Chapter 4

Bayesian Approach for Assessing Process Capability Based on Multiple Samples

4.1. Introduction

Most of the results obtained so far regarding the statistical properties of estimated capability indices are based on the assumption of a single sample. However, in practice and in much of the quality control literature, the process performance is monitored and controlled by periodically collecting subsamples of data (i.e. based on the concepts of rational subgrouping). To use estimators based on multiple samples and then interpret the results as if they were based on a single sample may result in incorrect conclusions (Vännman and Hubele (2003)). In order to use past in-control data from multiple samples to make correct decisions regarding process capability, the distribution of the estimated capability index based on multiple samples should be taken into account. Therefore, it is more practical to develop a procedure for assessing process capability for cases with multiple samples. Kirmani *et al.* (1991) considered the estimation of σ and C_p based on the sample standard deviations of the subsamples. Li *et al.* (1990) have investigated the distribution of estimators of C_p and C_{pk} based on the ranges of the subsamples. Under the assumption of normality, Vännman and Hubele (2003) considered the indices in the class defined by $C_p(u,v)$ and derived the distribution of the estimators of $C_p(u,v)$, when the process mean is estimated using the grand average and the process variance is estimated using the pooled variance from subsamples collected over time for an in-control process. Further, Hubele and Vännman (2004) considered the pooled and un-pooled estimators of the variance from subsamples, and gave the sampling distributions of the corresponding estimators of C_{pm}. Those methods assume that the estimation occurs after numerous subsamples have been collected, plotted on a control chart, and the process has been deemed to be in control. The un-pooled variance estimator is equivalent to the traditional "overall" or "long-term" variance estimator, whereas the pooled variance estimator is based on the control chart related "within" and "short-term" variance estimator. That is, when the process has undergone a change in variation, the un-pooled estimator captures all of the variation, whereas the pooled estimator captures the component of within subsamples variation (see, e.g., Cryer and Ryan (1990) and Hubele and Vännman (2004)).

However, those studies on PCIs are all based on the traditional frequentist point of view. In the following we consider the problem of estimating and testing process capability indices with multiple samples based on Bayesian approach, and propose accordingly a Bayesian procedure for testing process capability. We will assume that the process is in statistical control during the time period that the subsamples are taken. Assuming that the m samples are randomly taken from independent and identically distributed (*i.i.d.*) $N(\mu,\sigma^2)$, a normal distribution with mean μ and variance σ^2.

Denote the measures of the i-th sample as $\mathbf{x}_i = \{x_{i1}, x_{i2}, ..., x_{in_i}\}$ with sample size n_i, and $\mathbf{X} = \{x_{ij}, i = 1, 2, ..., m, j = 1, 2, ..., n_i\}$. Then, the likelihood function for μ and σ can be expressed as:

$$L(\mu,\sigma \mid \mathbf{X}) = \left(2\pi\sigma^2\right)^{\frac{-\sum_{i=1}^{m} n_i}{2}} \times \exp\left\{-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\mu)^2}{2\sigma^2}\right\}.$$

In practical situation, the choice of prior information is hard to justify. Therefore, in our investigation we adopt the following non-informative reference prior chosen by Cheng and Spiring (1989) and Shiau *et al.* (1999),

$$h(\mu,\sigma) = 1/\sigma, \quad -\infty < \mu < \infty, \quad 0 < \sigma < \infty.$$

The posterior probability density function (PDF), $f(\mu,\sigma \mid \mathbf{X})$ of (μ,σ) may be expressed as the following:

$$f(\mu,\sigma \mid \mathbf{X}) \propto L(\mu,\sigma \mid \mathbf{X}) \times h(\mu,\sigma) \propto \sigma^{-\left(\sum_{i=1}^{m} n_i + 1\right)} \times \exp\left\{-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\mu)^2}{2\sigma^2}\right\}.$$

Also

$$\int_0^\infty \int_{-\infty}^\infty \sigma^{-\left(\sum_{i=1}^{m} n_i + 1\right)} \times \exp\left\{-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\mu)^2}{2\sigma^2}\right\} d\mu\, d\sigma$$

$$= \int_0^\infty \sigma^{-\left(\sum_{i=1}^{m} n_i + 1\right)} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\int_{-\infty}^\infty \exp\left(-\frac{\sum_{i=1}^{m} n_i(\mu-\bar{\bar{x}})^2}{2\sigma^2}\right) d\mu\right] d\sigma$$

$$= \sqrt{\frac{\pi}{2\sum_{i=1}^{m} n_i}}\ \Gamma(\alpha)\beta^\alpha.$$

In order to satisfy the integration property that the probability over PDF is 1, the posterior PDF of (μ,σ) becomes

$$f(\mu,\sigma \mid \mathbf{X}) = \frac{2\sqrt{\sum_{i=1}^{m} n_i}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^\alpha} \sigma^{-(\sum_{i=1}^{m} n_i + 1)} \times \exp\left(-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\mu)^2}{2\sigma^2}\right). \quad (4.1)$$

where $\alpha = (\sum_{i=1}^{m} n_i - 1)/2$, $\beta = [\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{\bar{x}})^2/2]^{-1}$, $\bar{\bar{x}} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} x_{ij} / \sum_{j=1}^{n_i} x_{ij}$.

