf{m}_{st}^{12} & \dots & \mathbf{m}_{st}^{1N} \\ \mathbf{m}_{st}^{21} & \mathbf{m}_{st}^{22} & \dots & \mathbf{m}_{st}^{2N} \\ \dots & \dots & \dots & \dots \\ \mathbf{m}_{st}^{N1} & \mathbf{m}_{st}^{N1} & \dots & \mathbf{m}_{st}^{NN} \end{pmatrix}, \quad s = 1, 2, 3, 4; \quad t = 1, 2, \tag{5.2}$$

$$\begin{aligned} (\mathbf{m}_{11}^{pl})_{nm} &= \begin{cases} T_{AnR}^{(p)} \delta_{n,m}, & p = l, \\ -\text{Re} \eta_{mn}^{lp}, & p \neq l, \end{cases} \quad n, m = 1, 2, \dots, M, \\ (\mathbf{m}_{22}^{pl})_{nm} &= \begin{cases} T_{AnI}^{(p)} \delta_{n,m}, & p = l, \\ -\text{Re} \eta_{mn}^{lp}, & p \neq l, \end{cases} \\ (\mathbf{m}_{12}^{pl})_{nm} &= -(\mathbf{m}_{21}^{pl})_{nm} = \begin{cases} 0, & p = l, \\ \text{Im} \eta_{mn}^{lp}, & p \neq l, \end{cases} \\ (\mathbf{m}_{31}^{pl})_{1m} &= (\mathbf{m}_{42}^{pl})_{1m} = \begin{cases} 0, & p = l, \\ -\text{Re} \eta_{m0}^{lp}, & p \neq l, \end{cases} \\ (\mathbf{m}_{32}^{pl})_{1m} &= -(\mathbf{m}_{41}^{pl})_{1m} = \begin{cases} 0, & p = l, \\ -\text{Im} \eta_{m0}^{lp}, & p \neq l, \end{cases} \end{aligned} \tag{5.3}$$

$$\mathbf{b}_A = \begin{pmatrix} \begin{pmatrix} B_{1A}^{(1)} \\ B_{2A}^{(1)} \\ \vdots \\ B_{MA}^{(1)} \end{pmatrix} \\ \vdots \\ \begin{pmatrix} B_{1A}^{(N)} \\ B_{2A}^{(N)} \\ \vdots \\ B_{MA}^{(N)} \end{pmatrix} \end{pmatrix}_{NM \times 1}, \quad \mathbf{a}_{0A} = \begin{pmatrix} A_{0A}^{(1)} \\ A_{0A}^{(2)} \\ \vdots \\ A_{0A}^{(N)} \end{pmatrix}_{N \times 1}, \quad A = R, I, \tag{5.4}$$

$$\mathbf{u}_A = \begin{pmatrix} \begin{pmatrix} -E_A \frac{d_1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} -E_A \frac{d_N}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{pmatrix}_{NM \times 1}, \quad \mathbf{v}_R = \begin{pmatrix} -E_R \text{Re} Z_1 + E_I \text{Im} Z_1 \\ -E_R \text{Re} Z_2 + E_I \text{Im} Z_2 \\ \vdots \\ -E_R \text{Re} Z_N + E_I \text{Im} Z_N \end{pmatrix}_{N \times 1}, \tag{5.5}$$

$$\mathbf{v}_I = \begin{pmatrix} -E_I \text{Re} Z_1 - E_R \text{Im} Z_1 \\ -E_I \text{Re} Z_2 - E_R \text{Im} Z_2 \\ \vdots \\ -E_I \text{Re} Z_N - E_R \text{Im} Z_N \end{pmatrix}_{N \times 1},$$

and $\mathbf{0}$ is the zero matrix, \mathbf{I} is the identity matrix, and $\delta_{n,m}$ is the Kronecker delta symbol.

To check the correctness of our formulation, we first compute the potential fields of the special case of an infinite medium containing two circular inclusions subjected to a remotely applied uniform intensity. The numerical solutions will be compared with those derivable from the bipolar-coordinate transformation [9]. We check the potential contour for two identical circular inclusions by letting $I_y^{(p)} \rightarrow I_x^{(p)}$ under a unit intensity along the positive x direction (Fig. 4). The selected phase conductivities are $k_1/k_m = k_2/k_m = 10$. In numerical calculations, the linear systems (2.20) together with (2.21) are truncated at $M = 5$ and $M = 15$. We find that the numerical results remain unchanged till the four digits after the decimal point. A comparison of the potential field obtained from the present method and that by the bipolar coordinates method is illustrated in Fig. 5. The errors are within a satisfactory range of 10^{-3} for the potential, and the maximum error occurs at the interfaces, as

