Chapter 3

Bayesian Approach for Assessing Process Capability Based on Single Sample

3.1. Introduction

The usual practice of judging process capability by simply looking at the point estimates of capability indices, have a flaw since there is no assessment of the error of these estimates. Therefore, a simple point estimate of the index is highly unreliable in making decision in assessing process capability since the estimate does not represent the true index value. When the estimate is greater than a pre-specified value w, say 1.00, or 1.33, it does not guarantee that the index is greater than w and vice versa. It is therefore preferable to obtain an interval estimate for which we can assert, with a reasonable degree of certainty, that it contains the true index value. Existing methods for testing the capability indices have focused on using the traditional but long time been widely used distribution frequency approaches. However, the sampling distributions of those PCI estimators are usually complicated that it is very difficult to obtain exact interval estimates. Examples include Chou and Owen (1989), Chou *et al.* (1990), Li *et al.* (1990), Boyles (1991), Kushler and Hurley (1992), Kotz *et al.* (1993), Subbaiah and Taam (1993), Nagata and Nagahata (1994), Tang *et al.* (1997), Zimmer and Hubele (1997), Pearn *et al.* (1998), Hoffman (2001), Zimmer *et al*. (2001), Pearn and Lin (2002), Pearn and Shu (2003a, 2003b) and many others. Kotz and Johnson (2002) provided a compact survey with interpretations and comments on some 170 publications on process capability indices, which appeared during 1992-2000. Spiring *et al.* (2003) consolidated the research findings of process capability analysis for the period 1990–2002.

Bayesian statistical techniques are an alternative to the frequency approach. These techniques specify a prior distribution for the parameter of interest, in order to obtain the posterior distribution for the parameter. We then could make inferences about the parameter by using its posterior distribution given the sample data. It is not difficult to obtain the posterior distribution when a prior distribution is given, even when the form of the posterior distribution is complicated, as one can always use numerical methods or Monte Carlo methods (Berger (1985), Kalos and Whitlock (1986)) to obtain an approximate but quite accurate interval estimate. This is the advantage of the Bayesian approach over the traditional distribution frequency approach.

Assuming that the measures $\mathbf{x}=\{x_1,x_2,\cdots,x_n\}'$ are random sample taken from independent and identically distributed from $N(\mu,\sigma^2)$, a normal distribution with mean μ and variance σ^2. Then, the likelihood function for μ and σ is

$$L(\mu,\sigma\mid\mathbf{x})=\left(2\pi\sigma^2\right)^{-n/2}\times\exp\left\{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right\}. \tag{3.1}$$

The most important problem in Bayesian inference is how to specify an appropriate prior distribution. If prior information about the parameters is available, it should be incorporated in the prior density. If we have no prior information, we want a prior with minimal influence on the inference. There are mainly two types of priors: informative and non-informative. Ideally, a Bayesian should subjectively elicit a prior on the basis of available information, expert opinion or past experience. Informative prior distributions summarize the evidence about the parameters concerned from many sources and often have a considerable impact on the results. For an example of informative priors, conjugate priors, although being widely used, can only be justified if enough information is available to believe that the true prior distribution belongs to the specified family; otherwise, the main justification for using conjugate prior is their mathematical tractability.

On the other hand, non-informative prior, Bayesian analysis often leads to the procedures with approximate frequency validity while retaining the Bayesian flavor, thus allowing certain amount of reconciliation between the two conflicting paradigms of statistics and providing with mutual justification. Box and Tiao (1992) defined a non-informative prior as prior, which provides little information relative to the experiment. Bernardo and Smith (1993) use a similar definition, they say that non-informative prior have minimal effect relative to the idea, on the final inference. And Kass and Wasserman (1996) stated two interpretations of non-informative priors.

Therefore, the first step for the Bayesian approach is to find an appropriate prior. Usually, when there is only a little or no prior information is available, or only one parameter of interest, one of the most widely used non-informative priors is the so-called reference prior, which is a non-informative prior that maximizes the difference between information on the parameter provided by the prior and by the posterior. In other words, the reference prior allows the prior to provide information as little as possible about the parameter (see Bernardo and Smith (1993) for more details). For this reason, in our investigation we adopt the following non-informative reference prior chosen by Cheng and Spiring (1989) and Shiau *et al.* (1999a),

$$\pi(\mu,\sigma) = 1/\sigma\,,\quad -\infty < \mu < \infty\,,\quad 0 < \sigma < \infty\,. \tag{3.2}$$