4.2. Bayesian Approach for C_p Based on Multiple Samples

4.2.1. Estimation of C_p Based on Multiple Samples

For cases where data are collected as multiple samples, Kirmani *et al.* (1991) considered m samples each of size n and suggested the following estimator of C_p, where $\bar{x}_i$ is the *i*-th sample mean, and s_i is the *i*-th sample standard deviation:

$$\hat{C}_p^* = \frac{(USL - LSL)d_p}{6}, \quad \text{where}$$

$$d_p = \sqrt{\frac{m(n-1)-1}{m(n-1)}}\,\frac{\varepsilon_{m(n-1)-1}}{s_p},$$

$$\varepsilon_{m(n-1)-1} = \sqrt{\frac{2}{m(n-1)-1}}\,\Gamma\left(\frac{m(n-1)}{2}\right)\left[\Gamma\left(\frac{m(n-1)-1}{2}\right)\right]^{-1},$$

$$s_p^2 = \frac{1}{m(n-1)}\sum_{i=1}^{m}(n-1)s_i^2 = \frac{1}{m}\sum_{i=1}^{m}s_i^2,$$

noting that under normality assumption the statistic s_p/σ is distributed as $\chi_{m(n-1)-1}/[m(n-1)-1]^{1/2}$. Therefore, the estimator $\hat{C}_p^*$ is distributed as:

$$\hat{C}_p^* \sim \frac{\sqrt{m(n-1)-1}\,\varepsilon_{m(n-1)-1}}{\sqrt{\chi^2_{m(n-1)}}}C_p.$$

The estimator $\hat{C}_p^*$ is unbiased, and its probability density function can be obtained as the following, for $y > 0$, where $k = [m(n-1)-1]\,\varepsilon^2_{m(n-1)-1}C_p^2$, which can be expressed as a function of C_p.

$$g(y) = \frac{2k^{m(n-1)/2}}{2^{m(n-1)/2}\Gamma[m(n-1)/2]}y^{-[m(n-1)+1]}\exp\left[-\frac{k}{2}\left(\frac{1}{y^2}\right)\right].$$

The variance of $\hat{C}_p^*$ has been obtained as (see Kirmani *et al.* (1991)):

$$Var(\hat{C}_p^*) = E[(\hat{C}_p^*)^2] - [E(\hat{C}_p^*)]^2$$

$$= (USL-LSL)^2\,\varepsilon^2_{m(n-1)-1}\frac{[m(n-1)-1]}{36m(n-1)}E\left(s_p^2\right)^{-1} - C_p^2 \quad = C_p^2\left\{\left(\varepsilon^2_{m(n-1)-2}\right)^{-1} - 1\right\}.$$

In addition to being unbiased, Pearn and Yang (2003) investigated some statistical properties of $\hat{C}_p^*$ and showed that $\hat{C}_p^*$ is asymptotically efficient, consistent for C_p and $(mn)^{1/2}(\hat{C}_p^* - C_p)$ converges to $N(0, C_p^2/2)$ in distribution.

For cases where data are collected as subsamples with unequal sample size, we can consider the generalized estimator of C_p as below. In fact, the estimator $\hat{C}_p^*$ obtained from m samples each of size n_i remains unbiased.

$$\hat{C}_p^* = b_{\sum_{i=1}^{m}(n_i-1)} \times \frac{USL-LSL}{6s_p}, \quad (4.2)$$

where $s_p^2 = \frac{\sum_{i=1}^{m}(n_i-1)s_i^2}{\sum_{i=1}^{m}(n_i-1)}$, and $b_{\sum_{i=1}^{m}(n_i-1)} = \sqrt{\frac{2}{\sum_{i=1}^{m}(n_i-1)}}\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)}{2}\right)\left[\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)-1}{2}\right)\right]^{-1}$.

4.2.2. Posterior Probability p with C_p Based on Multiple Samples

As mentioned earlier, it is natural to consider the quantity Pr {process is capable | $\mathbf{X}$} in the Bayesian approach. That is, we want to obtain the posterior probability $p = \Pr\{C_p > w \mid \mathbf{X}\}$ given the multiple samples collected over time for an in-control process, for some fixed positive number w. Therefore, given a pre-specified precision level $w > 0$ and denote $a = (USL - LSL)/(6w)$, the posterior probability for index C_p based on multiple samples that a process is capable is given as

$$p = \Pr\{C_p > w \mid \mathbf{X}\} = \Pr\left\{\frac{USL-LSL}{6\sigma} > w \;\middle|\; \mathbf{X}\right\}$$

$$= \Pr\left\{\sigma < \frac{USL-LSL}{6w} \mid \mathbf{X}\right\} = \Pr\{\sigma < a \mid \mathbf{X}\}$$

$$= \int_0^a f(\sigma \mid \mathbf{X})d\sigma = \int_0^a \frac{2\sigma^{-\sum_{i=1}^{m} n_i}}{\Gamma(\alpha)\beta^{\alpha}} \times \exp\left(-\frac{1}{\beta\sigma^2}\right)d\sigma .$$

By changing the variable, let $y = (\beta\sigma^2)^{-1}$, then $dy = -2(\beta\sigma^3)^{-1}\, d\sigma$, the above posterior probability p expression may be rewritten as:

$$p = \int_{\frac{1}{t}}^{\infty} \frac{\sigma^{-(\sum_{i=1}^{m} n_i - 3)}}{\Gamma(\alpha)\beta^{\alpha-1}} \times \exp(-y)dy$$

$$= \int_{\frac{1}{t}}^{\infty} \frac{y^{\alpha-1}}{\Gamma(\alpha)} \times \exp(-y)dy = \frac{\Gamma(\alpha, 1/t)}{\Gamma(\alpha)}, \quad (4.3)$$

or equivalently,

$$p = 1 - Gamma\,(1/t,\, \alpha, 1), \quad (4.4)$$

where $\Gamma(\alpha,\, 1/t)$ is the value of the incomplete gamma function of $1/t$ with parameter α, $Gamma\,(1/t, \alpha, 1)$ is the cumulative probability at $1/t$ for the gamma distribution with parameters α and 1, and

$$t = \frac{2r}{\sum_{i=1}^{m}(n_i-1)}\left(\frac{\hat{C}_p^*}{w\, b_{\sum_{i=1}^{m}(n_i-1)}}\right)^2,$$

$$r=\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{\bar{x}})^2}=\frac{\sum_{i=1}^{m}(n_i-1)s_p^2}{\sum_{i=1}^{m}(n_i-1)s_p^2+\sum_{i=1}^{m}n_i(\bar{x}_i-\bar{\bar{x}})^2},$$

$$b_{\sum_{i=1}^{m}(n_i-1)}=\sqrt{\frac{2}{\sum_{i=1}^{m}(n_i-1)}}\,\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)}{2}\right)\left[\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)-1}{2}\right)\right]^{-1}.$$