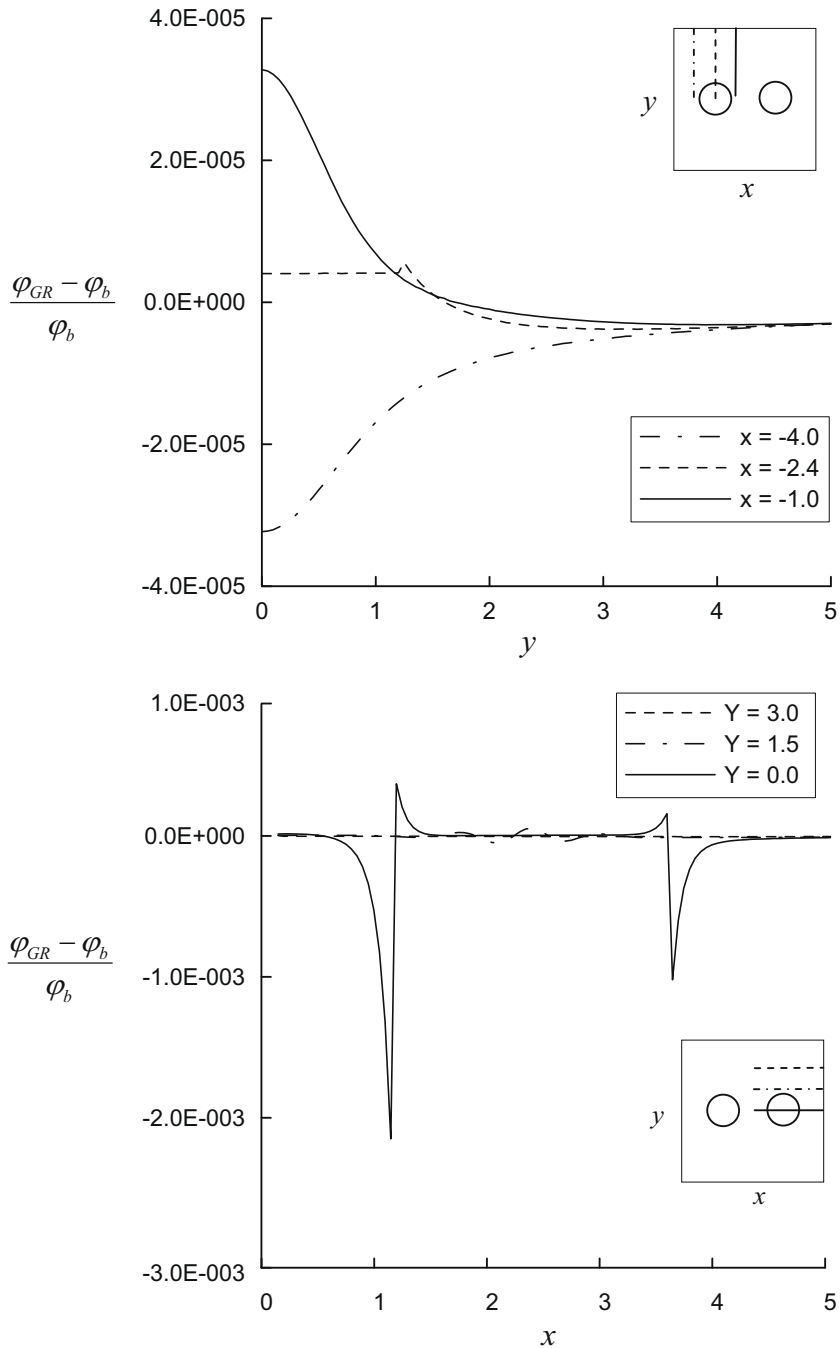


Fig. 5. Comparison of the potential obtained by generalized Rayleigh's method φ_{GR} ($M = 15$) and that by the bipolar transformation method φ_b ($M = 500$).

anticipated. Next, we present the field solution for a medium containing multiple elliptic inclusions. In Figs. 6 and 7, we demonstrate the potential contours for a medium containing three and five inclusions. Each of the elliptic inclusions has different