We note that the parameter space of the prior is infinite, hence the reference prior is improper, which means that it does not integrate to one. However, it is not always a serious problem, since the prior incorporated with ordinary likelihood will lead to proper posterior. Furthermore, the credible interval obtained from a non-informative prior has a more precise coverage probability than that obtained from any other priors. The posterior probability density function (PDF), $f(\mu,\sigma\,|\,\mathbf{x})$ of (μ,σ) may be expressed as the following:

$$f(\mu,\sigma\,|\,\mathbf{x}) \propto L(\mu,\sigma\,|\,\mathbf{x})\times\pi(\mu,\sigma) \propto \sigma^{-(n+1)}\times\exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right).$$

Also

$$\int_0^{\infty}\int_{-\infty}^{\infty} \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right) d\mu\, d\sigma$$

$$= \int_0^{\infty} \sigma^{-(n+1)} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\int_{-\infty}^{\infty} \exp\left(-\frac{n(\mu-\bar{x})^2}{2\sigma^2}\right) d\mu\right] d\sigma$$

$$= \int_0^{\infty} \sigma^{-(n+1)} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \sigma\sqrt{\frac{2\pi}{n}}\, d\sigma = \sqrt{\frac{2\pi}{n}} \times \int_0^{\infty} \sigma^{-n} \exp\left(-\frac{1}{\beta\sigma^2}\right) d\sigma$$

$$= \sqrt{\frac{\pi}{2n}} \times \int_0^{\infty} \left(\sigma^{-2}\right)^{\frac{n-3}{2}} \exp\left(-\frac{1}{\beta\sigma^2}\right) d\sigma^{-2}$$

$$= \sqrt{\frac{\pi}{2n}}\, \Gamma(\alpha)\, \beta^{\alpha},$$

where $\alpha = (n-1)/2$, $\beta = [\sum_{i=1}^{n}(x_i-\bar{x})^2/2]^{-1} = [(n-1)s^2/2]^{-1}$ and $\Gamma(k) = \int_0^{\infty} t^{k-1} e^{-t}\, dt$ is the Euler gamma function of k. Further, in order to satisfy the integration property that the probability over PDF is 1, a coefficient of $f(\mu,\sigma \mid \mathbf{x})$ can be obtained through some algebraic manipulations. Consequently, the posterior PDF of (μ,σ) can be expressed as:

$$f(\mu,\sigma \mid \mathbf{x}) = \frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^{\alpha}} \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right). \tag{3.3}$$

3.2. Bayesian Approach for C_{pk}

3.2.1. The Posterior Probability p with C_{pk}

Cheng and Spiring (1989) proposed a Bayesian procedure for assessing process capability index C_p. Chan *et al.* (1988) applied a similar Bayesian approach to index C_{pm} under the assumption that the process mean μ is equal to the target value T. Shiau *et al.* (1999b) derived the posterior distributions for C_p^2, C_{pm}^2 under the restriction that process mean μ equals to the target value T, and C_{pk}^2 under the restriction that the process mean μ equals to the midpoint of the two specification limits, M, with respect to the two priors (a non-informative and a Gamma prior). However, the restriction of $6\mu = T$ or $\mu = M$ is not a practical assumption for many industrial applications. A nice Bayesian procedure for assessing process capability index C_{pm} relaxing the restriction on the process mean was proposed by Shiau *et al.* (1999a). They also applied a similar Bayesian approach for testing the index C_{pk} but under the restriction $\mu = M$. We note that in this case C_{pk} reduces to C_p.

In the following, we consider a Bayesian procedure for the most popular capability index C_{pk} for general situation – no restriction on the process mean. Thus, the results obtained are more general and practical for real applications. We consider the quantity Pr{process is capable | $\mathbf{x}$} in the Bayesian approach. Since the index C_{pk} is our focus in this case, so we are interested in finding the posterior probability $p = \Pr\{C_{pk} > w \mid \mathbf{x}\}$ for some fixed positive number w.

Given a pre-specified capability level $w > 0$, the posterior probability based on index C_{pk} that a process is capable can be derived in the following way.

$$p = \Pr\{C_{pk} > w \mid \mathbf{x}\} = \Pr\left\{\frac{d - |\mu - M|}{3\sigma} > w \;\middle|\; \mathbf{x}\right\}$$

$$= \Pr\{|\mu - M| < d - 3\sigma w \mid \mathbf{x}\}$$

$$= \int_0^\infty \int_{M-d+3\sigma w}^{M+d-3\sigma w} f(\mu, \sigma \mid \mathbf{x})\, d\mu d\sigma$$