For the single sample, that is, $m=1$, $r=1$, and $s_p=s$, the results obtained in (4.3) and (4.4) can be reduced to those obtained in Cheng and Spiring (1989). Denoted $C^*=\hat{C}_p^*/w$. In fact, practitioners would be expected to find a bound which the true value of the process capability no less than the bound value with a certain level of confidence. When this happens, we have $\Pr\{C_p>w\mid\mathbf{X}\}>p$. Note that the posterior probability p depends on m, n_i, r, w and $\hat{C}_p$ only through m, n_i, r and $\hat{C}_p^*/w$. There is a one-to-one correspondence between p and C^* when m and n_i, are given, and by the fact that r and $\hat{C}_p^*$ can be calculated from the process data, we can find that the minimum value of C^* required to ensure the posterior probability p reaching a certain desirable level. Denote this minimum value as $C^*(p)$. Then, the value $C^*(p)$ satisfies

$$p=\Pr\{C_p>w\mid\mathbf{X}\}=\Pr\{C_p>\frac{\hat{C}_p^*}{C^*(p)}\mid\mathbf{X}\}.$$

Therefore, a $100p\%$ credible interval for C_p is $[\hat{C}_p^*/C^*(p),\infty)$, where p is a number between 0 and 1, say 0.95, for 95% confidence interval, which means that the posterior probability that the credible interval contains C_p is p. The credible interval (or called credible set) is a Bayesian analogue of the classical lower confidence interval. We say that the process is capable in a Bayesian sense if all the points in this credible interval are greater than a pre-specified value of w, say 1.00 or 1.33. In other words, to see if a process is capable (with capability level w and confidence level p), we only need to check if $\hat{C}_p^*>C^*(p)\times w$. If the estimated value $\hat{C}_p^*$ is greater than the critical value $C^*(p)\times w$, then we may conclude that the process meets the capability requirement ($C_p>w$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_p\le w$.

For users' convenience in applying our Bayesian procedure, we tabulate the minimum values $C^*(p)$ of $\hat{C}_p^*/w$, for various r with $m=2(2)10,15$, $n_i=n=10(5)30$ in Tables 4.1-4.3 to ensure $p=$ 0.99, 0.975, and 0.95, respectively. For example, if $w=1.33$ is the minimum capability requirement, then for $p=$ 0.95, with $m=$ 10 of each sample size $n_i=n=10$ and $r=0.90$, we can find $C^*(p)=$ 1.1297 by checking Table 4.3. Thus, the minimum value of $\hat{C}_p^*$ required for a capable process is $C^*(p)\times w$ = 1.1297×1.33 = 1.5026. That is, if $\hat{C}_p^*$ is greater than 1.5026, we say that the process is capable in Bayesian sense.

4.3. Bayesian Approach for C_{pk} Based on Multiple Samples

4.3.1. Estimation of C_{pk} Based on Multiple Samples

For the case when the studied characteristic of the process is normally distributed and we have m subsamples, where the sample size of the i-th subsample is n_i. For each i, $i = 1, 2, ..., m$, let x_{ij}, $j = 1, 2, ..., n_i$, be a random sample from a normal distribution with mean μ and variance σ^2 measuring the studied characteristic. We will assume that the process is in statistical control during the time period that the subsamples are taken. Consider the process is monitored using a $\bar{X}$ -chart together with a S -chart. Then, for each subsample, let $\bar{x}_i$ and s_i denote the sample mean and sample variance, respectively, of the i-th sample and let N denote the total number of observations, i.e.

$$\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}, \quad s_i^2 = \frac{1}{n_i - 1}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2 \text{, and } N = \sum_{i=1}^{m} n_i .$$

Furthermore, let $N_1 = \sum_{i=1}^{m}(n_i - 1) = N - m$. When all subsamples have the same size n, then $N = mn$ and $N_1 = m(n-1)$. As an estimator of μ and σ^2, we use the overall sample mean and the pooled sample variance, which are the unbiased estimators, i.e.

$$\hat{\mu} = \bar{\bar{x}} = \frac{1}{N}\sum_{j=1}^{n_i} n_i \bar{x}_i , \quad \hat{\sigma}^2 = s_p^2 = \frac{1}{N_1}\sum_{j=1}^{n_i}(n_i - 1)s_i^2 .$$

For the commonly used C_{pk} index, the estimator based on multiple samples can be considered as follows,

$$\hat{C}_{pk}^* = \min\left\{\frac{USL - \bar{\bar{x}}}{3s_p}, \frac{\bar{\bar{x}} - LSL}{3s_p}\right\} = \frac{d - |\bar{\bar{x}} - M|}{3s_p}. \tag{4.5}$$