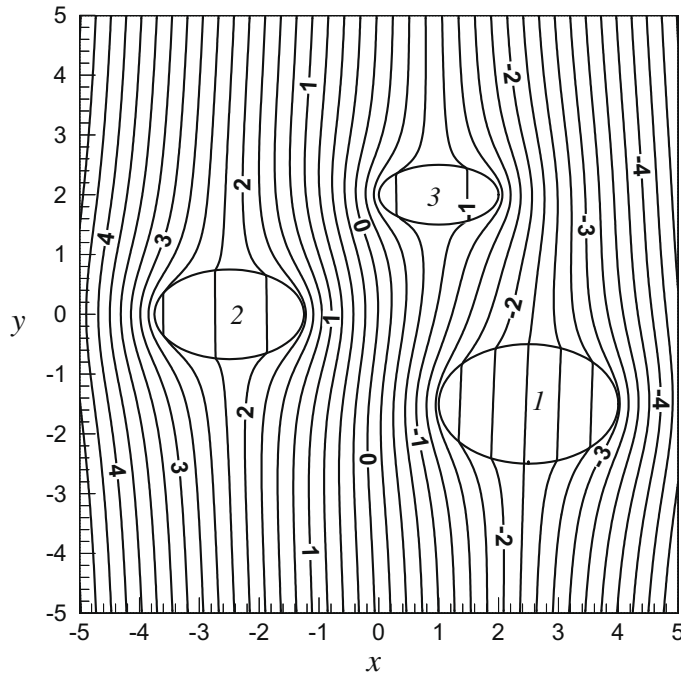


Fig. 6. Potential contours for a system under a unit intensity along the positive x -axis. Phase properties are $k_1/k_m = 4$, $k_2/k_m = 8$, $k_3/k_m = 12$, $l_x^{(1)} = 1.5$, $l_x^{(2)} = 1.25$, $l_x^{(3)} = 1$, $l_y^{(1)} = 1$, $l_y^{(2)} = 0.75$, $l_y^{(3)} = 0.5$. The centroids of the ellipses locate at $O_1 = (2.5, -1.5)$, $O_2 = (-2.5, 0)$, and $O_3 = (1, 2)$.

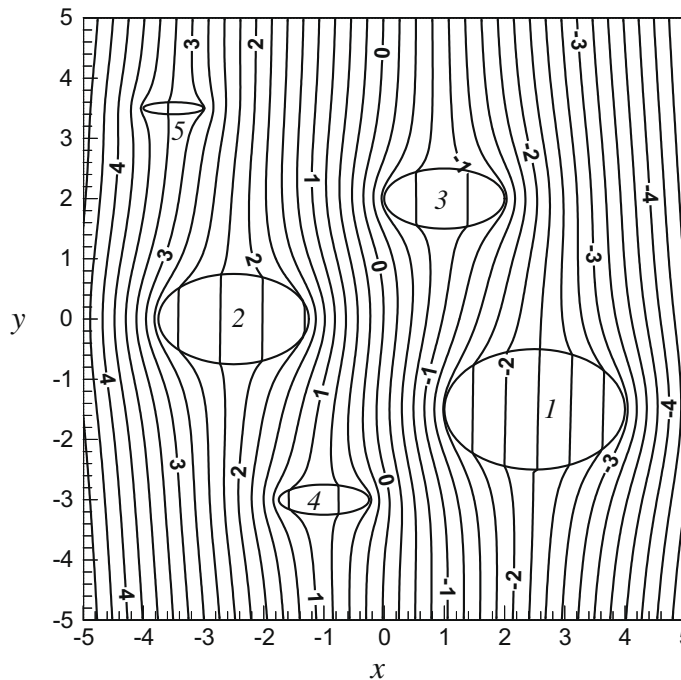


Fig. 7. Potential contours for a system under a unit intensity along the positive x -axis. Phase properties are $k_1/k_m = 4$, $k_2/k_m = 6$, $k_3/k_m = 8$, $k_4/k_m = 10$, $k_5/k_m = 12$, $l_x^{(1)} = 1.5$, $l_x^{(2)} = 1.25$, $l_x^{(3)} = 1$, $l_x^{(4)} = 0.75$, $l_x^{(5)} = 0.5$, and $l_y^{(1)} = 1$, $l_y^{(2)} = 0.75$, $l_y^{(3)} = 0.5$, $l_y^{(4)} = 0.25$, $l_y^{(5)} = 0.1$. The centroids of the ellipses locate at $O_1 = (2.5, -1.5)$, $O_2 = (-2.5, 0)$, $O_3 = (1, 2)$, $O_4 = (-1, -3)$, and $O_5 = (-3.5, 3.5)$, respectively.

