Appendix I

Derivation of expression (4.6)

For the multiple samples, we have the posterior probability density function (PDF), $f(\mu,\sigma \mid \mathbf{X})$ of (μ,σ) expressed in (4.1) as

$$f(\mu,\sigma \mid \mathbf{X}) = \frac{2\sqrt{\sum_{i=1}^{m} n_i}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^{\alpha}}\sigma^{-(\sum_{i=1}^{m} n_i+1)} \times \exp\left(-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\mu)^2}{2\sigma^2}\right).$$

Therefore, given a pre-specified capability level $w > 0$, the posterior probability based on index C_{pk} that a process is capable is given as

$$p = \Pr\{C_{pk} > w \mid \mathbf{X}\} = \Pr\left\{\frac{d-|\mu-M|}{3\sigma} > w \;\middle|\; \mathbf{X}\right\}$$

$$= \Pr\{|\mu - M| < d - 3\sigma w \mid \mathbf{X}\}$$

$$= \Pr\{M - d + 3\sigma\omega < \mu < M + d - 3\sigma w \mid \mathbf{X}\}$$

$$= \int_0^{\infty}\int_{M-d+3\sigma w}^{M+d-3\sigma w} f(\mu,\sigma \mid \mathbf{X})\, d\mu d\sigma$$

$$= \int_0^{\infty}\int_{M-d+3\sigma w}^{M+d-3\sigma w} \frac{2\sqrt{\sum_{i=1}^{m} n_i}}{\sqrt{2\pi}\Gamma(\alpha)\beta^{\alpha}}\sigma^{-(\sum_{i=1}^{m} n_i+1)} \times \exp\left(-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\mu)^2}{2\sigma^2}\right) d\mu d\sigma$$

$$= \int_0^{\infty} \frac{2\sqrt{\sum_{i=1}^{m} n_i}}{\sqrt{2\pi}\Gamma(\alpha)\beta^{\alpha}}\sigma^{-(\sum_{i=1}^{m} n_i+1)} \times \exp\left(-\frac{1}{\beta\sigma^2}\right)\int_{M-d+3\sigma w}^{M+d-3\sigma w} \exp\left(-\frac{\sum_{i=1}^{m} n_i(\mu-\bar{\bar{x}})^2}{2\sigma^2}\right) d\mu\, d\sigma$$

$$= \int_0^{\infty} \frac{2\sigma^{-\sum_{i=1}^{m} n_i}}{\Gamma(\alpha)\beta^{\alpha}}\exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{M+d-3\sigma w-\bar{\bar{x}}}{\sigma/\sqrt{\sum_{i=1}^{m} n_i}}\right) - \Phi\left(\frac{M-d+3\sigma w-\bar{\bar{x}}}{\sigma/\sqrt{\sum_{i=1}^{m} n_i}}\right)\right] d\sigma$$

$$= \int_0^{\infty} \frac{2\sigma^{-\sum_{i=1}^{m} n_i}}{\Gamma(\alpha)\beta^{\alpha}}\exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{d+(M-\bar{\bar{x}})}{s_p/\sqrt{\sum_{i=1}^{m} n_i}} \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^{m} n_i}\, w\right)\right.$$

$$\left. -\Phi\left(\frac{-d+(M-\bar{\bar{x}})}{s_p/\sqrt{\sum_{i=1}^{m} n_i}} \times \frac{s_p}{\sigma} + 3\sqrt{\sum_{i=1}^{m} n_i}\, w\right)\right] d\sigma$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(\frac{d-(\bar{\bar{x}}-M)}{s_p / \sqrt{\sum_{i=1}^m n_i}} \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^m n_i w} \right) + \Phi\left(\frac{d-(M-\bar{\bar{x}})}{s_p / \sqrt{\sum_{i=1}^m n_i}} \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^m n_i w} \right) - 1 \right] d\sigma. \tag{A.1}$$

where $\alpha = (\sum_{i=1}^m n_i - 1)/2$, $\beta = [\sum_{i=1}^m \sum_{j=1}^{n_i} (x_{ij} - \bar{\bar{x}})^2 / 2]^{-1}$, $\bar{\bar{x}} = \sum_{i=1}^m \sum_{j=1}^{n_i} x_{ij} / \sum_{j=1}^{n_i} x_{ij}$, $s_p^2 = \sum_{i=1}^m (n_i - 1)s_i^2 / \sum_{i=1}^m (n_i - 1)$ is the pooled variance of subsamples and $\Phi(\cdot)$ is the CDF of $N(0,1)$. Next, we consider the two cases for derivation of posterior probability p as follows:

[Case 1]:

If $\bar{\bar{x}} \ge M$, then $\hat{C}_{pk}^* = \frac{d-(\bar{\bar{x}}-M)}{3s_p}$ and $\frac{d-(M-\bar{\bar{x}})}{3s_p} = \frac{d+(\bar{\bar{x}}-M)}{3s_p} = \hat{C}_{pk}^* + \frac{2}{3}\delta$,

where $\delta = |\bar{\bar{x}} - m| / s_p$. Thus, the expression (A.1) can be rewritten as

$$p = \Pr\{C_{pk} > w \mid \mathbf{X}\}$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(3\sqrt{\sum_{i=1}^m n_i} \hat{C}_{pk}^* \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^m n_i w} \right) + \Phi\left(3\sqrt{\sum_{i=1}^m n_i} \left(\hat{C}_{pk}^* + \frac{2}{3}\delta \right) \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^m n_i w} \right) - 1 \right] d\sigma. \tag{A.2}$$

[Case 2]:

If $\bar{\bar{x}} < M$, then $\frac{d-(\bar{\bar{x}}-M)}{3s_p} = \frac{d+(M-\bar{\bar{x}})}{3s_p} = \hat{C}_{pk}^* + \frac{2}{3}\delta$ and $\frac{d-(M-\bar{\bar{x}})}{3s_p} = \hat{C}_{pk}^*$,

where $\delta = |\bar{\bar{x}} - m| / s_p$. Thus, the expression (A.1) can be rewritten as

$$p = \Pr\{C_{pk} > w \mid \mathbf{X}\}$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left[\Phi\left(3\sqrt{\sum_{i=1}^m n_i} \left(\hat{C}_{pk}^* + \frac{2}{3}\delta \right) \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^m n_i w} \right) + \Phi\left(3\sqrt{\sum_{i=1}^m n_i} \hat{C}_{pk}^* \times \frac{s_p}{\sigma} - 3\sqrt{\sum_{i=1}^m n_i w} \right) - 1 \right] d\sigma. \tag{A.3}$$

From both cases expressed in (A.2) and (A.3), we obtain that

$$p = \Pr\{C_{pk} > w \mid \mathbf{X}\}$$

$$= \int_0^\infty \frac{2\sigma^{-\sum_{i=1}^m n_i}}{\Gamma(\alpha)\beta^\alpha} \exp\left(-\frac{1}{\beta\sigma^2}\right) \times \left\{ \Phi\left[3\sqrt{\sum_{i=1}^m n_i} \left(\hat{C}_{pk}^* \times \frac{s_p}{\sigma} - w \right) \right] + \Phi\left[3\sqrt{\sum_{i=1}^m n_i} \left((\hat{C}_{pk}^* + \frac{2}{3}\delta) \times \frac{s_p}{\sigma} - w \right) \right] - 1 \right\} d\sigma. \tag{A.4}$$

By changing the variable, we let $y = \beta\sigma^2$. Then, $dy = 2\beta\sigma\, d\sigma$, and $s_p / \sigma = \sqrt{2r / \sum_{i=1}^{m}(n_i - 1)y}$. Thus, the posterior probability p for multiple samples, which given in (A.4) can be simplified to:

$$p = \Pr\{\text{the process is capable} \mid \mathbf{X}\} = \Pr\{C_{pk} > w \mid \mathbf{X}\}$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \left\{ \Phi\left[3\sqrt{\sum_{i=1}^{m} n_i} \left(\hat{C}_{pk}^* \times \sqrt{\frac{2r}{\sum_{i=1}^{m}(n_i - 1)y}} - w \right) \right] \right.$$

$$\left. + \Phi\left[3\sqrt{\sum_{i=1}^{m} n_i} \left((\hat{C}_{pk}^* + \frac{2}{3}\delta) \times \sqrt{\frac{2r}{\sum_{i=1}^{m}(n_i - 1)y}} - w \right) \right] - 1 \right\} dy$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) y^{\alpha+1}} \exp\left(-\frac{1}{y}\right) \times \{ \Phi[b_1(y)] + \Phi[b_2(y)] - 1 \}\, dy ,$$

where

$$\alpha = (\sum_{i=1}^{m} n_i - 1)/2 , \quad \delta = \frac{|\bar{\bar{x}} - m|}{s_p} , \quad r = \frac{\sum_{i=1}^{m}\sum_{j=1}^{ni}(x_{ij} - \bar{x}_i)^2}{\sum_{i=1}^{m}\sum_{j=1}^{ni}(x_{ij} - \bar{\bar{x}})^2} = \frac{\sum_{i=1}^{m}(n_i - 1)s_p^2}{\sum_{i=1}^{m}(n_i - 1)s_p^2 + \sum_{i=1}^{m} n_i(\bar{x}_i - \bar{\bar{x}})^2} ,$$

$$b_1(y) = 3\sqrt{\sum_{i=1}^{m} n_i} \left(\hat{C}_{pk}^* \times \sqrt{\frac{2r}{\sum_{i=1}^{m}(n_i - 1)y}} - w \right) ,$$

$$b_2(y) = 3\sqrt{\sum_{i=1}^{m} n_i} \left((\hat{C}_{pk}^* + \frac{2}{3}\delta) \times \sqrt{\frac{2r}{\sum_{i=1}^{m}(n_i - 1)y}} - w \right) .$$

Therefore, the posterior probability p that a process is capable based on the index C_{pk} for multiple samples, which given in (4.6) can be derived.