4.3.2. Posterior Probability p with C_{pk} Based on Multiple Samples

Subsequently, we consider the quantity $p = \Pr\{\text{the process is capable} \mid \mathbf{X}\}$ in the Bayesian approach. Since the index C_{pk} is our main concern in this case, so we are interested in finding the posterior probability $p = \Pr\{C_{pk} > w \mid \mathbf{X}\}$ for some fixed positive number w. Therefore, given a pre-specified capability level $w > 0$, the posterior probability for index C_{pk} based on multiple samples that a process is capable is given as the following,

$$p = \Pr\{\text{the process is capable} \mid \mathbf{X}\} = \Pr\{C_{pk} > w \mid \mathbf{X}\}$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \{\Phi[b_1(y)] + \Phi[b_2(y)] - 1\}\, dy , \tag{4.6}$$

where

$$\alpha = (\sum_{i=1}^{m} n_i - 1)/2\,,\quad \delta = \frac{|\bar{\bar{x}} - m|}{s_p},\quad r = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{\bar{x}})^2} = \frac{\sum_{i=1}^{m}(n_i-1)s_p^2}{\sum_{i=1}^{m}(n_i-1)s_p^2 + \sum_{i=1}^{m} n_i(\bar{x}_i - \bar{\bar{x}})^2},$$

$$b_1(y) = 3\sqrt{\sum_{i=1}^{m} n_i}\left(\hat{C}_{pk}^{*} \times \sqrt{\frac{2r}{\sum_{i=1}^{m}(n_i-1)y}} - w\right),$$

$$b_2(y) = 3\sqrt{\sum_{i=1}^{m} n_i}\left((\hat{C}_{pk}^{*} + \frac{2}{3}\delta) \times \sqrt{\frac{2r}{\sum_{i=1}^{m}(n_i-1)y}} - w\right).$$

Detailed derivations of the posterior probability $p = \Pr\{C_{pk} > w \mid \mathbf{X}\}$ in (4.6) are presented in Appendix I. When m, n_i and w are given, $\hat{C}_{pk}$, δ and r are calculated from the observed data, we can find the minimum value of $\hat{C}_{pk}$, $C^{*}(p)$, required to ensure the posterior probability p reaching a certain desirable level w using the similar technique as single sample. Therefore, suppose for this particular process under consideration to be capable, the process index must reach at least a certain level w, say, 1.00 or 1.33. Based on the observed process data, we have $p = \Pr\{C_{pk} > w \mid \mathbf{x}\}$. Further, to see if a process is capable (with capability level w and confidence level p), we only need to check if $\hat{C}_{pk}^{*} > C^{*}(p)$. Thus, if $\hat{C}_{pk}^{*} > C^{*}(p)$, then we say that the process is capable in a Bayesian sense. Otherwise, we do not have sufficient information to conclude that the process meets the preset capability requirement, and then we tend to believe that the process is incapable in this case. To make this Bayesian procedure practical for in-plant applications, we calculate the values of $C^{*}(p)$ for various values of r = 0.7(0.1)1.0 and δ = 0(0.5)2.0 with m = 2(2)10, $n_i = n$ = 10, w = 1.00 in Tables 4.4(a)-4.4(b) for p = 0.95, 0.99, respectively. And the values of $C^{*}(p)$ for r = 0.7(0.1)1.0, δ = 0(0.5)2.0, m = 2(2)10, $n_i = n$ = 15, w = 1.00 are displayed in Tables 4.5(a)-4.5(b) for p = 0.95, 0.99, respectively. Tables 4.6(a)-4.6(b) and 4.7(a)-4.7(b) summarized the values of $C^{*}(p)$ for r = 0.7(0.1)1.0, δ = 0(0.5)2.0, w = 1.33 with m = 2(2)10, $n_i = n$ = 10 and 15, respectively. For example, if w = 1.00 is the minimum capability requirement, then for p = 0.95, with m = 10 of each sample size n = 15 and $r = 0.8$, $\delta = 0.5$ we can find $C^{*}(p)$ = 1.2480 by checking Table 4.5(b). Thus, the value of $\hat{C}_{pk}^{*}$ calculated from sample data must satisfy $\hat{C}_{pk}^{*}$ ≥ 1.2480 to conclude that C_{pk} ≥ 1.00 (process is capable).

4.4. Bayesian Approach for C_{pm} Based on Multiple Samples

4.4.1. Estimation of C_{pm} Based on Multiple Samples

For single sample, Boyles (1991) showed that $\hat{\tau}^2 = s_n^2 + (\bar{x} - T)^2$ is the unbiased estimator of $\sigma^2 + (\mu - T)^2$. Therefore, for cases where the data are collected as multiple samples, we consider m samples each of size n_i and suggest the following estimator of C_{pm}, where $\bar{x}_i$ is the i-th sample mean, and s_i is the i-th sample standard deviation,

$$\hat{C}^{*}_{pm} = \frac{d}{3\hat{\tau}'^2}, \quad \hat{\tau}'^2 = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-T)^2}{\sum_{i=1}^{m} n_i}. \tag{4.7}$$

First, by taking the expectation of the numerator of $\hat{\tau}'^2$, we obtain that

$$\begin{aligned}
E\left(\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-T)^2\right) &= E\left(\sum_{i=1}^{m}\sum_{j=1}^{n_i}x_{ij}^2\right) - 2T\times E\left(\sum_{i=1}^{m}\sum_{j=1}^{n_i}x_{ij}\right) + E\left(\sum_{i=1}^{m}\sum_{j=1}^{n_i}T^2\right)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n_i}E\left(x_{ij}^2\right) - 2T\times\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\left(x_{ij}\right) + \sum_{i=1}^{m}n_i T^2\\
&= \sum_{i=1}^{m}n_i\left(\mu^2+\sigma^2\right) - 2T\times\sum_{i=1}^{m}n_i\mu + \sum_{i=1}^{m}n_i T^2\\
&= \sum_{i=1}^{m}n_i\left[\sigma^2+(\mu-T)^2\right].
\end{aligned}$$