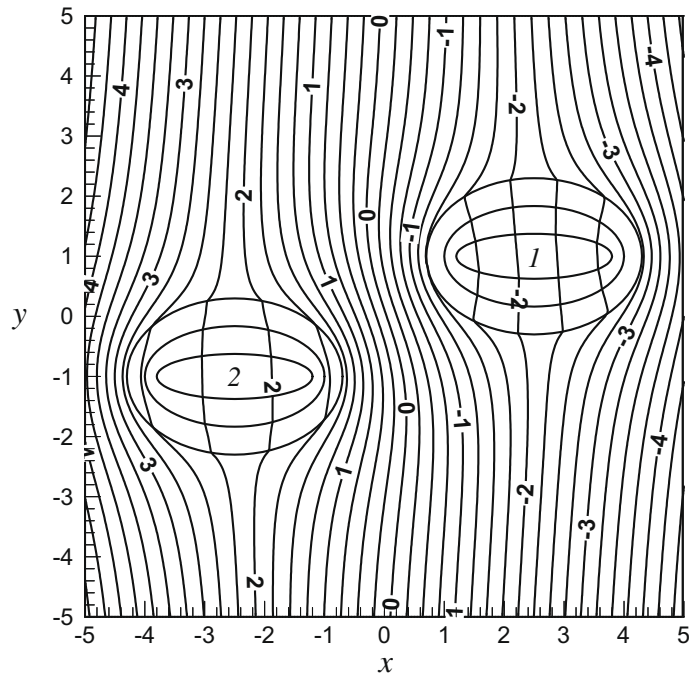


Fig. 8. Potential contours for a system under a unit intensity along the positive x -axis. Phase properties are $k_1^{(1)}/k_0 = 10$, $k_1^{(2)}/k_0 = 5$, $k_1^{(3)}/k_0 = 2$, $k_2^{(1)}/k_0 = 5$, $k_2^{(2)}/k_0 = 10$, $k_2^{(3)}/k_0 = 15$, $l_x^{(1,1)} = l_x^{(2,1)} = 1.8$, $l_x^{(1,2)} = l_x^{(2,2)} = 1.5$, $l_x^{(1,3)} = l_x^{(2,3)} = 1.3$, and $d_1 = d_2 = 1.245$. The centroids of the ellipses locate at $O_1 = (2.5, 1)$ and $O_2 = (-2.5, -1)$, respectively.

elliptic radius, aspect ratio and conductivity. The potential fields inside the inclusions are nearly linear with respect to x . It should be mentioned that higher-order terms are, in fact, non-vanishing, but its effect is rather minor compared with the linear term. Finally, a potential plot for a system with two doubly confocally coated ellipses are illustrated in Fig. 8. The constituent properties are $k_1^{(1)}/k_0 = 10$, $k_1^{(2)}/k_0 = 5$, $k_1^{(3)}/k_0 = 2$, $k_2^{(1)}/k_0 = 5$, $k_2^{(2)}/k_0 = 10$, $k_2^{(3)}/k_0 = 15$, $l_x^{(1,1)} = l_x^{(2,1)} = 1.8$, $l_x^{(1,2)} = l_x^{(2,2)} = 1.5$, $l_x^{(1,3)} = l_x^{(2,3)} = 1.3$, and $d_1 = d_2 = 1.245$. It is seen that the electrostatic potential is continuous across the interface, and the main difference of the contour pattern appears in the inclusions. One tends to deviate from the origin, and the other moves towards the centroid of the ellipse.

6. Conclusions

In summary, we have applied Rayleigh's formalism to estimate the conductive field and effective behavior of composites consisting of multicoated elliptic inclusions dispersing arbitrarily or in periodic arrays. This extension is a hybrid technique: the admissible potentials for the matrix and inclusions are expanded in complex planes, while the interface conditions are directly satisfied by using elliptic coordinates. Detailed formulations are presented for homogeneous and confocally multicoated elliptic cylinders. For the multicoated case, we derive a recurrence procedure valid for any number of coatings. The present procedure can also apply to a composite with imperfect interfaces with slight modifications. Compared to the integral equations method which often involves a discretization along the interface over the domain, the interface conditions are directly fulfilled in the formulation, and thus the computation time is greatly reduced. In the future, the described formulation can be incorporated with some refined numerical algorithms in order to treat problems with large numbers of inclusions. We will also study the parallel formulation to an analogous three-dimensional problem of an infinite medium containing an arbitrary dispersion of ellipsoids.

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