$$= \int_0^\infty \frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(n+1)} \times \exp\left(-\frac{1}{\beta\sigma^2}\right) \int_{M-d+3\sigma w}^{M+d-3\sigma w} \exp\left(-\frac{n(\mu - \bar{x})^2}{2\sigma^2}\right) d\mu\, d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{M + d - 3\sigma w - \bar{x}}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{M - d + 3\sigma w - \bar{x}}{\sigma/\sqrt{n}}\right)\right] d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{d + (M - \bar{x})}{s/\sqrt{n}} \times \frac{s}{\sigma} - 3\sqrt{n}w\right) - \Phi\left(\frac{-d + (M - \bar{x})}{s/\sqrt{n}} \times \frac{s}{\sigma} + 3\sqrt{n}w\right)\right] d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{d - (\bar{x} - M)}{s/\sqrt{n}} \times \frac{s}{\sigma} - 3\sqrt{n}w\right) + \Phi\left(\frac{d - (M - \bar{x})}{s/\sqrt{n}} \times \frac{s}{\sigma} - 3\sqrt{n}w\right) - 1\right] d\sigma. \quad (3.4)$$

Next, we consider the two cases for derivation of posterior probability p as follows:

[CASE I]: $\bar{x} \ge M$

If $\bar{x} \ge M$, then $\hat{C}_{pk} = \frac{d - (\bar{x} - M)}{3s}$ and $\frac{d - (M - \bar{x})}{3s} = \frac{d - (\bar{x} - M) + 2(\bar{x} - M)}{3s}$

$= \hat{C}_{pk} + \frac{2}{3}\delta$, where $\delta = |\bar{x} - M| / s$. Thus, the expression (3.4) can be rewritten as

$$p = \Pr\{C_{pk} > w \mid \mathbf{x}\}$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(3\sqrt{n}\hat{C}_{pk} \times \frac{s}{\sigma} - 3\sqrt{n}w\right) + \Phi\left(3\sqrt{n}\left(\hat{C}_{pk} + \frac{2}{3}\delta\right) \times \frac{s}{\sigma} - 3\sqrt{n}w\right) - 1\right] d\sigma. \quad (3.5)$$

[CASE II]: $\bar{x} < M$

If $\bar{x} < M$, then $\frac{d - (\bar{x} - M)}{3s} = \frac{d + (M - \bar{x})}{3s} = \hat{C}_{pk} + \frac{2}{3}\delta$ and $\frac{d - (M - \bar{x})}{3s} = \hat{C}_{pk}$ where

$\delta = |\bar{x} - M| / s$. Thus, the expression (3.4) can be rewritten as

$$p = \Pr\{C_{pk} > w \mid \mathbf{x}\}$$
$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(3\sqrt{n}\left(\hat{C}_{pk} + \frac{2}{3}\delta\right) \times \frac{s}{\sigma} - 3\sqrt{n}w\right) + \Phi\left(3\sqrt{n}\hat{C}_{pk} \times \frac{s}{\sigma} - 3\sqrt{n}w\right) - 1\right] d\sigma. \quad (3.6)$$

From both cases expressed in (3.5) and (3.6), we obtain that

$$p = \Pr\{C_{pk} > w \mid \mathbf{x}\}$$
$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(3\sqrt{n}\hat{C}_{pk} \times \frac{s}{\sigma} - 3\sqrt{n}w\right) + \Phi\left(3\sqrt{n}\left(\hat{C}_{pk} + \frac{2}{3}\delta\right) \times \frac{s}{\sigma} - 3\sqrt{n}w\right) - 1\right] d\sigma. \quad (3.7)$$

Consequently, by changing the variable, let $y = \beta\sigma^2$, then $dy = 2\beta\sigma\, d\sigma$, and $s/\sigma = \sqrt{2/(n-1)y}$. Therefore, the posterior probability p in (3.7) can be calculated as:

$$p = \Pr\{\text{the process is capable} \mid \mathbf{x}\} = \Pr\{C_{pk} > w \mid \mathbf{x}\}$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \left\{\Phi\left[3\sqrt{n}\left(\hat{C}_{pk} \times \sqrt{\frac{2}{(n-1)y}} - w\right)\right]\right.$$
$$\left. + \Phi\left[3\sqrt{n}\left((\hat{C}_{pk} + \frac{2}{3}\delta) \times \sqrt{\frac{2}{(n-1)y}} - w\right)\right] - 1\right\} dy$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \{\Phi[b_1(y)] + \Phi[b_2(y)] - 1\}\, dy, \quad (3.8)$$

where $\alpha = (n-1)/2$, $\delta = |\bar{x} - M| / s$,

$$b_1(y) = 3\sqrt{n}\left(\hat{C}_{pk} \times \sqrt{\frac{2}{(n-1)y}} - w\right), \quad b_2(y) = 3\sqrt{n}\left((\hat{C}_{pk} + \frac{2}{3}\delta) \times \sqrt{\frac{2}{(n-1)y}} - w\right),$$

and $\Phi(\cdot)$ is the CDF of the standard normal distribution. Note that the posterior probability p depends on n, w, δ and $\hat{C}_{pk}$.