Thus, the estimator $\hat{\tau}'^2$, such that $E(\hat{\tau}'^2)=\sigma^2+(\mu-T)^2$, is the unbiased estimator of $\sigma^2+(\mu-T)^2$. However, for cases with multiple samples, we need to consider the variation between and within multiple samples. Thus, we define the ratio of total within sample variation (SSW) and total sum of square variation (SST) as

$$r = \frac{\text{SSW}}{\text{SST}} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{\bar{x}})^2} = \frac{\sum_{i=1}^{m}(n_i-1)s_p^2}{\sum_{i=1}^{m}(n_i-1)s_p^2 + \sum_{i=1}^{m}n_i(\bar{x}_i-\bar{\bar{x}})^2}, \tag{4.8}$$

where $s_p^2 = \sum_{i=1}^{m}(n_i-1)s_i^2 / \sum_{i=1}^{m}(n_i-1)$ is the pooled variance of these samples. Moreover, the total sample variation about target value T can be decomposed as:

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-T)^2 &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2 + \sum_{i=1}^{m}\sum_{j=1}^{n_i}(\bar{x}_i-\bar{\bar{x}})^2 + \sum_{i=1}^{m}\sum_{j=1}^{n_i}(\bar{\bar{x}}-T)^2\\
&= \sum_{i=1}^{m}(n_i-1)s_p^2 + \frac{1-r}{r}\sum_{i=1}^{m}(n_i-1)s_p^2 + \sum_{i=1}^{m}n_i\delta^2 s_p^2 = \left(\frac{1}{r}\sum_{i=1}^{m}(n_i-1)+\sum_{i=1}^{m}n_i\delta^2\right)s_p^2.
\end{aligned}$$

Thus, the generalization of the estimator of C_{pm} for multiple samples defined in (4.7) can be rewritten as:

$$\hat{C}^{*}_{pm} = \frac{d}{3s_p\sqrt{\frac{\sum_{i=1}^{m}(n_i-1)}{r\sum_{i=1}^{m}n_i}+\delta^2}}, \quad \delta = \frac{|\bar{\bar{x}}-T|}{s_p}. \tag{4.9}$$

For the single sample, that is, $m = 1$, $r = 1$, and $s_p = s$, the estimator of C_{pm}, $\hat{C}^*_{pm} = d/(3s\sqrt{(n-1)/n+\delta^2})$, which can be reduced to the estimator $\hat{C}_{pm}$ defined in Boyles (1991). Therefore, we may view the estimator $\hat{C}^*_{pm}$ as a direct extension of $\hat{C}_{pm}$.

4.4.2. Posterior Probability p with C_{pm} Based on Multiple Samples

Shiau *et al.* (1999) proposed a Bayesian procedure for the index C_{pm} on cases with one single sample. As we mentioned earlier, it is practical to develop a procedure for assessing process capability for cases with multiple samples. Therefore, we now are interested in finding the posterior probability $p = \Pr\{C_{pm} > w \mid \mathbf{X}\}$ based on multiple samples collected from a in-control process. Given a pre-specified precision level $w > 0$, the posterior probability based on index C_{pm} that a process is capable is given as follows, where $\Phi(\cdot)$ is the CDF of the standard normal distribution, and r and δ are defined as in (4.8) and (4.9).

$$p = \Pr\{C_{pm} > w \mid \mathbf{X}\}$$
$$= \int_0^t \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times [\Phi(b_1(y)+b_2(y)) - \Phi(b_1(y)-b_2(y))]\, dy, \qquad (4.10)$$

$$t = \frac{2}{\sum_{i=1}^m (n_i-1)} \left(\frac{\hat{C}^*_{pm}}{w}\right)^2 \left(\frac{\sum_{i=1}^m (n_i-1)}{\sum_{i=1}^m n_i} + r\,\delta^2\right),$$

$$b_1(y) = \sqrt{\frac{2r\sum_{i=1}^m n_i}{\sum_{i=1}^m (n_i-1)y}}\,\delta,$$

$$b_2(y) = \sqrt{\sum_{i=1}^m n_i}\left(\frac{t}{y}-1\right)^{1/2}.$$

All the derivations of (4.10) are given in Appendix II. The results obtained in our investigation, is a generalization of those obtained in Shiau *et al.* (1999) from one single sample case to multiple samples case based on control chart data. Note that the posterior probability p depends on m, n_i, r, w and $\hat{C}^*_{pm}$ only through m, n_i, r, δ and $\hat{C}^*_{pm}/w$. If C^* is denoted by $C^* = \hat{C}^*_{pm}/w$ in (4.10), we can see that there is a one-to-one correspondence between p and C^* when m and n_i are given, and r, δ and $\hat{C}^*_{pm}$ are calculated from the process data. The minimum value of C^* can be found to ensure the posterior probability p reaching a certain desirable level. This minimum value is denoted by $C^*(p)$. Thus, the value of $C^*(p)$ satisfies

$$p = \Pr\{C_{pm} > w \mid \mathbf{X}\} = \Pr\{C_{pm} > \frac{\hat{C}^*_{pm}}{C^*(p)} \mid \mathbf{X}\}.$$

A 100p% credible interval for C_{pm} is $[\hat{C}^*_{pm}/C^*(p), \infty)$, where p is a number between 0 and 1, say 0.95, for a 95% confidence interval. This means that the posterior probability that the credible interval contains C_{pm} is p. Therefore, for practitioners' convenience in applying our Bayesian procedure, we tabulate the minimum values $C^*(p)$ of $\hat{C}^*_{pm}/w$ for various values of r = 0.7(0.1)1.0 and δ = 0(0.5)2.0 with n = 5(5)20, m = 2(2)10 in Tables 4.8(a)-4.8(d) and Tables 4.9(a)-4.9(d) for p = 0.95, 0.99, respectively. For example, if w = 1.33 is the minimum capability requirement, then for p = 0.95, with m = 10 of each sample size $n_i = n$ = 10 and $r = 0.9$, $\delta = 0.5$ we can find $C^*(p)$ = 1.1569 from Table 4.8(b). Thus, the minimum value of $\hat{C}^*_{pm}$ required for the process is capable is $C^*(p)\times w$ = 1.1569×1.33 = 1.5387. That is, if $\hat{C}^*_{pm}$ is greater than 1.5387, we say that the process is capable in a Bayesian sense.

From these tables we observe that for each fixed p, m, n and r the value of $C^*(p)$ decreases as δ increases. This phenomenon can be explained by the relationship of $\hat{C}^*_{pm}$ in (4.9). For a fixed $\hat{C}^*_{pm}$, s_p becomes smaller when δ becomes larger, and a smaller s_p means that it is plausible that the underlying process is tighter (i.e. with smaller σ). Since the estimation is usually more accurate with the data drawn from a tighter process, it is then plausible that the estimate $\hat{C}^*_{pm}$ is more accurate with a smaller s_p and the required minimum value $C^*(p)$ is smaller, since we need only a smaller $C^*(p)$ to account for the smaller uncertainty in the estimation. Intuitively, if the estimation error in our estimate is potentially large, then it is reasonable that we need a large $\hat{C}^*_{pm}$ to be able to claim that the process is capable, and this means that the corresponding minimum value $C^*(p)$ should be large as well. Thus, the value of $C^*(p)$ decreases as δ increases, and this pattern is consistent with Shiau *et al.* (1999) for single sample. On the other hand, according to the definition of r as (4.8) becomes larger, the variation between these multiple samples will become smaller when the other conditions are fixed. And the smaller the variation is between these multiple samples, the more stable the process. Thus, we need only a smaller $C^*(p)$ to assess the process capability. Another observation from the tables is that the value of $C^*(p)$ decreases as n and/or m increases for fixed δ, r and p. This can also be explained by the same reasoning as above, since the estimation will be more accurate with a larger sample size.

4.5. Bayesian Approach for C_{PU} and C_{PL} Based on Multiple Samples

4.5.1. Estimations of C_{PU} and C_{PL} Based on Multiple Samples

For cases where the data are collected as multiple control samples, we considered m samples each of size n_i and suggested the following unbiased estimators of C_{PU} and C_{PL}, where $\bar{x}_i$ is the i-th sample mean, and s_i is the i-th sample standard deviation, $N = \sum_{i=1}^m n_i$ is the number of total sample:

$$\tilde{C}^*_{PU} = \frac{b_{N-m}(USL - \bar{\bar{x}})}{3s_p}, \quad \tilde{C}^*_{PL} = \frac{b_{N-m}(\bar{\bar{x}} - LSL)}{3s_p}, \tag{4.11}$$

where $b_g = (2/g)^{1/2}\Gamma(g/2)/\Gamma[(g-1)/2]$, $g = N - m$, $\bar{\bar{x}} = \sum_{i=1}^m n_i \bar{x}_i / N$ is the overall sample mean, and $s_p^2 = \sum_{i=1}^m (n_i - 1)s_i^2/(N-m)$ is the pooled sample variance, which are the unbiased estimators of μ and σ^2 respectively.

4.5.2. Posterior Probability p for C_{PU} and C_{PL} Based on Multiple Samples

For cases with multiple samples, the extension of posterior probability p for C_I may be derived using the similar technique for cases with one single sample as:

$$p = \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \Phi\left[3\sqrt{N}\left(\frac{\tilde{C}_I^*}{b_{N-m}} \times \sqrt{\frac{2r}{(N-m)y}} - w\right)\right] dy, \quad (4.12)$$

where $\alpha = (N-1)/2$, $N = \sum_{i=1}^{m} n_i$ and r is defined as (4.8), $\tilde{C}_I^*$ denotes either $\tilde{C}_{PU}^*$ or $\tilde{C}_{PL}^*$. A brief derivation of the posterior probability p for C_I is given in the Appendix III. We note that for cases with one single sample, the ratio $r = 1$, $N - m = n - 1$, $\alpha = (n-1)/2$, then the estimator of C_I, $\tilde{C}_I^*$, and the posterior probability p for C_I can be reduced to the results obtained before.

As noted earlier, the posterior probability p depends on n, w and $\tilde{C}_I^*$. When n and w are given, and by the fact that $\tilde{C}_I^*$ and r can be calculated from the process data. Therefore, we can find that the minimum value of $C^*(p)$ required to ensure the posterior probability p reaching a certain desirable level. Thus, to see if a process is capable (with capability level w and confidence level p), we only need to check if $\tilde{C}_I^* > C^*(p)$. For users' convenience in applying this Bayesian procedure with multiple samples, we provide tables of the minimum values of $\tilde{C}_I^*$, $C^*(p)$ for confidence levels p = 0.95 and 0.99, with commonly used capability requirements C_I = 1.00, 1.25, 1.45, 1.60 in Tables 4.10(a)-4.13(b) respectively for various values of r = 0.7(0.1)1.0 and δ = 0(0.5)2.0 with m = 2(2)10, n = 5(5)15.

4.6. Application Examples

4.6.1. Example 1: Assessing the Resistor Process Capability

To illustrate the application of assessing process capability for multiple samples collected over time from a in-control process, we consider a real example on an electronic component manufacturer, which developing passive and active components for the personal computers, telecommunications, industrial controls, automotive parts, and avionics. The factory manufactures various types of the resistors. For a particular model of the resistors investigated, the target value is set to T = 10.0 mil, and the tolerance of thickness is 2.0 mil, that is, the lower and upper specification limit are set to LSL = 8.0 mil and USL = 12.0 mil. If the characteristic data does not fall within the tolerance $[LSL, USL]$, the lifetime or reliability of the resistors will be discounted. The collected sample data (10 samples of size 15 each) are displayed in Table 4.14.