3.2.2. Bayesian Procedure for Testing C_{pk}

As we can see it is rather complicated to compute posterior probability p in (3.8). However, there is a one-to-one correspondence between p and $C^*(p)$ when n and w are given, and by the fact that $\hat{C}_{pk}$ and δ can be calculated from the process data. While $C^*(p)$ is the minimum value of $\hat{C}_{pk}$ required to ensure the posterior probability p reaching a certain desirable level w. Thus, we can find the value of $C^*(p)$ satisfies equation (3.8) for various p. Suppose for this particular process under consideration to be capable, the process index must reach at least a certain level w, say, 1.00 or 1.33. From expression (3.8) we have the probability $p = \Pr\{C_{pk} > w \mid \mathbf{x}\}$ based on the observed process data. Moreover, to see if a process is capable (with capability level w and confidence level p), we only need to check if $\hat{C}_{pk} > C^*(p)$. Thus, if $\hat{C}_{pk} > C^*(p)$, then we say that the process is capable in a Bayesian sense. Otherwise, we do not have sufficient information to conclude that the process meets the preset capability requirement, and then we tend to believe that the process is incapable in this case.

To make this Bayesian procedure practical for in-plant applications, we calculate the values of $C^*(p)$ for various values of n = 10(5)160 and δ = 0(0.5)2.0 with posterior probability p = 0.90, 0.95, 0.99 and w = 1.00, 1.33, 1.50, 2.00. Tables 3.1(a)-(c) summarize the values of $C^*(p)$ with w = 1.00, for p = 0.90, 0.95 and 0.99, respectively. Tables 3.2(a)-(c) summarize the values of $C^*(p)$ with w = 1.33, for p = 0.90, 0.95, and 0.99, respectively. Tables 3.3(a)-(c) summarize the values of $C^*(p)$ with w = 1.50, for p = 0.90, 0.95 and 0.99, respectively. And the values of $C^*(p)$ with w = 2.00, for p = 0.90, 0.95 and 0.99 are displayed in Tables 3.4(a)-(c), respectively. For example, if w = 1.33 is the minimum capability requirement, then for p = 0.95, n = 100, δ = 0.5, $C^*(p)$ = 1.5173 by checking Table 3.2(b). Thus, the value $\hat{C}_{pk}$ calculated from sample data must satisfy $\hat{C}_{pk}$ ≥ 1.5173 to conclude that C_{pk} ≥ 1.33 (process is capable). From these tables we observe that the value of $C^*(p)$ decreases as δ increases for each fixed p and n. Figures 3.1-3.4 display the value of $C^*(p)$ versus $\delta = |\bar{x} - M|/s$ for sample size n = 10(10)50 from top to bottom in plots, with p = 0.95 and w = 1.00, 1.33, 1.50 and 2.00, respectively. This phenomenon can be explained by the following argument. For a fixed $\hat{C}_{pk}$, since

$$\hat{C}_{pk} = \frac{d - |\bar{x} - M|}{3s} = \frac{d/s - \delta}{3}, \tag{3.10}$$

then s becomes smaller when δ becomes larger, and a smaller s means that it is plausible that the underlying process is tighter (i.e. with smaller σ). Since the estimation is usually more accurate for data drawn from a tighter process, it is then plausible that the estimate $\hat{C}_{pk}$ is more accurate with a smaller s. In this case the required minimum value is smaller, so we need only a smaller $C^*(p)$ to account for the smaller uncertainty in the estimation. Intuitively, if the estimation error in our estimate is potentially large, then it is reasonable that we need a large $C^*(p)$ to be able to claim that the process is capable, and this means that the corresponding minimum value $C^*(p)$ should be large as well. Thus, the value of $C^*(p)$ decreases as δ increases. Another observation from the tables is that the value of $C^*(p)$ decreases as n increases for fixed δ and p. This can also be explained by the same reasoning as above, since a larger n implies that $\hat{C}_{pk}$ is more accurate.

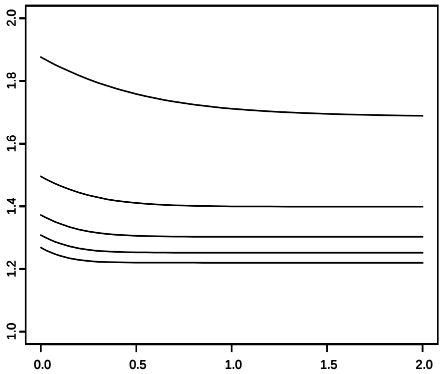

Figure 3.1 Plots of $C^*(p)$ versus δ for w = 1.00, p = 0.95, and n = 10, 20, 30, 40, and 50 (top to bottom in plot).