In order to make the estimation of these capability indices meaningful, we would check if the manufacturing process is under statistical control and the distribution is normal. For those 10 samples of size 15 each, the Shapiro-Wilk test for normality confirms this with p-value > 0.1. That is, it is reasonable to assume that the process data collected from the factory is normally distributed. We then construct the $\bar{X} - S$ charts to check if the process is in control. The $\bar{X} - S$ charts based on the collected samples are displayed in Figures 4.1-4.2. The $\bar{X} - S$ control charts show that the process seems to be in-control since all the sample points are within the control limits without any special pattern. Therefore, the basic assumptions are justified so we could proceed with the capability calculations.

Table 4.14 The 10 subsamples of 15 observations for the thickness of resistor (unit: mil).

Samples									
1	2	3	4	5	6	7	8	9	10
10.21	9.66	9.80	9.48	10.74	10.71	10.00	10.09	10.58	10.23
10.19	10.36	9.96	9.91	9.72	10.36	10.12	10.12	10.42	10.44
9.88	10.55	10.04	9.94	10.34	10.17	10.29	9.99	9.58	9.86
10.73	10.31	9.99	9.93	10.88	10.53	9.62	10.57	10.44	10.16
10.59	9.72	10.35	10.08	10.48	10.15	9.98	10.50	10.39	10.14
10.21	10.00	9.94	9.59	10.01	10.09	10.00	9.43	10.87	9.99
10.61	10.34	10.96	10.01	10.71	10.14	10.12	10.60	9.56	11.12
10.68	9.77	10.33	9.85	10.15	9.76	9.97	9.86	10.26	10.10
9.86	10.12	10.39	10.50	10.46	10.15	10.56	9.90	10.16	10.00
10.69	10.40	10.63	9.77	10.38	10.36	10.60	9.84	10.46	9.97
10.12	11.11	9.13	9.97	10.39	10.28	9.76	10.31	9.83	10.50
10.62	10.25	10.57	10.03	10.33	10.05	9.78	10.03	10.09	10.47
9.73	11.03	10.24	10.02	10.33	9.50	9.74	9.53	10.43	10.30
10.35	10.23	10.65	10.37	10.15	10.29	10.48	9.72	10.38	10.17
10.51	9.98	10.70	9.81	10.26	10.29	9.79	10.56	10.27	10.04

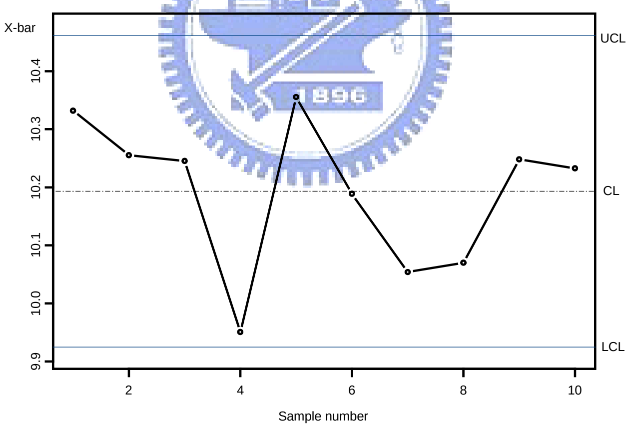

Figure 4.1 $\bar{X}$ control chart of the process.

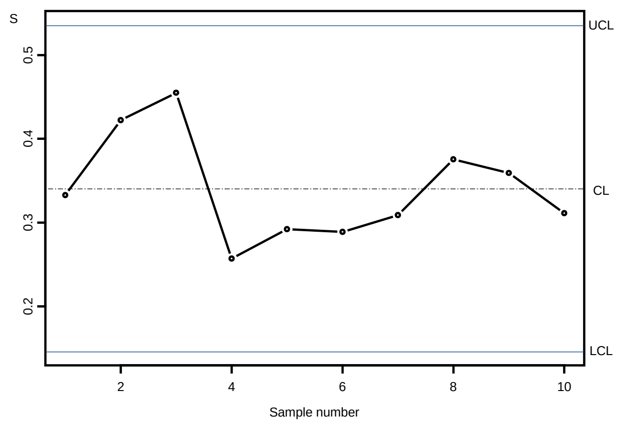

Figure 4.2 S control chart of the process.

The calculated sample mean $\bar{x}_i$ and the sample variance s_i^2 for the ten samples of size 15 are summarized in Table 4.15. Thus,

$$\bar{\bar{x}} = 10.1929,\quad s_p^2 = \sum_{i=1}^{m} s_i^2 / m = 0.1192,$$

$$r = \frac{m(n-1)s_p^2}{m(n-1)s_p^2 + n\sum_{i=1}^{m}(\bar{x}_i - \bar{\bar{x}})^2} = 0.8816, \qquad \delta = \frac{|\bar{\bar{x}} - T|}{s_p} = 0.5587,$$

$$\hat{C}_{pm}^* = \frac{d}{3s_p\sqrt{\frac{n-1}{rn} + \delta^2}} = 1.6489.$$

Suppose for this particular process under consideration to be capable, the process index must reach at least a certain level w, say, 1.33. Thus, by solving the equation (4.10) the critical value of $\hat{C}_{pm}^*$ can be obtained as $\hat{C}^*(p) \times w = 1.1069 \times 1.33 = 1.4722$ based on p = 0.95, m = 10, n =15. Since the calculated $\hat{C}_{pm}^*$ from the samples, 1.6489, is greater than the critical value $C^*(p) \times w =$ 1.4722, we may conclude, with 95 % confidence level, that the process meets the capability requirement "$C_{pm} > 1.33$" in this case. In addition to know that the process is capable, we also have the probability $\Pr\{C_{pm} > w \mid \mathbf{X}\} = 0.99976$ and the 95% lower confidence bound of C_{pm} is $\hat{C}_{pm}^* / C^*(p) = 1.6489 / 1.1069 = 1.4897$ based on collected samples.