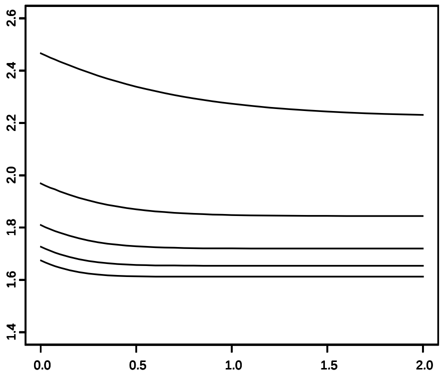

Figure 3.2 Plots of $C^*(p)$ versus δ for w = 1.33, p = 0.95, and n = 10, 20, 30, 40, and 50 (top to bottom in plot).

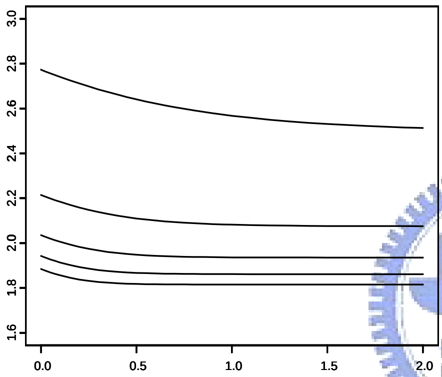

Figure 3.3 Plots of $C^*(p)$ versus δ for w = 1.50, p = 0.95, and n = 10, 20, 30, 40, and 50 (top to bottom in plot).

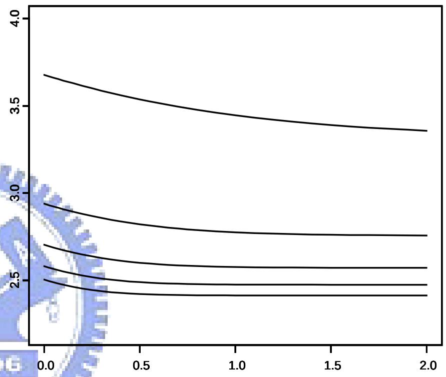

Figure 3.4 Plots of $C^*(p)$ versus δ for w = 2.00, p = 0.95, and n = 10, 20, 30, 40, and 50 (top to bottom in plot).

3.3. Bayesian Approach for C_{PU} and C_{PL}

3.3.1. Posterior Probability p with C_{PU} and C_{PL}

As mentioned before, C_{PU} and C_{PL} have been designed particularly for processes with one-sided manufacturing specifications (which require only USL or LSL, but not both). The index C_{PU} measures the capability of a smaller-the-better process with an upper specification limit, whereas the index C_{PL} measures the capability of a larger-the-better process with a lower specification limit.

In the following we derive a Bayesian approach using the indices C_{PU} and C_{PL}, for measuring the process capability in which the specifications are one sided rather than two sided. Since the index C_{PU} and C_{PL} is our major concern in this case, so we are interested in finding the posterior probability $p = \Pr\{C_{PU} > w \mid \mathbf{x}\}$ or $p = \Pr\{C_{PL} > w \mid \mathbf{x}\}$ for some fixed positive number w. Therefore, given a pre-specified capability level $w > 0$, the posterior probability based on index C_{PU} that a process is capable is given as

$$p = \Pr\{C_{PU} > w \mid \mathbf{x}\} = \Pr\left\{\frac{USL-\mu}{3\sigma} > w \;\middle|\; \mathbf{x}\right\} = \Pr\{\mu + 3\sigma w < USL \mid \mathbf{x}\}$$

$$= \int_0^\infty \int_{-\infty}^{USL-3\sigma w} \frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}\right) d\mu d\sigma$$

$$= \int_0^\infty \frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(n+1)} \times \exp\left(-\frac{1}{\beta\sigma^2}\right) \int_{-\infty}^{USL-3\sigma w} \exp(-\frac{n(\mu-\bar{x})^2}{2\sigma^2})\, d\mu\, d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \Phi\left(\frac{USL - 3\sigma w - \bar{x}}{\sigma/\sqrt{n}}\right) d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \Phi\left(3\sqrt{n}(\hat{C}_{PU} \times \frac{s}{\sigma} - w)\right) d\sigma\,.$$

By changing variable, let $y = \beta\sigma^2$. Then, $dy = 2\beta\sigma\, d\sigma$ and $s/\sigma = \sqrt{2/(n-1)y}$. Therefore, the posterior probability p may be rewritten as:

$$p = \Pr\{C_{PU} > w \mid \mathbf{x}\}$$
$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \Phi\left(3\sqrt{n}(\frac{\tilde{C}_{PU}}{b_{n-1}} \times \sqrt{\frac{2}{(n-1)y}} - w)\right) dy\,, \tag{3.11}$$

where $\alpha = (n-1)/2$, $\beta = [\sum_{i=1}^n (x_i - \bar{x})^2/2]^{-1} = [(n-1)s^2/2]^{-1}$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$, $b_{n-1} = [2/(n-1)]^{1/2}\Gamma[(n-1)/2]/\Gamma[(n-2)/2]$ and $\Phi(\cdot)$ is the CDF of the standard normal distribution.

On the other hand, the posterior probability based on index C_{PL} that a process is capable is given as

$$p = \Pr\{C_{PL} > w \mid \mathbf{x}\} = \Pr\left\{\frac{\mu - LSL}{3\sigma} > w \;\middle|\; \mathbf{x}\right\} = \Pr\{\mu - 3\sigma w > LSL \mid \mathbf{x}\}$$

$$= \int_0^\infty \int_{LSL+3\sigma w}^{\infty} \frac{2\sqrt{n}}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha} \sigma^{-(n+1)} \times \exp\left(-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^2}\right) d\mu d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[1 - \Phi\left(\frac{LSL + 3\sigma w - \bar{x}}{\sigma/\sqrt{n}}\right)\right] d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-n}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \Phi\left(3\sqrt{n}(\hat{C}_{PL} \times \frac{s}{\sigma} - w)\right) d\sigma$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \Phi\left(3\sqrt{n}(\frac{\tilde{C}_{PL}}{b_{n-1}} \times \sqrt{\frac{2}{(n-1)y}} - w)\right) dy\,. \tag{3.12}$$

3.3.2. Bayesian Procedures for Testing C_{PU} and C_{PL}

For convenience of presentation, we let C_I be either C_{PU} or C_{PL} and $\tilde{C}_I$ denote as either $\tilde{C}_{PU}$ or $\tilde{C}_{PL}$, then from the equations (3.11) and (3.12), the posterior probability based on the one-sided indices C_{PU} and C_{PL} can be rewritten as

$$p = \Pr\{\text{the process is capable } | \mathbf{x}\} = \Pr\{C_I > w \mid \mathbf{x}\}$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \Phi\left(3\sqrt{n}\left(\frac{\tilde{C}_I}{b_{n-1}} \times \sqrt{\frac{2}{(n-1)y}} - w\right)\right) dy\,, \qquad (3.13)$$

where $\alpha = (n-1)/2$, $b_{n-1} = [2/(n-1)]^{1/2}\Gamma[(n-1)/2]/\Gamma[(n-2)/2]$ and $\Phi(\cdot)$ is the CDF of the standard normal distribution. Note that the posterior probability p depends on n, w and $\tilde{C}_I$. From expression (3.13), by noticing that there is a one-to-one correspondence between p and $C^*(p)$ when n and w are given, and by the fact that $\tilde{C}_I$ can be calculated from the process data. While $C^*(p)$ is the minimum value of $\tilde{C}_I$ required to ensure the posterior probability p reaching a certain desirable level w. Thus, we can find the value of $C^*(p)$ for various p, which can be useful in assessing process capability. Figures 3.5-3.7 plot the probability p versus $C^*(p)$ from (3.13) for n = 10(30)100 with w = 1.25, 1.45, 1.60, respectively. From these figures, we can see that the larger is the sample size, the steeper the curve. That is, the larger is the sample size, the smaller the critical value $C^*(p)$. To make this Bayesian procedure practical for in-plant applications, we tabulated the minimum values $C^*(p)$ of $\tilde{C}_I$ for reaching desirable confidence levels p with various capability requirements w. Tables 3.5-3.7 display the values of $C^*(p)$ with w = 1.25, 1.45, 1.60, n = 10(10)300 for p = 0.95, 0.975 and 0.99, respectively. For example, if w = 1.25 is the minimum capability requirement, then for p = 0.95, n = 50, $C^*(p)$ = 1.493. Thus, the value $\tilde{C}_I$ calculated from sample data must satisfy $\tilde{C}_I \geq 1.493$ to conclude that C_{PU} (or C_{PL}) $\geq$ 1.25 (process is capable).

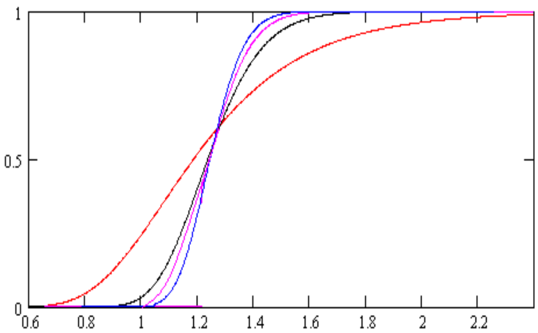

Figure 3.5 Probability p versus $C^*(p)$ for n = 10(30)100, w = 1.25.