Table 4.15 The calculated sample mean and the sample variance for the 10 subsamples.

Sample i	1	2	3	4	5	6	7	8	9	10
$\bar{x}_i$	10.332	10.255	10.245	9.951	10.354	10.188	10.053	10.070	10.247	10.233
s_i^2	0.110	0.178	0.207	0.066	0.085	0.083	0.096	0.141	0.129	0.097

4.6.2. Example 2: Assessing the Single Coupler Process Capability

The rapid development of optical and photonic technologies for a variety of applications has resulted in a similarly rapid need for all-optical systems, and thus the need for passive optical components. The number of stations or nodes on an all-optical fiber data bus is limited by the total allowable system loss. Consider a company devoted to the optical fiber component module manufacturing, such as Collimator, Isolator, Coupler, DWDM, CWDM, and EDFA, etc. The insertion loss (IL) is the most critical quality characteristic the company focuses on, which has significant impact to product quality. For the insertion loss of a model of single coupler with coupling ratio 50/50 (%), the upper specification limit, USL, is set to 3.5 dB. The capability requirement for this particular model of single mode couplers was defined as "Satisfactory" if $C_{PU} > 1.25$.

We note that to make the estimation of the capability indices meaningful, it is essential to check whether the manufacturing process is under statistical control and the normally distributed. The process has been justified to be well in-control and near normally distributed. A sample data collection procedure is implemented in the factory on a daily basis to monitor/control process quality. The collected data of 15 samples each of size 10 are taken from a stable process (under statistical control), and displayed in Table 4.16.

Table 4.16 The 15 subsamples of 10 observations with calculated sample statistics.

Sample i	Observations										$\bar{x}_i$	s_i^2
1	3.37	3.28	3.30	3.38	3.37	3.34	3.26	3.29	3.34	3.35	3.328	0.001796
2	3.37	3.38	3.37	3.40	3.35	3.32	3.34	3.31	3.30	3.25	3.339	0.002010
3	3.37	3.30	3.35	3.33	3.36	3.34	3.36	3.36	3.30	3.32	3.339	0.000654
4	3.35	3.31	3.29	3.32	3.31	3.32	3.34	3.27	3.34	3.35	3.320	0.000689
5	3.35	3.32	3.31	3.29	3.40	3.29	3.36	3.32	3.32	3.33	3.329	0.001121
6	3.39	3.33	3.35	3.35	3.34	3.33	3.44	3.33	3.38	3.40	3.364	0.001382
7	3.36	3.32	3.37	3.35	3.33	3.37	3.29	3.32	3.36	3.30	3.337	0.000846
8	3.35	3.30	3.31	3.26	3.31	3.34	3.32	3.32	3.35	3.33	3.319	0.000721
9	3.30	3.34	3.31	3.36	3.29	3.34	3.32	3.36	3.35	3.34	3.331	0.000610
10	3.37	3.32	3.35	3.33	3.30	3.31	3.34	3.41	3.29	3.29	3.331	0.001454
11	3.29	3.38	3.27	3.36	3.33	3.34	3.38	3.34	3.34	3.32	3.335	0.001250
12	3.40	3.31	3.40	3.31	3.30	3.25	3.30	3.36	3.33	3.35	3.331	0.002232
13	3.38	3.33	3.29	3.40	3.32	3.29	3.32	3.31	3.35	3.34	3.333	0.001290
14	3.29	3.29	3.29	3.30	3.28	3.29	3.32	3.35	3.34	3.30	3.305	0.000561
15	3.33	3.34	3.41	3.28	3.34	3.35	3.35	3.29	3.33	3.26	3.328	0.001818

The individual observation plot of each sample with respect to the upper specification limit is displayed in Figure 4.3. The calculated sample mean $\bar{x}_i$ and the sample variance s_i^2 for the fifteen samples are summarized in the last two columns of Table 4.16. Thus, we have $s_p^2 = \sum_{i=1}^{m} s_i^2 / m = 0.001229$, $\bar{\bar{x}} = 3.3313$ $r = 0.8813$, and $\tilde{C}_{PU}^* = b_{m(n-1)} \times (USL - \bar{\bar{x}})/(3s_p)$ = 1.5931 based on these samples.

By solving the posterior probability in (4.12), the minimum value of $\tilde{C}_{PU}^*$ is found to be $C^*(p)$ = 1.4025 based on p = 0.95 with $m = 15$, $n = 10$ and w = 1.25. Since $\tilde{C}_{PU}^*$ = 1.5931 is larger than the critical value $C^*(p)$ = 1.4025 in this case, we therefore conclude that with 95% confidence the single coupler manufacturing process satisfies the requirement "$C_{PU} > 1.25$". i.e., the produced single couplers are conformed to the manufacturing specifications with fraction of nonconformities 88 PPM.

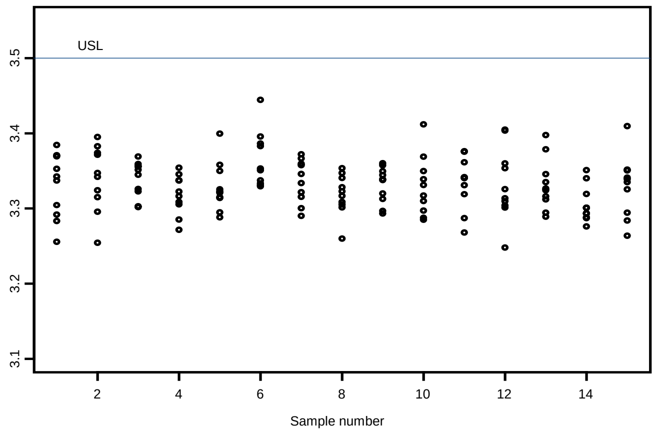

Figure 4.3 Individual observations plot of each sample.