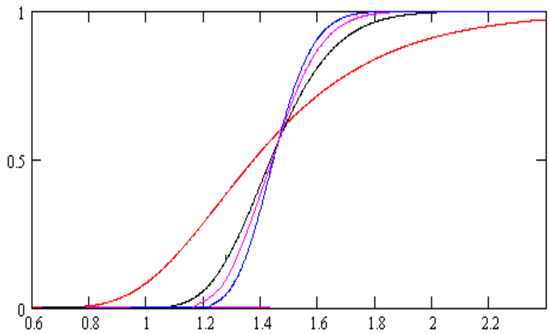

Figure 3.6 Probability p versus $C^*(p)$ for n = 10(30)100, w = 1.45.

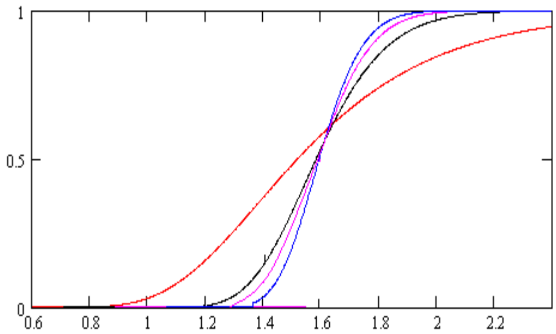

Figure 3.7 Probability p versus $C^*(p)$ for n = 10(30)100, w = 1.60.

As a result, to judge if a given process meets the capability requirement, we first determine the pre-specified value w, the capability requirement, and the confidence level p (or the α-risk) for the interval. Check the appropriate table or solve the equation (3.13), the critical value $C^*(p)$ based on given values of p, w and n can be obtained and next to calculate $\tilde{C}_I$ from sample data. If the estimated value $\tilde{C}_I$ is greater than the critical value $C^*(p)$, then we may conclude that the process meets the capability requirement ($C_I > w$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_I \le w$.

3.4. Application Examples

3.4.1. Example 1: Assessing Oil-hydraulic Cylinder Process Capability

To illustrate how we apply the proposed procedure to actual data collected from the factory. We consider the following example taken from a company engaged mainly in making oil-hydraulic cylinder components and oil-hydraulic cylinder (oil-hydraulic propeller) assembly. Oil-hydraulic equipment is required for automation and oil-hydraulic cylinders are the main component of such equipment. The pistons are one of the most critical parts of oil-hydraulic cylinders. The manufacturing specifications for the grooves of the piston are set to USL = 13.25 mm and LSL = 13.15 mm. The capability requirement for this particular model of oil-hydraulic cylinder was defined as "Satisfactory" if C_{pk} > 1.33. i.e., the requirement for the process yield is no more than 66 NCPPM.

Table 3.8 A total of 150 observations for the grooves of the piston.

13.207	13.194	13.186	13.204	13.202	13.222	13.172	13.200	13.197	13.192
13.178	13.190	13.215	13.199	13.196	13.205	13.203	13.195	13.194	13.206
13.184	13.215	13.199	13.182	13.207	13.203	13.206	13.184	13.184	13.194
13.208	13.212	13.207	13.200	13.191	13.206	13.195	13.203	13.194	13.200
13.215	13.211	13.187	13.211	13.207	13.189	13.215	13.203	13.198	13.206
13.184	13.218	13.201	13.198	13.207	13.214	13.199	13.197	13.206	13.208
13.192	13.203	13.207	13.193	13.209	13.201	13.196	13.213	13.198	13.211
13.194	13.207	13.190	13.207	13.202	13.209	13.206	13.192	13.209	13.208
13.204	13.218	13.191	13.209	13.191	13.187	13.200	13.190	13.209	13.212
13.198	13.186	13.197	13.187	13.205	13.193	13.196	13.210	13.199	13.199
13.207	13.184	13.208	13.202	13.199	13.203	13.190	13.195	13.189	13.199
13.206	13.212	13.207	13.210	13.205	13.208	13.222	13.203	13.196	13.203
13.205	13.218	13.208	13.196	13.208	13.199	13.190	13.189	13.218	13.193
13.181	13.194	13.197	13.213	13.187	13.212	13.212	13.189	13.206	13.198
13.205	13.190	13.211	13.217	13.190	13.196	13.214	13.207	13.200	13.190

Historical data based on routine process monitoring shows that the process is under statistical control and the process distribution is justified and is shown to be fairly close to the normal distribution. The sample data collected from the stable process (a total of 150 observations) are displayed in Table 3.8. The sample mean and sample standard deviation are calculated as $\bar{x}$ = 13.201 and s = 0.00969. And we then calculate the value of the estimator $\hat{C}_{pk} = (d - |\bar{x} - M|)/(3s) =$ 1.6925, and $\delta = |\bar{x} - M|/s =$ 0.103. By solving the posterior probability in (3.9), the minimum value of $\hat{C}_{pk}$ required to ensure the probability p = 0.95 with w = 1.33 and n = 150, is found to be $C^*(p)$ = 1.4869. Since $\hat{C}_{pk}$ = 1.6925 is greater than the critical value $C^*(p)$ = 1.4869 in this case, it is therefore concluded with 95% confidence (α = 0.05) that the grooves of the piston manufacturing process satisfies the requirement "$C_{pk} > 1.33$". That is, at least 99.9934% of the produced oil-hydraulic cylinders are conformed to the manufacturing specifications.

3.4.2. Example 2: Assessing EEPROM Chip Process Capability

Electrically Erasable Programmable Read-Only Memory (EEPROM) chip is a user-modifiable read-only memory chip that can be erased and reprogrammed (written onto) repeatedly through the application of higher electrical voltage. It is usually used in portable phones, PHS phones, compact portable terminals, consumer products (such as cordless phones and audio systems); industrial equipment including measuring instruments and PLCs; OA products such as printers and scanners, in-house telephone switches, and other communication equipment. The output leakage current (OLC) is an essential product quality characteristic, which has significant impact to product quality. For the output leakage current of a particular model of EEPROM, the upper specification limit, USL, is set to 5 μA. We consider the following example taken from a company located on the Science-Based Industrial Park in Taiwan, which manufacturing and designing standard Flash Memory EEPROM and Mixed-Signal products, such as, PLL, ADC DAC, and many others. The manufacturing specifications for a 128-bit EEPROM chip, has an upper specification limit $USL = 5\mu\text{A}$ for the output leakage current. If the OLC is greater than 5 μA, then the EEPROM chip is considered to be nonconforming product, and will not be used to make the EEPROM chip of that particular model. The capability requirement for this particular model of EEPROM chip was defined as "Capable" if $C_{PU} > 1.45$.

A total of 100 observations were collected from a stable process in the factory are displayed in Table 3.9. From the Shapiro-Wilk test for normality confirms this with *p*-value > 0.1, it is evident to consider the data collected from the factory are normal distributed. The calculated sample mean $\bar{x}$ = 2.987, the sample standard deviation s = 0.382, the value of $\hat{C}_{PU} = (USL - \bar{x})/3s$ = 1.757 and b_{n-1} = 0.992 based on the 100 sample data. The critical value $C^*(p)$ = 1.640 by checking Table 3.6 based on w = 1.45, p = 0.95, and n = 100. Since $\tilde{C}_{PU} = b_{n-1} \times \hat{C}_{PU} = 1.743$ is larger than the critical value $C^*(p)$ = 1.640 in this case, we therefore conclude that with 95% confidence the 128-bit EEPROM chip manufacturing process satisfies the requirement " $C_{PU} > 1.45$ ". Furthermore, we have the probability $\Pr\{C_{PU} > 1.45 \mid \mathbf{x}\} = 0.9916$ from the equation (3.13). That is, we have a 99.16% confidence to conclude that the produced EEPROM chips are conformed to the manufacturing specifications with fraction of nonconformities 13.614 PPM.

Table 3.9 A total of 100 observations of EEPROM chips.

2.74	2.25	2.98	3.14	3.08	2.85	3.21	2.51	3.19	2.75
3.68	3.23	2.90	3.05	2.58	3.31	2.52	3.16	2.62	2.95
2.85	2.80	3.03	3.05	2.54	2.44	2.82	3.01	2.93	3.39
2.47	3.08	2.40	3.22	2.77	3.05	4.15	2.59	3.28	3.56
2.75	3.38	3.49	2.54	2.28	2.93	3.54	3.49	3.09	3.17
3.17	2.66	3.35	2.77	2.68	3.15	3.23	2.77	2.30	2.17
3.35	2.76	2.20	2.75	3.58	2.70	2.78	2.99	3.63	3.44
2.91	2.67	3.56	2.73	2.90	2.41	3.20	3.86	3.02	3.39
3.26	3.60	2.89	3.18	3.03	2.60	2.70	3.25	3.32	2.67
2.61	3.09	3.07	2.89	3.49	3.14	2.96	2.87	2.97	3.26